SOME RELATIONS BETWEEN THE $\mu$-PARAMETERS OF REGULAR GRAPHS

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We consider undirected, simple, finite, connected graphs. Some relations between the $\mu$-parameters are obtained for the case of regular graphs.

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Introduction. We consider finite, undirected, connected graphs without loops and multiple edges containing at least one edge. For any graph $G$ we denote by $V(G)$ and $E(G)$ the sets of vertices and edges of $G$, respectively. For any $x \in V(G)$ $d_G(x)$ denotes the degree of the vertex $x$ in $G$. For a graph $G$ $\delta(G)$ and $\Delta(G)$ denote the minimum and the maximum degree of a vertex in $G$, respectively.

An arbitrary nonempty finite subset of consecutive integers is called an interval. An interval with the minimum element $p$ and the maximum element $q$ is denoted by $[p, q]$.

A function $\varphi : E(G) \rightarrow [1, t]$ is called a proper edge $t$-coloring of a graph $G$, if each of $t$ colors is used, and adjacent edges are colored differently.

The minimum value of $t$ for which there exists a proper edge $t$-coloring of a graph $G$, is denoted by $\chi'(G)$ [1].

For any graph $G$ and for any $t \in [\chi'(G), |E(G)|]$ we denote by $\alpha(G, t)$ the set of all proper edge $t$-colorings of $G$.

Let us also define a set $\alpha(G)$ of all proper edge colorings of a graph $G$:

$$\alpha(G) = \bigcup_{t=\chi'(G)}^{\lfloor E(G) \rfloor} \alpha(G, t).$$

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If $\varphi \in \alpha(G)$ and $x \in V(G)$, then the set $\{\varphi(e)/e \in E(G), e \text{ is incident with } x\}$ is called a spectrum of the vertex $x$ of the graph $G$ at the proper edge coloring $\varphi$ and is denoted by $S_G(x, \varphi)$.

If $G$ is a graph, $\varphi \in \alpha(G)$, then set

$$V_{in}(G, \varphi) \equiv \{x \in V(G)/S_G(x, \varphi) \text{ is an interval}\}$$

and

$$f_G(\varphi) \equiv |V_{in}(G, \varphi)|.$$  

A proper edge coloring $\varphi \in \alpha(G)$ is called an interval edge coloring \cite{2-4} of the graph $G$, if and only if $f_G(\varphi) = |V(G)|$. The set of all graphs having an interval edge coloring is denoted by $\mathcal{N}$. The terms and concepts, which are not defined can be found in \cite{5}.

For a graph $G$ and for any $t \in [\chi'(G), |E(G)|]$, we set \cite{6}:

$$\mu_1(G,t) \equiv \min_{\varphi \in \alpha(G,t)} f_G(\varphi), \quad \mu_2(G,t) \equiv \max_{\varphi \in \alpha(G,t)} f_G(\varphi).$$

For any graph $G$, we set \cite{6}:

$$\mu_{11}(G) \equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G,t), \quad \mu_{12}(G) \equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_1(G,t),$$

$$\mu_{21}(G) \equiv \min_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G,t), \quad \mu_{22}(G) \equiv \max_{\chi'(G) \leq t \leq |E(G)|} \mu_2(G,t).$$

Clearly, the $\mu$-parameters are correctly defined for an arbitrary graph. Some remarks on their interpretations in games are given in \cite{7}.

The exact values of the parameters $\mu_{11}$, $\mu_{12}$, $\mu_{21}$ and $\mu_{22}$ are found for simple paths, simple cycles and simple cycles with a chord \cite{8,9}, “Möbius ladders” \cite{6,10}, complete graphs \cite{11}, complete bipartite graphs \cite{12,13}, prisms \cite{10,14}, $n$-dimensional cubes \cite{14,15} and the Petersen graph \cite{7}. The exact values of $\mu_{11}$ and $\mu_{22}$ for trees are found in \cite{16}. The exact value of $\mu_{12}$ for an arbitrary tree is found in \cite{17} (see also \cite{18,19}).

In this paper some relations between the $\mu$-parameters of regular graphs are obtained.

The Main Results. In the rest part of this paper we admit an additional condition: an arbitrary graph $G$ satisfies the inequality $\delta(G) \geq 2$.

