M a t h e m a t i c s

ON THE BOUNDEDNESS OF A CLASS OF THE FIRST ORDER LINEAR DIFFERENTIAL OPERATORS

V. Zh. DUMANYAN[*]

Chair of Numerical Analysis and Mathematical Modeling, YSU

In the present article the first order linear differential operators with unbounded coefficients are investigated. The boundedness of the operators under consideration was proved.

***Keywords*:** bounded differential operator, first order differential operator.

Let $Q \subset R^n$, $n \geq 2$, be a bounded domain with smooth boundary $\partial Q \in C^1$. Consider the first order differential expression

$$Tu \equiv (\bar{b}(x), \nabla u(x)) - div(\bar{c}(x)u(x)) + d(x)u(x), \quad u \in \overset{\circ}{W}_2^1(Q),$$

with coefficients $\bar{b}(x) = (b^{(1)}(x),...,b^{(n)}(x))$, $\bar{c}(x) = (c^{(1)}(x),...,c^{(n)}(x))$ and $d(x)$ that are measurable and bounded on each strong inner subdomain of the domain Q.

For an arbitrary $u, v \in \overset{\circ}{W}_2^1(Q)$ define

$$\langle Tu, v \rangle \equiv \int_Q ((\bar{b}(x), \nabla u(x))v(x) + (\bar{c}(x)u(x), \nabla v(x)) + d(x)u(x)v(x))dx.$$

The aim of this article is to obtain conditions to be imposed on the coefficients $\bar{b}(x), \bar{c}(x)$ and $d(x)$, for which T is a linear bounded operator acting from $\overset{\circ}{W}_2^1(Q)$ into $\overset{\circ}{W}_2^{-1}(Q)$. This property has important applications in studying the problems of mathematical physics (see, for example, [1, 2]).

The following theorem is proved.

T h e o r e m . Let the following conditions hold

$$\left|\bar{b}(x)\right| = O\left(\frac{1}{r(x)}\right) \text{ as } r(x) \to 0, \tag{1}$$

where $r(x)$ is the distance of a point $x \in Q$ from the boundary ∂Q,

$$\int_0 tC^2(t)dt < \infty, \text{ where } C(t) = \sup_{r(x) \geq t}\left|\bar{c}(x)\right|, \tag{2}$$

$$\int_0 t^3 D^2(t)dt < \infty, \text{ where } D(t) = \sup_{r(x) \geq t}\left|d(x)\right|. \tag{3}$$

---

[*] E-mail: duman@ysu.am

Then the operator T is a bounded linear operator from $\overset{\circ}{W}{}_2^1(Q)$ into $\overset{\circ}{W}{}_2^{-1}(Q)$.

Proof of Theorem. Let $x^0 \in \partial Q$ be an arbitrary point of the boundary ∂Q of the domain Q, (x', x_n) be a local coordinate system with the origin x^0 and the x_n axis directed along the inner normal $\nu(x^0)$ to ∂Q at the point x^0. Since, $\partial Q \in C^1$, there exists a positive number $r_{x^0} > 0$ and a function $\varphi_{x^0} \in C^1(R^{n-1})$ with properties

$$\varphi_{x^0}(0) = 0, \ \nabla \varphi_{x^0}(0) = 0 \ \text{ and } \ \left|\nabla \varphi_{x^0}(x')\right| \le \frac{1}{2} \ \text{ for all } \ x' \in R^{n-1},$$

such that the intersection of the domain Q with the ball $U_{x^0}^{(r_{x^0})} = \{x : \left|x - x^0\right| < r_{x^0}\}$ of radius r_{x^0} and the centre x^0 has the form $Q \bigcap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \bigcap \{(x', x_n) : x_n > \varphi_{x^0}(x')\}$.

