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On a Class of $\mathcal{L}$-Wiener-Hopf Operators

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Abstract—By replacement in the definition of the convolution operator of Fourier transform by a spectral transform of a selfadjoint Sturm-Liouville operator on the axis $\mathcal{L}$, the concepts of $\mathcal{L}$-convolution and $\mathcal{L}$-Wiener-Hopf operators are introduced. The case of the reflectorless potentials with a single eigenvalue is considered. A relationship between the Wiener-Hopf and $\mathcal{L}$-Wiener-Hopf operators is established. In the case of piecewise continuous symbol the Fredholm property and invertibility of the $\mathcal{L}$-Wiener-Hopf operator are investigated.

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1. $\mathcal{L}$-CONVOLUTION AND $\mathcal{L}$-WIENER-HOPF OPERATORS

Let $\mathcal{L}$ be the maximal symmetric operator generated by the differential relation $(\ell y)(x) = -y'' + q(x)y(x)$ with a real potential q satisfying the condition: $(1 + |x|)|q(x)| \in L_1(\mathbb{R})$, and let $u^-(x,\lambda)$, $u^+(x,\lambda)$ $(x,\lambda \in \mathbb{R})$ be the solutions of the equation $\ell y = \lambda^2 y$, being the eigenfunctions of the left and right scattering problems, respectively, and representing a complete orthonormal system of eigenfunctions of the continuous spectrum (see [1], [2]).

In what follows we will use the following notation. By $m(a)$ $(a \in L_\infty(\mathbb{R}))$ and τ we denote the operators acting in the spaces $L_p(\mathbb{R})$ $(1 \leqslant p < \infty)$ according to the formulas $m(a)y = ay$ and $(\tau y)(x) = y(-x)$, $x \in \mathbb{R}$, respectively. By I we denote the identity operator.

Define the operators $U_\mp, U : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ as follows:

$$(U_\mp y)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^\mp(x,\lambda) y(x)\, dx, \quad \lambda \in \mathbb{R}$$
$$U = m(\chi_+)U_- + m(\chi_-)\tau U_+,$$

where $\chi_\pm$ stand for the characteristic functions of the sets $\mathbb{R}_\pm$ ($\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_- = \mathbb{R} \setminus \mathbb{R}_+$), and the integrals should be understood in the sense of convergence in the norm of $L_2(\mathbb{R})$. Observe that $U_\mp$, U are bounded operators. Also, the operator U is a partial isometry and satisfies the following equalities:

$$U^*U = I - P, \qquad UU^* = I, \tag{1.1}$$

where P is a projector in $L_2(\mathbb{R})$ onto the proper subspace H, corresponding to the discrete spectrum of the operator $\mathcal{L}$ (see [1], [3], [4]).

Denote by $\mathcal{M}_{p,\mathcal{L}}$, $1 \leqslant p \leqslant \infty$, the set of all functions $a \in L_\infty(\mathbb{R})$ possessing the following properties: if $y \in L_2(\mathbb{R}) \cap L_p(\mathbb{R})$, then $U^*m(a)Uy \in L_p(\mathbb{R})$ and $\|U^*m(a)Uy\| \leqslant c_p \|y\|_p$, where the constant c_p does not depend on y. For $a \in \mathcal{M}_{p,\mathcal{L}}$, the operator $W^0_{\mathcal{L}}(a)$ defined by $W^0_{\mathcal{L}}(a) = U^*m(a)U$ is bounded in

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$L_2(\mathbb{R})$ and can continuously be continued from $L_2(\mathbb{R}) \cap L_p(\mathbb{R})$ to a bounded operator acting in $L_2(\mathbb{R})$. The resulting operator we denote by $W^0_{\mathcal{L}}(a)$, and call $\mathcal{L}$-convolution operator with $\mathcal{L}$-symbol a.

Next, we define the operators $\pi_\pm : L_p(\mathbb{R}) \to L_p(\mathbb{R}_\pm)$ and $\pi^0_\pm : L_p(\mathbb{R}_\pm) \to L_p(\mathbb{R})$ $(1 \leqslant p \leqslant \infty)$ by formulas: $(\pi_\pm y)(x) = y(x)$, $x \in \mathbb{R}_\pm$, $(\pi^0_\pm y)(x) = y(x)$ for $x \in \mathbb{R}_\pm$ and $(\pi^0_\pm y)(x) = 0$ for $x \in \mathbb{R}_\mp$.

