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Two types of universal proof systems for all variants of many-valued logics and some properties of them

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Abstract
Two types of propositional proof systems are described in this paper such that proof system for each variant of propositional many-valued logic can be presented in both of the described forms. The first of introduced systems is a Gentzen-like system, the second one is based on the generalization of the notion of determinative disjunctive normal form, formerly defined by first coauthor. Some generalization of Kalmar's proof of deducibility for two-valued tautologies in the classical propositional logic gives us a possibility to suggest the easy method of proving the completeness for first type of described systems. The completeness of the second type one is received from its construction automatically. The introduced proof systems are "weak" ones with a "simple strategist" of proof search and we have investigated the quantitative properties, related to proof complexity characteristics in them as well. In particular, for some class of many-valued tautologies simultaneously optimal bounds (asymptotically the same upper and lower bounds for each proof complexity characteristic) are obtained in the systems, considered for some versions of many-valued logic.

Keywords Many-valued propositional logic · Gentzen-like system · Determinative conjunct · Determinative disjunctive normal form · Elimination rule · Proof complexity

1 Introduction

It is known that many-valued logic (MVL) was created and developed in 1920 first by Łukasiewicz [1], who used a third truth value for "possible" (or "unknown"). Essentially parallel to the Łukasiewicz approach, Post introduced in 1921 the basic idea of additional truth degrees, and applied it to problems of functions presentation [2]. Later on many others continued investigation in this area. In the earlier years of development, this caused some doubts about the usefullness of MVL. In the mean time many interesting applications of MVL were found in such fields as Logic, Mathematics, Formal Verification, Artificial Intelligence, Operations Research, Computational Biology, Cryptography, Data Mining, Machine Learning, Hardware Design etc., therefore, the investigations in area of MVL are very actual. The main theoretical results concern several properties of formal systems, which can present different variants of MVL and, in particular, issues on logical completeness of defined systems. The completeness of some type of constructed proof systems is proved by hard, many-stepped operations of immersion into two-valued logic, that for the other systems it is proved by reducing to the completeness of the first type.

The current research refers to the problem of constructing some universal proof systems for all versions of propositional MVL. Two types of propositional proof systems are described in this paper such that propositional proof system for every variant of MVL can be presented in both of the described forms. The first of introduced systems is a Gentzen-like system, the second one is based on the generalization of the notion of determinative disjunctive normal forms, defined in [3] by first coauthor. Some generalization of Kalmar's proof of deducibility for two-valued tautologies in the classical propositional logic (see for example in [4]) gives us a possibility to suggest some simple method for proving the completeness for first type of described systems. The completeness of the second type one obviously follows from its construction. The introduced proof systems

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have the "simple strategist" of proof search and we have investigated also the quantitative properties, related to proof complexity characteristics in described systems. In particular, simultaneously optimal bounds (asymptotically the same upper and lower bounds for each proof complexity characteristics: length, size, space and width are obtained for some class of many-valued tautologies in the systems, considered for some versions of MVL.

This article consists of the following main sections: Introduction, Preliminaries, in which the main notions, materials and methods are given, Main Results, Discussion and Conclusion.

2 Preliminaries (main notions, materials and methods)

2.1 Main notions of k-valued logic

Let E_k be the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$. We used the well-known notions of propositional formula, which is defined as usual from propositional variables with values from E_k, (may be also propositional constants), parentheses (,), and logical connectives &, ∨, ⊃, ¬, all of which can be defined by different modes. Additionally, we used two modes of exponential function p^{σ} and introduce the additional notion of formula: for every formulas A and B the expression $\boldsymbol{A^B}$ (for both modes) is formula also.

In the considered logics either only **1** or all values of $\frac{1}{2} \leq \frac{\boldsymbol{i}}{\mathbf{k-1}} \leq \mathbf{1}$ can be fixed as ***designated values.***

Definitions of main logical functions are:

$\boldsymbol{p} \vee \boldsymbol{q} = \max(p, q)$	(1) disjunction or
$\boldsymbol{p} \vee \boldsymbol{q} = [(k-1)(p+q)] \pmod k / (k-1)$	(2) disjunction,
$\boldsymbol{p}\&q = \min(p, q)$	(1) conjunction or
$\boldsymbol{p}\&q = \max(\boldsymbol{p} + q - 1, 0)$	(2) conjunction

Sometimes (1) conjunction is denoted by ∧.

For implication we have two following versions:

$$\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, & for \quad p \leq q \\ 1 - p + q, & for \quad p > q \end{cases} \quad \text{(1) Łukasiewicz's implication or}$$

$$\mathbf{p} \supset \boldsymbol{q} = \begin{cases} 1, & for \quad p \leq q \\ q, & for \quad p > q \end{cases} \quad \text{(2) Gödel's implication}$$

And for negation two versions also:

$\neg\boldsymbol{p} = 1 - p$	(1) Łukasiewicz's negation or
$\neg\boldsymbol{p} = ((k-1)\,p + 1)\,(mod\ k) / (k-1)$	(2) cyclically permuting negation.

