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Mathematics

ON FUNCTIONS SEMI-ANALYTICAL IN THE POLYDISK

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In the present paper the class of semi-analytical functions in the polydisk $U^n \subset \mathbb{C}^n$ is introduced. This class is an extension of the set of holomorphic functions. For $n=1$ the concept of semi-analyticity coincides with analyticity. The Dirichlet problem with values given on the distinguished boundary of the polydisk always has a solution in the set of real parts of semi-analytical functions. Therefore, to investigate semi-analytical functions one can apply the potential theory methods, like one does it for the one-dimensional case. In the present paper the Schwarz type integral representation for the above-mentioned functions is obtained.

Keywords: polydisk, *n*-harmonic function, pluriharmonic function, the Schwarz formula.

Introduction. Let function u be pluriharmonic in the domain $D \subset \mathbb{C}^n$, i.e. it satisfies the conditions

$$\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial^2 u}{\partial y_i \partial y_j} = 0, \qquad \frac{\partial^2 u}{\partial x_i \partial y_j} - \frac{\partial^2 u}{\partial x_j \partial y_i} = 0 \tag{1}$$

($i, j = 1, \ldots, n$, if $i = j$ the equations of second group are trivial). The function u is called n-harmonic (doubly harmonic in the case $n = 2$), if only for $i = j$ the conditions (1) are satisfied, i.e.

$$\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial y_i^2} = 0, \quad i = 1, \ldots, n. \tag{2}$$

The conditions (2) imply harmonicity in each variable $z_i = x_i + iy_i$.

The pluriharmonic functions are connected with holomorphic functions of several variables the same way as harmonic functions defined on plane domain with holomorphic functions of one variable. In the case of one variable this connection enables to apply methods of the potential theory in the complex analysis. In the multidimensional case the situation is more difficult: the class of pluriharmonic (unlike harmonic) functions is too narrow in sense that the Dirichlet problem is not always solvable in this class. This fact makes it difficult to generalize some one-dimensional results for higher dimension case.

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S. Bergman [1] offered the following idea for $n=2$: to put into correspondence with any doubly harmonic function $u(z_1,z_2)$ a complex-valued function $f(z_1,z_2)=u(z_1,z_2)+iv(z_1,z_2)$ so that:

1) $f(z_1,z_2)$ is holomorphic in z_1 for a fixed z_2;

2) $v(0,z_2)\equiv 0$.

The obtained class of functions was called *the extended class of complex functions*. These functions may be studied with the help of potential theory methods, since the Dirichlet's problem with values given on the distinguished boundary is always solvable in the class of all doubly harmonic functions. However this class is not an extension of the class of holomorphic functions, as holomorphic functions do not necessarily satisfy the condition 2.

In [2, 3] a modified version of the Bergman class (*class of semi-analytical functions*) is introduced, which has the following advantage: in that special case, when the real part of semi-analytical function is pluriharmonic, the function itself is holomorphic. Thus, any holomorphic function is also semi-analytical.

In the present paper the concept of semi-analyticity is defined for functions of arbitrary number of variables (Definition 1). For such functions we obtain an integral representation, which is an analogue of the well-known Schwarz integral representation for one variable case.

Notation. We use the following notations:

$$U^n=\left\{z=\left(z_1,\dots,z_n\right)\in\mathbb{C}^n:\ \left|z_k\right|<1,\ k=1,\dots,n\right\}$$

is the unit polydisk in n-dimensional complex space $\mathbb{C}^n$,

$$T^n=\left\{z\in\mathbb{C}^n:\ \left|z_k\right|<1,\ k=1,\dots,n\right\}$$

is its distinguished boundary, $P\left(z\right)=\dfrac{1-|z|}{|1-z|}$ is one-dimensional Poisson kernel, $Q(z)$ is the function harmonically conjugate to $P(z)$,

$$S\left(z\right)=P\left(z\right)+iQ\left(z\right)=\frac{1-z}{1+z}$$

is the Schwarz kernel.