The operator $W_{\mathcal{L}}(a) = \pi_+ W^0_{\mathcal{L}}(a)\pi^0_+ : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is called $\mathcal{L}$-Wiener-Hopf integral operator.

Observe that in the case $q = 0$, the operator U coincides with the Fourier transform F:

$$(Fy)(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda s} y(s)\, ds, \quad \lambda \in \mathbb{R},\ \ y \in L_2(\mathbb{R}).$$

And, in the case $q = 0$, the operators $W^0_{\mathcal{L}}(a)$ and $W_{\mathcal{L}}(a)$ coincide with the convolution operator $W(a)$ and the Wiener-Hopf operator, respectively (see [5]). Also, in the case $q = 0$, the set $\mathcal{M}_{p,\mathcal{L}}$ we will denote by $\mathcal{M}_p$. It is known (see [5]) that $\mathcal{M}_p$ is a Banach algebra with norm $\|a\|_{\mathcal{M}_p} = \left\|W^0(a)\right\|_{\mathcal{L}(L_p)}$.

2. REFLECTORLESS POTENTIALS

In this paper we study the Fredholm property and invertibility of the operator $W_{\mathcal{L}}(a)$ in the space $L_p(\mathbb{R}_+)$, $1 < p < \infty$, in the case of the operator $\mathcal{L}$, corresponding to a reflectorless potential with a discrete spectrum consisting of a single eigenvalue $(i\mu)^2$ and with a right norming coefficient c_+. Observe that in the considered case, the passage coefficient $t(\lambda)$ is determined by formula $t(\lambda) = (\lambda + i\mu)(\lambda - i\mu)^{-1}$ (see [2, Theorem 3.5.1].

It is known (see [1], [2]) that the transformation operators defined by formulas:

$$(\mathcal{K}_+ y)(x) = y(x) + \int_x^{\infty} K_+(x,t)y(t)\, dt, \quad (\mathcal{K}_- y)(x) = y(x) + \int_{-\infty}^{x} K_-(x,t)y(t)\, dt,$$

are bounded operators respectively in the spaces $L_p(\gamma,\infty)$ and $L_p(-\infty,\gamma)$, $1 \leqslant p \leqslant \infty$, for all $\gamma \in \mathbb{R}$. The kernels $K_\pm$ are determined from Gel'fand-Levitan-Marchenko equations, and in our case the following equalities hold (see [6]):

$$K_\pm(x,t) = -\varphi(x)\psi_\pm(t),$$

where $\varphi(x) = \sqrt{\mu/2}\,\mathrm{ch}^{-1}\mu(x-\xi)$, $\psi_\pm(t) = \sqrt{2\mu}\, e^{\mp\mu(t-\xi)}$, and $\xi = \mu^{-1}\ln\left[c_+(2\mu)^{-1/2}\right]$.

Observe that the potential is determined by equality (see [6]):

$$q(x) = 2\frac{d}{dx}(K^+(x,e)) = -\frac{2\mu^2}{\mathrm{ch}^2\mu(x-\xi)},$$

and $\varphi(x)$ is the normed eigenfunction corresponding to the eigenvalue $(i\mu)^2$. Besides, we have

$$u^-(x,\lambda) = t(\lambda)\left(1 - \frac{1}{\mu - i\pi}\psi_+(x)\varphi(x)\right) e^{i\lambda x}, \quad u^+(x,\lambda) = t(\lambda)u^-(x,-\lambda). \tag{2.1}$$

Consider the operators $V_\pm, \Gamma_\pm : L_p(\mathbb{R}_\pm) \to L_p(\mathbb{R}_\pm)$, $\Gamma : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, defined by formulas:

$$(V_+y)(x) = \int_0^x y(t)\, dt, \qquad (V_-y)(x) = \int_x^0 y(t)\, dt,$$
$$\Gamma_\pm = I - m(\psi_\pm)V_\pm m(\varphi), \qquad \Gamma = \pi^0_+\Gamma_+\pi_+ + \pi^0_-\Gamma_-\pi_-,$$

and observe that $\Gamma_\pm = \mathcal{K}^*_\pm$.