Then $\partial Q \bigcap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \bigcap \{(x', x_n) : x_n = \varphi_{x^0}(x')\}$. Let $l_{x^0} = \dfrac{r_{x^0}}{\sqrt{2}}$. From the covering $\{U_{x^0}^{(l_{x^0})}, x^0 \in \partial Q\}$ of the boundary ∂Q select a finite subcovering $U_{x^m}^{(l_{x^m})}$, $m = 1, ..., p$. Denote for simplicity $U_{x^m}^{(l_{x^m})}$ by U_m, r_{x^m} by r_m, l_{x^m} by l_m, φ_{x^m} by φ_m, $m = 1, ..., p$. Now set $h = \dfrac{1}{3}\left(\dfrac{2}{\sqrt{5}} - \dfrac{\sqrt{2}}{2}\right) \min(r_1, ..., r_p)$. Then each of the curvilinear "cylinders "

$$\Pi_m^{l_m, h} = \{(x', x_n) : \left|x'\right| < l_m, \varphi_m(x') < x_n < \varphi_m(x') + h\}, \ m = 1, ..., p,$$

is contained in the corresponding ball U_m, as well as in $U_m \bigcap Q$ (recall that (x', x_n) are the coordinates of a point in a local system of coordinates with origin at x^m). Let $l_0 < h$ be a positive number such that the complement of the domain $Q_{l_0} = \{x \in Q : r(x) = \text{dist}(x, \partial Q) > l_0\}$ in Q is contained in the union of the "cylinders" $\Pi_m^{l_m, h}$, $m = 1, ..., p$, i.e. $Q^{l_0} = \{x \in Q : r(x) = \text{dist}(x, \partial Q) \le l_0\} \subset \bigcup\limits_{m=1}^{p} \Pi_m^{l_m, h}$.

It is easily verified that for all $x = (x', x_n) \in \Pi_m^{l_m, h}$, $m = 1, ..., p$,

$$r(x) \le x_n - \varphi_m(x') \le \frac{\sqrt{5}}{2} r(x).$$

Now fix some number m, $1 \le m \le p$, and take a local coordinate system with origin at x^m.

We define mappings L and L_{-1} of the space R^n onto itself using relations $L(x) = (x', x_n - \varphi_m(x'))$, where $x = (x', x_n)$ and $L_{-1}(y) = (y', y_n + \varphi_m(y'))$, $y = (y', y_n)$. The image of $\Pi_m^{l_m, h}$ under the mapping L will be denoted by $\tilde{\Pi}_m^{l_m, h}$:

$$L\left(\Pi_m^{l_m, h}\right) = \tilde{\Pi}_m^{l_m, h}.$$

Now take arbitrary functions $u \in \overset{\circ}{W}{}_2^1(Q)$ and $\eta \in C_0^\infty(Q)$ and make the notations $u(y', y_n + \varphi(y')) = \tilde{u}(y)$, $\eta(y', y_n + \varphi(y')) = \tilde{\eta}(y)$.

In view of (1), (2) and (3) we have

$$\left|\langle Tu,\eta\rangle\right|\le K\int_Q\frac{|\nabla u(x)||\eta(x)|}{r(x)}dx+\int_Q C(r(x))|u(x)||\nabla\eta(x)|dx+\int_Q D(r(x))|u(x)||\eta(x)|dx,$$

where K is a constant.

Let us estimate

$$I_1=\int_Q\frac{|\nabla u(x)||\eta(x)|}{r(x)}dx\le\int_{Q^{l_0}}\frac{|\nabla u(x)||\eta(x)|}{r(x)}dx+\frac{1}{l_0}\int_{Q_{l_0}}|\nabla u(x)||\eta(x)|dx\le$$

$$\le\sum_{m=1}^p\int_{\Pi_m^{l_m,h}}\frac{|\nabla u(x)||\eta(x)|}{r(x)}dx+\frac{1}{l_0}\|u\|_{\overset{\circ}{W}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}.$$