Definition 1. A function $f(z)=f\left(z_1,z_2,\dots,z_n\right)$ defined in U^n is called semi-analytical, if

a) the function $\operatorname{Re} f(z)$ is n-harmonic;

b) for fixed $z_{k+1},\dots,z_n$ the functions $f\left(0,\dots,0,z_k,z_{k+1},\dots,z_n\right)$ are holomorphic in the disk $\left|z_k\right|<1$, $k=1,\dots,n$.

Note that in the one-dimensional case (i.e. n=1) the n-harmonicity is simply harmonicity and the semi-analyticity coincides with analyticity.

Integral representation. The following theorem is an analogue of the Schwarz formula for semi-analytical functions.

Theorem 1. Let the function $f(z)$ be semi-analytical in the unit polydisk U^n, and $\rho_1,\dots,\rho_n$ be arbitrary numbers from $(0,1)$. Then for any $z\in\left\{z=\left(z_1,\dots,z_n\right):\ \left|z_k\right|<\rho_k,\ k=1,\dots,n\right\}$ the following formula

$$f(z_1,\dots,z_n)=iv(0,\dots,0)+\frac{1}{(2\pi)^n}\int_{T^n}S_n\left(\frac{z_1}{\rho_1}e^{-i\theta_1},\dots,\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{-i\theta_1},\dots,\rho_ne^{-i\theta_n}\right)d\theta_1\cdots d\theta_n$$

is true, where

$$S_n(z_1,\dots,z_n)=\prod_{j=1}^{n}P(z_j)+i\sum_{j=1}^{n}Q(z_j)\prod_{k=j+1}^{n}P(z_k) \qquad (3)$$

is the n-dimensional analogue of the Schwarz kernel. Note that for $n=1$ the function $S_n(z_1,\dots,z_n)$ coincides with the usual Schwarz kernel: $S_n=S$.

Proof. For fixed z_k, $|z_k|<1$, $k=2,\dots,n$, the function $f(z_1,z_2,\dots,z_n)$ is analytical in the disk $|z_1|<1$. By the Schwarz formula (see, for instance, [3]) we obtain

$$f(z_1,\dots,z_n)=iv(0,z_2,\dots,z_n)+\frac{1}{2\pi}\int_0^{2\pi}S\left(\frac{z_1}{\rho_1}e^{-i\theta_1}\right)u\left(\rho_1e^{i\theta_1},z_2,\dots,z_n\right)d\theta_1\,. \qquad (4)$$

Then for a fixed z_k, $|z_k|<1$, $k=3,\dots,n$, the function $f(0,z_2,\dots,z_n)$ is analytical in the disk $|z_2|<1$. Hence

$$f(0,z_2,\dots,z_n)=iv(0,0,z_3,\dots,z_n)+\frac{1}{2\pi}\int_0^{2\pi}S\left(\frac{z_2}{\rho_2}e^{-i\theta_2}\right)u\left(0,\rho_2e^{i\theta_2},z_3,\dots,z_n\right)d\theta_2\,. \qquad (5)$$

Reasoning by analogy, on the j-th step we obtain

$$f(0,\dots,0,z_j,\dots,z_n)=iv(0,\dots,0,z_{j+1},\dots,z_n)+$$
$$+\frac{1}{2\pi}\int_0^{2\pi}S\left(\frac{z_j}{\rho_j}e^{-i\theta_j}\right)u\left(0,\dots,0,\rho_je^{i\theta_j},z_{j+1},\dots,z_n\right)d\theta_j,\quad j=1,2,\dots,n.$$

Since the function u is n-harmonic, hence for the fixed $\rho_1e^{i\theta_1}$ we have

$$u\left(\rho_1e^{i\theta_1},z_2,\dots,z_n\right)=\frac{1}{(2\pi)^{n-1}}\int_{T^{n-1}}P\left(\frac{z_2}{\rho_2}e^{-i\theta_2}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\dots,\rho_ne^{i\theta_n}\right)d\theta_2\cdots d\theta_n.$$

The obtained result and (4) imply

$$f(z_1,\dots,z_n)=iv(0,z_2,\dots,z_n)+$$
$$+i\frac{1}{2\pi}\int_{T^n}Q\left(\frac{z_1}{\rho_1}e^{-i\theta_1}\right)P\left(\frac{z_2}{\rho_2}e^{-i\theta_2}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\dots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n+ \qquad (6)$$
$$+\frac{1}{(2\pi)^n}\int_{T^n}P\left(\frac{z_1}{\rho_1}e^{-i\theta_1}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\dots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n\,.$$