**Theorem 1.** \cite{8,9}. For any integer $k \geq 2$ the following equalities hold:

1. $\mu_{12}(C_{2k}) = \mu_{22}(C_{2k}) = 2k$,
2. $\mu_{21}(C_{2k}) = 2k - 1$;
3. $\mu_{11}(C_{2k}) = \left\{\begin{array}{ll} 1, & \text{if } k = 2, \\ 0, & \text{if } k \geq 3. \end{array}\right.$
Theorem 2. For any positive integer $k$ the following equalities hold:

1. $\mu_{12}(C_{2k+1}) = 2$;
2. $\mu_{21}(C_{2k+1}) = \mu_{22}(C_{2k+1}) = 2k$;
3. $\mu_{11}(C_{2k+1}) = \begin{cases} 2, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2. \end{cases}$

Corollary 1. For any integer $k \geq 2$ the inequalities $\mu_{21}(C_2k) < \mu_{12}(C_{2k})$ and $\mu_{22}(C_{2k+1}) < \mu_{21}(C_{2k+1})$ hold.

Theorem 3. For any graph $G$ the inequalities $\mu_{11}(G) \leq \mu_{12}(G) \leq \mu_{22}(G)$, $\mu_{11}(G) \leq \mu_{21}(G) \leq \mu_{22}(G)$ hold.

Corollary 1 means that there are graphs $G$, for which $\mu_{21}(G) < \mu_{12}(G)$, and there are also graphs $G$, for which $\mu_{12}(G) < \mu_{21}(G)$.

Theorem 4. If $G$ is a regular graph with $\chi'(G) = \Delta(G)$, then $\mu_{12}(G) = |V(G)|$.

Theorem 5. If $G$ is an $r$-regular graph and $\varphi \in \alpha(G, |E(G)|)$, then

$$|V_{int}(G, \varphi)| \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.$$

Corollary 2. If $G$ is an $r$-regular graph, then

$$\mu_{2}(G, |E(G)|) \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.$$

Corollary 3. If $G$ is an $r$-regular graph, then

$$\mu_{21}(G) \leq \left\lfloor \frac{r \cdot |V(G)| - 2}{2 \cdot (r-1)} \right\rfloor.$$

Proposition. For arbitrary integers $r \geq 2$ and $n \geq 1$ the inequality

$$\left\lfloor \frac{r \cdot n - 2}{2 \cdot (r-1)} \right\rfloor \leq n - 1$$

holds.

Proof.\[\left\lfloor \frac{rn - 2}{2 \cdot (r-1)} \right\rfloor = \left\lfloor \frac{n - 2}{2 \cdot (r-1)} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n-2}{2} \right\rfloor = n - 1. \square\]

Corollary 4. If $G$ is a regular graph, then $\mu_{21}(G) \leq |V(G)| - 1$. From Corollary 4 and Theorem 3 we obtain:

Corollary 5. For an arbitrary regular graph $G$ with $\chi'(G) = \Delta(G)$ the inequality $\mu_{21}(G) < \mu_{12}(G)$ holds.

Theorem 6. For an arbitrary regular graph $G$ the following four statements are equivalent:

1. $\chi'(G) = \Delta(G)$,
2. $G \in \mathcal{M}$,
3. $\mu_{22}(G) = |V(G)|$,
4. $\mu_{12}(G) = |V(G)|$. 
Proof. The equivalence between 1) and 2) was proved in [2–4]. The equivalence between 2) and 3) is evident.
Let us show the equivalence between 1) and 4).
If $\chi'(G) = \Delta(G)$, then by Theorem 4 we have the equality $\mu_{12}(G) = |V(G)|$. It means that 1) $\Rightarrow$ 4).
Now suppose that $\mu_{12}(G) = |V(G)|$. By Theorem 3 we have also the equality $\mu_{22}(G) = |V(G)|$. Consequently, using the equivalence between 2) and 3), we have also the relation $G \in \mathcal{F}$. Finally, using the equivalence between 1) and 2), we have also the equality $\chi'(G) = \Delta(G)$. Thus, 4) $\Rightarrow$ 1).

Theorem 6 implies that the problem of determined whether $\mu_{12}(G) = |V(G)|$ for a given regular graph $G$ is NP-complete.

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