Lemma 2.1. *The operators $\Gamma_\pm, \Gamma$ are invertible, and the corresponding inverse operators are given by the following formulas:*

$$\Gamma^{-1}_\pm = I + m(\varphi)V_\pm m(\psi_\pm), \qquad \Gamma^{-1} = \pi^0_+\Gamma^{-1}_+\pi_+ + \pi^0_-\Gamma_-\pi_- \,. \tag{2.2}$$

Proof. Let the operators $\Gamma_{\pm}^{-1}$ be defined by (2.2). Using the equalities $(2\mu)^{-1}\varphi\psi_{\pm}^2 = \psi_{\pm} - \varphi$ and $(2\mu)^{-1}(\psi_{\pm}^2)' = \mp\psi_{\pm}^2$, and integration by parts, it is easy to check that

$$m(\varphi)V_{\pm}m(\psi_{\pm}^2)V_{\pm}m(\varphi) = -m(\psi_{\pm})V_{\pm}m(\varphi)V_{\pm}m(\psi_{\pm}),$$

implying that $\Gamma_{\pm}^{-1}\Gamma_{\pm} = I$.

Denote by $\hat{\varphi}$ the antiderivative of φ^2. Interchanging the integrals in the relation $V_{\pm}m(\varphi^2)V_{\pm}m(\psi_{\pm})y$, where y is a continuous finite function defined on $\mathbb{R}_{\pm}$, we can write

$$V_{\pm}m(\varphi^2)V_{\pm}m(\psi_{\pm}) = \pm m(\hat{\varphi})V_{\pm}m(\psi_{\pm}) \mp V_{\pm}m(\psi_{\pm}\hat{\varphi}).$$

Therefore,

$$\Gamma_{\pm}\Gamma_{\pm}^{-1} = I + m\left(\varphi \mp \psi_{\pm}\hat{\varphi}\right)V_{\pm}m(\psi_{\pm}) - m(\psi_{\pm})m\left(\varphi \mp \psi_{\pm}\hat{\varphi}\right).$$

It is easy to see that $\left(\varphi\psi_{\pm}^{-1} - \hat{\varphi}\right)' = 0$, that is, $\varphi \mp \hat{\varphi}\psi_{\pm} = d_{\pm}\psi_{\pm}$, where $d_{\pm}$ are numbers, and hence $\Gamma_{\pm}\Gamma_{\pm}^{-1} = I$. Thus, the first formula in (2.2) is proved. The second formula in (2.2) obviously follows from the definition of Γ. Lemma 2.1 is proved.

Denote by S the singular integral operator defined by formula:

$$(Sy)(x) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{y(s)}{s-x}\,ds, \qquad x \in \mathbb{R},$$

where the integral is in the sense of Cauchy principal value. It is known (see [7]), that the operator S is bounded in the space $L_p(\mathbb{R})$, $1 < p < \infty$. Let $P_{\pm} = (I \pm S)/2$. From the results of [4] it follows that

$$U_- = (m(t)P_+ + P_-)F\Gamma, \quad U_+ = \tau(P_+ + m(\bar{t})P_-)F\Gamma, \tag{2.3}$$

$$U = \left(m(t\chi_+ + \chi_-)P_+ + m(\chi_+ + \bar{t}\chi_-)P_-\right)F\Gamma. \tag{2.4}$$

In particular, the operators $U_{\pm}$, U are related by the relations $U = m(\chi_+ + \bar{t}\chi_-)U_-$, $U = \tau(\chi_+ + \bar{t}\chi_-)U_+$, implying that

$$W_{\mathcal{L}}^0(a) = U_-^* m(a)U_- = U_+^* m(a)U_+\,. \tag{2.5}$$

Also, in view of (1.1) and the equality $r^* = \tau$, we have

$$U_-^*U_- = U_+^*U_+ = I - P, \qquad U_-U_-^* = U_+U_+^* = I\,. \tag{2.6}$$

Using formulas (2.1), (2.3)-(2.6), it is easy to check that in the case $a = 1 + F_k$, where $k \in L_1(\mathbb{R})$, we have

$$\left(W_{\mathcal{L}}^0(a)y\right)(x) = y(x) - \varphi(x)\int_{-\infty}^{\infty}\varphi(s)y(s)\,ds + \int_{-\infty}^{\infty}K(x,s)y(s)\,ds, \quad x \in \mathbb{R},$$

where

$$K(x,s) = k(x-s) + \int_{-\infty}^{\infty}\operatorname{sgn}(x-s-s')e^{\mu s'\cdot\operatorname{sgn}(x-s-s')}k(s')\,ds'\varphi(x)\varphi(s)\,.$$

Similarly, we obtain

$$\left(W_{\mathcal{L}}(a)y\right)(x) = y(x) - \varphi(x)\int_0^{\infty}\varphi(s)y(s)\,ds + \int_0^{\infty}K(x,s)y(s)\,ds.$$

3. THE MAIN RESULTS

The next theorem establishes a relationship between the operators $W_{\mathcal{L}}(a)$ and $W(a)$.