For $m=1,...,p$ the following estimate holds:

$$\int_{\Pi_m^{l_m,h}}\frac{|\nabla u(x)||\eta(x)|}{r(x)}dx\le\sqrt{\frac{5}{2}}\int_{\tilde\Pi_m^{l_m,h}}\frac{|\nabla\tilde u(y)||\tilde\eta(y)|}{y_n}dy\le\sqrt{\frac{5}{2}}\left(\int_{\tilde\Pi_m^{l_m,h}}|\nabla\tilde u(y)|^2dy\right)^{1/2}\left(\int_{\tilde\Pi_m^{l_m,h}}\frac{\tilde\eta^2(y)}{y_n^2}dy\right)^{1/2}\le$$

$$\le\sqrt5\left(\int_{\Pi_m^{l_m,h}}|\nabla u(x)|^2dx\right)^{1/2}\left(\int_{\tilde\Pi_m^{l_m,h}}\frac{\tilde\eta^2(y)}{y_n^2}dy\right)^{1/2}\le\text{const}\|u\|_{\overset{\circ}{W}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}.$$

We used here the Hardy inequality (see, for example, [3]), in virtue of which

$$\int_{\tilde\Pi_m^{l_m,h}}\frac{\tilde\eta^2(y)}{y_n^2}dy=\int_0^h dy_n\int_{|y'|<l_m}dy'\frac{\tilde\eta^2(y',y_n)}{y_n^2}\le\text{const}\int_{\tilde\Pi_m^{l_m,h}}|\nabla\tilde\eta(y)|^2\,dy.$$

Thus,

$$I_1\le\text{const}\|u\|_{\overset{\circ}{W}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}_2^1(Q)},\tag{4}$$

where the constant does not depend on u and η. Next,

$$I_2=\int_Q C(r(x))|u(x)||\nabla\eta(x)|dx\le\int_{Q^{l_0}}C(r(x))|u(x)||\nabla\eta(x)|dx+C(l_0)\int_{Q_{l_0}}|u(x)||\nabla\eta(x)|dx\le$$

$$\le\sum_{m=1}^p\int_{\Pi_m^{l_m,h}}C(r(x))|u(x)||\nabla\eta(x)|dx+C(l_0)\|u\|_{\overset{\circ}{W}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}.$$

For $m=1,...,p$ we have

$$\int_{\Pi_m^{l_m,h}}C(r(x))|u(x)||\nabla\eta(x)|dx\le\left(\int_{\Pi_m^{l_m,h}}C^2(r(x))u^2(x)dx\right)^{1/2}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}\le$$

$$\le\left(\int_{\tilde\Pi_m^{l_m,h}}C^2\left(\frac{2}{\sqrt5}y_n\right)\tilde u^2(y)dy\right)^{1/2}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}\le\left(\int_{\tilde\Pi_m^{l_m,h}}C^2\left(\frac{2}{\sqrt5}y_n\right)y_n\int_0^{y_n}|\nabla\tilde u(y',\tau)|^2d\tau dy\right)^{1/2}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}\le$$

$$\le\left(\int_0^h dy_n C^2\left(\frac{2}{\sqrt5}y_n\right)y_n\int_{|y'|<l_m}dy'\int_0^h d\tau|\nabla\tilde u(y',\tau)|^2\right)^{1/2}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}\le$$

$$\le\sqrt2\left(\int_0^h C^2\left(\frac{2}{\sqrt5}y_n\right)y_n dy_n\right)^{1/2}\|u\|_{\overset{\circ}{W}_2^1(Q)}\|\eta\|_{\overset{\circ}{W}_2^1(Q)}.$$

Thus, we obtain

$$I_2 \leq \mathrm{const} \, \|u\|_{\overset{\circ}{W}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}_2^1(Q)}, \tag{5}$$

where the constant does not depend on u and η.