Equating the imaginary parts in (5), we get

$$v(0,z_2,\dots,z_n)=v(0,0,z_3,\dots,z_n)+\frac{1}{2\pi}\int_0^{2\pi}Q\left(\frac{z_2}{\rho_2}e^{-i\theta_2}\right)u\left(0,\rho_2e^{i\theta_2},z_3,\dots,z_n\right)d\theta_2. \qquad (7)$$

Using the integral representation of the function $f(0,z_2,\dots,z_n)$ (which is holomorphic in z_2) and separating again the imaginary parts, we obtain

$$v(0,0,z_3,\dots,z_n)=v(0,0,0,z_4,\dots,z_n)+\frac{1}{2\pi}\int_0^{2\pi}Q\left(\frac{z_3}{\rho_3}e^{-i\theta_3}\right)u\left(0,0,\rho_3e^{i\theta_3},z_4,\dots,z_n\right)d\theta_3.$$

Continuing in a similar way and substituting the obtained formulas consecutively in (7), we get

$$v\left(0,z_2,\ldots,z_n\right)=v\left(0,0,\ldots,0\right)+\frac{1}{2\pi}\sum_{k=2}^{n}\int_0^{2\pi}Q\left(\frac{z_k}{\rho_k}e^{-i\theta_k}\right)u\left(0,\ldots,0,\rho_ke^{i\theta_k},z_{k+1},\ldots,z_n\right)d\theta_k. \tag{8}$$

Taking into account the n-harmonicity of u, we receive

$$u\left(0,\ldots,0,\rho_ke^{i\theta_k},z_{k+1},\ldots,z_n\right)=$$

$$=\frac{1}{(2\pi)^{n-k}}\int_{T^{n-k}}P\left(\frac{z_{k+1}}{\rho_{k+1}}e^{-i\theta_{k+1}}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(0,\ldots,0,\rho_ke^{i\theta_k},\ldots,\rho_ne^{i\theta_n}\right)d\theta_{k+1}\cdots d\theta_n.$$

Due to the mean value theorem

$$u\left(0,\ldots,0,\rho_ke^{i\theta_k},\ldots,\rho_ne^{i\theta_n}\right)=\frac{1}{(2\pi)^{k-1}}\int_{T^{k-1}}u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_{k-1}.$$

From the last two equalities it follows that

$$u\left(0,\ldots,0,\rho_ke^{i\theta_k},z_{k+1},\ldots,z_n\right)=$$

$$=\frac{1}{(2\pi)^{n-1}}\int_{T^{n-1}}P\left(\frac{z_{k+1}}{\rho_{k+1}}e^{-i\theta_{k+1}}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_{k-1}d\theta_{k+1}\cdots d\theta_n.$$

Substituting this equality into (8), we obtain

$$v\left(0,z_2,\ldots,z_n\right)=v\left(0,0,\ldots,0\right)+\sum_{k=2}^{n}\frac{1}{2\pi}\int_0^{2\pi}Q\left(\frac{z_k}{\rho_k}e^{-i\theta_k}\right)\frac{1}{(2\pi)^{n-1}}\int_{T^{n-1}}P\left(\frac{z_{k+1}}{\rho_{k+1}}e^{-i\theta_{k+1}}\right)\cdots$$

$$\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n,$$

or

$$v\left(0,z_2,\ldots,z_n\right)=v\left(0,0,\ldots,0\right)+$$

$$+\frac{1}{(2\pi)^n}\sum_{k=2}^{n}\int_{T^n}Q\left(\frac{z_k}{\rho_k}e^{-i\theta_k}\right)P\left(\frac{z_{k+1}}{\rho_{k+1}}e^{-i\theta_{k+1}}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n.$$