Theorem 3.1. *For $a \in \mathcal{M}_{p,\mathcal{L}}$, $1 \leqslant p < \infty$, the following equality holds:*

$$W_{\mathcal{L}}(a) = \mathcal{K}_+W(a)\Gamma_+. \tag{3.1}$$

Proof. Let $y \in L_p(\mathbb{R}_+) \cap L_2(\mathbb{R}_+)$. Using the identities $P_\pm F = Fm(\chi_\pm)$ (see [5]), and formulas (2.3)-(2.5), we can write

$$\begin{aligned} W_{\mathcal{L}}(a)y &= \pi_+ U_-^* m(a)\pi_+^0 y \\ &= \pi_+\Gamma^*\left(m(\chi_+)W^0(a)m(\chi_+) + m(\chi_-)W^0(at)m(\chi_+) + m(\chi_+)W^0(\bar{t}a)m(\chi_-)\right. \\ &\quad \left.+m(\chi_-)W^0(a)m(\chi_-)\right)\Gamma\pi_+^0 y \\ &= \mathcal{K}_+\pi_+(m(\chi_+)+m(\chi_-))\begin{pmatrix} W^0(a) & W^0(\bar{t}a) \\ W^0(ta) & W^0(a)\end{pmatrix}\pi_+^0\Gamma_+ y \\ &= \mathcal{K}_+\pi_+W^0(a)\pi_+^-\Gamma_+ y = \mathcal{K}_+ W(a)\Gamma_+ y. \end{aligned}$$

Thus, the operator $W(a)$ together with $W_{\mathcal{L}}(a)$ admits a continuous continuation into $L_p(\mathbb{R})$, and the result follows. Theorem 3.1 is proved.

Corollary 3.1. *The set $\mathcal{M}_{p,\mathcal{L}}$, $1 \leqslant p < \infty$, coincides with $\mathcal{M}_p$.*

It is known (see [5]) that any piecewise continuous function defined on $\mathbb{R}$ with a finite number of discontinuities belongs to $\mathcal{M}_p$, $1 < p < \infty$. The closure of the algebra of step functions in the algebra $\mathcal{M}_p$ we denote by PC_p. We have the following inclusion $PC_p \subset PC_2 = PC$ (see [5]), where PC stands for the class of functions a, having limiting values $a(x \pm 0)$ at each point $x \in \mathbb{R}$, including infinity: $a(\infty \pm 0) = \lim\limits_{x\to\mp\infty} a(x)$. On the other hand, we have $C_p \subset PC_p$, where C_p is the closure of the Wiener algebra $W(\mathbb{R}) = \{c + F_k; c \in \mathbb{C}, k \in L_1(\mathbb{R})\}$ in $\mathcal{M}_p$. Besides, PC_p contains functions of bounded variations (see [5]).

Let $\dot{\mathbb{R}} = R \cup \{\infty\}$ and $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\})$ be the one-point and two-point compactifications of $\mathbb{R}$, respectively. To each function $a \in PC_p$ we associate a function $a_p : \dot{\mathbb{R}} \times \bar{\mathbb{R}} \to \mathbb{C}$ defined by (see [5]):

$$a_p(x,\xi) = \frac{1}{2}(a(x-0)+a(x+0)) + \frac{1}{2}(a(x-0)-a(x+0))\mathrm{cth}\pi\left(\frac{i}{p}+\xi\right).$$

Observe that the function a can have at most countable number of discontinuities $a(x_k-0) \neq a(x_k+0)$ ($x_k \in \mathbb{R}, k \in \mathbb{N}$). The function $a_p(x,\xi)$ is continuous in the following sense (see [5]): the points $a(x_k-0)$ and $a(x_k+0)$ are connected by an arc of the circle, from which the segment connecting the points $a(x_k-0)$ and $a(x_k+0)$ is visible at angle $2\pi/\max\{p,p'\}$ ($p' = p/p-1$) located to the right (left) of the segment for $p > 2$ (for $p > 2$), and for $p = 2$ the arc coincides with the segment connecting that points. In the case $a_p(x,\xi) \neq 0$, the index $\mathrm{ind}a_p$ is defined to be the increment of the argument $(2\pi)^{-1}\mathrm{arg}a_p(x,\xi)$, when x runs in $\mathbb{R}$, and at discontinuity points $x_k \in \mathbb{R}$ the parameter ξ ran from $-\infty$ to ∞.