Similarly we obtain

$$I_3 = \int_Q D(r(x)) |u(x)||\eta(x)| dx \leq \int_{Q^{l_0}} D(r(x)) |u(x)||\eta(x)| dx + D(l_0) \int_{Q_{l_0}} |u(x)||\eta(x)| dx \leq$$

$$\leq \sum_{m=1}^p \int_{\Pi_m^{l_m,h}} D(r(x)) |u(x)||\eta(x)| dx + D(l_0) \|u\|_{\overset{\circ}{W}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}_2^1(Q)}.$$

Finally, for $m = 1,\ldots,p$ we have

$$\int_{\Pi_m^{l_m,h}} D(r(x)) |u(x)||\eta(x)| dx \leq \int_{\tilde{\Pi}_m^{l_m,h}} D\left(\frac{2}{\sqrt{5}} y_n\right) |\tilde{u}(y)||\tilde{\eta}(y)| dy \leq$$

$$\leq \left(\int_{\tilde{\Pi}_m^{l_m,h}} D^2\left(\frac{2}{\sqrt{5}} y_n\right) y_n^2 \tilde{u}^2(y) dy\right)^{1/2} \left(\int_{\tilde{\Pi}_m^{l_m,h}} \frac{\tilde{\eta}^2(y)}{y_n^2} dy\right)^{1/2} \leq$$

$$\leq \mathrm{const} \left(\int_{\tilde{\Pi}_m^{l_m,h}} D^2\left(\frac{2}{\sqrt{5}} y_n\right) y_n^3 \int_0^{y_n} |\nabla \tilde{u}(y',\tau)|^2 d\tau dy\right)^{1/2} \|\eta\|_{\overset{\circ}{W}_2^1(Q)} \leq$$

$$\leq \mathrm{const} \left(\int_0^h D^2\left(\frac{2}{\sqrt{5}} y_n\right) y_n^3 dy_n\right)^{1/2} \|u\|_{\overset{\circ}{W}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}_2^1(Q)}.$$

Thus,

$$I_3 \leq \mathrm{const} \, \|u\|_{\overset{\circ}{W}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}_2^1(Q)}, \tag{6}$$

where the constant is independent of u and η.

Therefore, in view of (4)–(6) the following estimate holds $|\langle Tu,\eta \rangle| \leq \mathrm{const} \, \|u\|_{\overset{\circ}{W}_2^1(Q)} \|\eta\|_{\overset{\circ}{W}_2^1(Q)}$, where the constant is independent of u and η.

Since the functions $\eta(x)$ from $C_0^\infty(Q)$ are dense everywhere in $\overset{\circ}{W}_2^1(Q)$, the proof of the Theorem immediately follows from the established estimate. The Theorem is proved.

REFERENCES

1. **Ladyzhenskaya O.A., Uraltseva N.N.** Linear and Quasilinear Elliptic Equations. Academic Press, 1968.
2. **Mikhailov V.P.** Partial Differential Equations. 1st ed. M.: Nauka, 1976; English transl. of 1st ed. M.: Mir, 1978.
3. **Maz'ya V.G.** Sobolev Spaces. Leningrad: Leningrad. Univ. Press, 1985 (in Russian); English translation from the Russian by T.O. Shaposhnikova. Berlin–New York: Springer-Verlag, 1985.
4. **Mikhailov V.P., Gushchin A.K.** Advanced Topics of "Equations of Mathematical Physics". Lecture courses of SEC. M.: Steklov Mathematical Institute of RAS, 2007.

Վ. Ժ. Դումանյան

Առաջին կարգի գծային դիֆերենցիալ օպերատորների սահմանափակության մասին

Հոդվածում ուսումնասիրվում են առաջին կարգի անսահմանափակ գործակիցներով գծային դիֆերենցիալ օպերատորներ: Ապացուցվում է այդ օպերատորների սահմանափակությունը:

В. Ж. Думанян

Об ограниченности линейных дифференциальных операторов первого порядка

.

В настоящей работе исследуются линейные дифференциальные операторы первого порядка с неограниченными коэффициентами. Установлена ограниченность рассматриваемых операторов.