From here and from (6) we have

$$f\left(z_1,\ldots,z_n\right)=iv\left(0,0,\ldots,0\right)+$$

$$+i\frac{1}{(2\pi)^n}\sum_{k=2}^{n}\int_{T^n}Q\left(\frac{z_k}{\rho_k}e^{-i\theta_k}\right)P\left(\frac{z_{k+1}}{\rho_{k+1}}e^{-i\theta_{k+1}}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n+$$

$$+i\frac{1}{(2\pi)^n}\int_{T^n}Q\left(\frac{z_1}{\rho_1}e^{-i\theta_1}\right)P\left(\frac{z_2}{\rho_2}e^{-i\theta_2}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n+$$

$$+\frac{1}{(2\pi)^n}\int_{T^n}P\left(\frac{z_1}{\rho_1}e^{-i\theta_1}\right)\cdots P\left(\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1e^{i\theta_1},\ldots,\rho_ne^{i\theta_n}\right)d\theta_1\cdots d\theta_n.$$

Taking into account (3) from the last equality, we obtain the required integral representation. The Theorem 1 is proved.

Theorem 2. The imaginary part of an arbitrary semi-analytical function $f = u + iv$ is a n-harmonic function.

Proof. Separating the imaginary parts of the left-hand and right-hand sides in the formula of Theorem 1, we obtain

$$v(z_1,\ldots,z_n) = v(0,\ldots,0) + \frac{1}{(2\pi)^n}\int_{T^n} \operatorname{Im} S_n\left(\frac{z_1}{\rho_1}e^{-i\theta_1},\ldots,\frac{z_n}{\rho_n}e^{-i\theta_n}\right)u\left(\rho_1 e^{-i\theta_1},\ldots,\rho_n e^{-i\theta_n}\right)d\theta_1\cdots d\theta_n. \tag{9}$$

According to the definition (3) of the kernel S_n, we have

$$\operatorname{Im} S_n\left(\frac{z_1}{\rho_1}e^{-i\theta_1},\ldots,\frac{z_n}{\rho_n}e^{-i\theta_n}\right) = \sum_{j=1}^{n} Q\left(\frac{z_j}{\rho_j}e^{-i\theta_j}\right)\prod_{k=j+1}^{n} P\left(\frac{z_k}{\rho_k}e^{-i\theta_k}\right), \tag{10}$$

whence it is clear that the left-hand side of this formula is n-harmonic in the poly-disk $\{z = (z_1,\ldots,z_n) :\ |z_k| < \rho_k,\ k = 1,\ldots,n\}$. Taking into account that $\rho_1,\ldots,\rho_n$ are arbitrary numbers from (0, 1), we obtain the assertion of the Theorem from (9) and (10).

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Պոլիդիսկում կիսաանալիտիկ ֆունկցիաների մասին

Աշխատանքում ներմուծվում է $U^n \subset \mathbb{C}^n$ միավոր պոլիդիսկում կիսաանալիտիկ ֆունկցիաների դաս, որը հոլոմորֆ ֆունկցիաների դասի ընդլայնում է: $n=1$ դեպքում կիսաանալիտիկության հասկացությունը համընկնում է անալիտիկության հետ: Պոլիդիսկի հենքի վրա տրված արժեքների համար Դիրիխլեի խնդիրը միշտ ունի լուծում կիսաանալիտիկ ֆունկցիաների իրական մասերի բազմության մեջ: Ուստի, ինչպես և միաչափ դեպքում, կիսաանալիտիկ ֆունկցիաները հետազոտելիս մենք կարող ենք կիրառել պոտենցիալի տեսության մեթոդները: Հոդվածում այդ ֆունկցիաների համար ստացված է Շվարցի ինտեգրալային բանաձևը:

О функциях, полуаналитических в полидиске

В работе вводится класс полуаналитических в полидиске $U^n \subset \mathbb{C}^n$ функций, являющийся расширением множества аналитических функций. В случае $n=1$ понятие полуаналитичности совпадает с аналитичностью. Проблема Дирихле с заданными на основе полидиска значениями всегда имеет решение в классе вещественных частей полуаналитических функций. Поэтому, как и в одномерном случае, при исследовании полуаналитических функций можно применять методы теории потенциалов. Для этих функций в работе получено интегральное представление Шварца.