By a generalized p-factorization of a function $a \in L_\infty(\mathbb{R})$ we mean the representation:

$$a(\lambda) = a_-(\lambda)(\lambda-i)^{\varkappa}(\lambda+i)^{-\varkappa}a_+(\lambda),$$

where $(\lambda-i)^{-2/p}a_- \in P_-(L_p(\mathbb{R}))$, $(\lambda-i)^{-2/p'}a_-^{-1} \in P_-(L_{p'}(\mathbb{R}))$, $(\lambda+i)^{-2/p'}a_+ \in P_+(L_{p'}(\mathbb{R}))$, $(\lambda+i)^{-2/p}a_+^{-1} \in P_+(L_p(\mathbb{R}))$, and $\varkappa$ is an integer, called the p-index of function a.

In view of Theorem 3.1 and results from [5], Sec. 4, we obtain the following theorems.

Theorem 3.2. *Let $a \in PC_p$, $1 < p < \infty$. The operator $W_{\mathcal{L}}(a)$ is normally solvable in the space $L_p(\mathbb{R}_+)$ if and only if* $\inf|a_p(\lambda,\xi)| \neq 0$ ($\lambda \in \dot{\mathbb{R}}$, $\xi \in \bar{\mathbb{R}}$)*. Under this condition, the operator $W_{\mathcal{L}}(a)$ is invertible, left-invertible, or right-invertible if the number* $\mathrm{ind}\, a_p$ *is equal to zero, is positive, or is negative, respectively. Besides,* $\mathrm{Ind}\, W_{\mathcal{L}}(a) = -\mathrm{ind}\, a_p$.

Theorem 3.3. *Let $a \in PC_p$, $1 < p < \infty$, $p' = p/(p-1)$, and* $\inf|a_p(\lambda,\xi)| \neq 0$ ($\lambda \in \dot{\mathbb{R}}$, $\xi \in \bar{\mathbb{R}}$)*. Then a admits a generalized p'-factorization $a = a_- r_\varkappa a_+$, $r_\varkappa(\lambda) = (\lambda-i)^{\varkappa}(\lambda+i)^{-\varkappa}$, $\varkappa = \mathrm{ind}\, a_p$, and the left-inverse (right-inverse) of $W_{\mathcal{L}}(a)$ on a dense set in $L_p(\mathbb{R}_+)$ for $\varkappa \geqslant 0$ (for $\varkappa \leqslant 0$) can be written in the form:*

$$(W_{\mathcal{L}}(a))_\ell^{-1} = \Gamma_+^{-1}W(r_{-\varkappa})W\left(a_+^{-1}\right)W\left(a_-^{-1}\right)\mathcal{K}_+^{-1}$$

$$\left((W_{\mathcal{L}}(a))_r^{-1} = \Gamma_+ W\left(a_+^{-1}\right) W\left(a_-^{-1}\right) W(r_{-\varkappa})\right).$$

If $\varkappa < 0$*, then* $\ker W_{\mathcal{L}}(a) = \operatorname{span}\left\{\Gamma_+^{-1}\pi_+ F^{-1} g_k; k = 1, \dots, -\varkappa\right\}$*, where* $g_k(\lambda) = (1 - i\lambda)^{-k}$*. If* $\varkappa > 0$*, then the equation* $W_{\mathcal{L}}(a)\varphi = f$ *has a solution in* $L_p(\mathbb{R}_+)$ *if and only if*

$$\int_0^\infty \left(\mathcal{K}_+^{-1} f\right)(t)\overline{h_k(t)}\, dt, \quad k = 1, \dots, \varkappa, \quad \textit{where} \quad h_k = \pi_+ F^{-1}\left(\bar{a}_-^{-1} g_{+k}\right).$$

Note that $\mathcal{M}_1 = W(\mathbb{R})$ (see [5]), and hence, $a \in W(\mathbb{R})$ is a natural requirement when we study the operator $W_{\mathcal{L}}(a)$ in the space $L_1(\mathbb{R})$. Also, based on the results of [8] and formula (3.1), similar results can be stated in this case as well.

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