Sometimes we can use the notation $\bar{\boldsymbol{p}}$ instead of $\neg\boldsymbol{p}$.

For propositional variable $\boldsymbol{p}$ and $\boldsymbol{\delta} = \frac{i}{\mathrm{k}-1} (0 \leq i \leq k-1)$ we define additionally "exponent" functions:

$\boldsymbol{p}^{\delta}$	as $(p \supset \delta)\&(\delta \supset p)$ with (1) implication	(1) exponent,
$\boldsymbol{p}^{\delta}$	as p with $(k-1) - i$ (2) negations.	(2) exponent.

Note, that both (1) exponent and (2) exponent are no new logical functions.

If we fix "**1**" (every values of $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) as designated value, so a formula φ with variables $p_1, p_2, \ldots p_n$ is called ***1-k*-tautology**(***≥1/2-k-tautology***) if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning $\delta_j (1 \leq j \leq n)$ to each p_j gives the value 1 (or some value $\frac{1}{2} \leq \frac{i}{\mathrm{k}-1} \leq 1$) of φ.

Sometimes we call ***1-k*-tautology** or ***≥1/2-k-tautology*** simply ***k*-tautology.**

2.2 Determinative disjunctive normal form for MVL

The notions of determinative conjunct and determinative disjunctive normal forms are introduced at first in [3] and then are described in more detail in [5]. Based on these notions some new proof system for classical propositional logic, dual to resolution system, was defined. Then the analogous systems were given for intuitionistic, minimal, monotone, positive and some other two-valued propositional logics in [6-8]. The notions of determinative conjunct and determinative disjunctive normal form are generalized for some variants of 3-valued logic in [9]. Here we generalize this notion for all variants of MVL.

For every propositional variable p in k-valued logic $p^0, p^{1/\mathrm{k}-1}, \ldots, p^{\mathrm{k}-2/\mathrm{k}-1}$ and p^1 in sense of both exponent modes are the literals. The conjunct K (term) can be represented simply as a set of literals (no conjunct contains a variable with different measures of exponents simultaneously), and DNF can be represented as a set of conjuncts.

As in [5] we call a *replacement-rule* each of the following trivial identities for a propositional formula $\boldsymbol{\psi}$:

for both conjunction and (1) disjunction

$$\varphi \,\&\, 0 = 0 \,\&\, \varphi = 0,$$
$$\varphi \vee 0 = 0 \vee \varphi = \varphi,$$
$$\varphi \& 1 = 1 \& \varphi = \varphi,$$
$$\varphi \vee 1 = 1 \vee \varphi = 1,$$

for (2) disjunction

$$\left(\varphi \vee \frac{i}{k-1}\right) = \left(\frac{i}{k-1} \vee \varphi\right) = \overbrace{\neg\neg \ldots \neg}^{i} \varphi \;\; (0 \leq i \leq k-1),$$

for (1) implication

$$\varphi \supset 0 = \bar{\varphi} \text{ with (1) negation,}$$
$$0 \supset \varphi = 1, \quad \varphi \supset 1 = 1, \quad 1 \supset \varphi = \varphi,$$

for (2) implication

$\varphi \supset 1 = 1, \quad 0 \supset \varphi = 1,$
$\varphi \supset 0 = \overline{sg}\varphi,$ where $\overline{sg}\varphi$is 0for $\varphi > 0$ and 1 for $\varphi = 0,$

for (1) negation

$\neg(\mathrm{i/k-1}) = 1 - \mathrm{i/k-1}\ (0 \le \mathrm{i} \le \mathrm{k-1}), \neg\boldsymbol{\psi} = \boldsymbol{\psi},$

for (2) negation

$$\neg(\mathrm{i/k-1}) = \mathrm{i+1/k-1}\ (0 \le i \le \mathrm{k-2}),$$
$$\neg\mathbf{1} = \mathbf{0}, \overbrace{\neg\neg\ldots\neg}^{\boldsymbol{k}}\boldsymbol{\psi} = \boldsymbol{\psi}.$$

Application of a replacement-rule to some word consists replacing of its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

We introduce the following *auxiliary relations for replacement* as well:

for both variants of conjunction

$$\left(\varphi\,\&\,\frac{i}{k-1}\right) = \left(\frac{i}{k-1}\,\&\,\varphi\right) \le \frac{i}{k-1} \quad (1 \le i \le k-2),$$

for (1) implication

$$\left(\varphi \supset \frac{i}{k-1}\right) \ge \frac{i}{k-1} \text{ and}$$
$$\left(\frac{i}{k-1} \supset \varphi\right) \ge \frac{k-(i+1)}{k-1} \ (1 \le i \le k-2),$$

for (2) implication

$$\left(\varphi \supset \frac{i}{k-1}\right) \ge \frac{i}{k-1} \ (1 \le i \le k-2),$$
$$\left(\frac{i}{k-1} \supset \varphi\right) \ge \varphi \ (1 \le i \le k-1).$$

Let φ be a propositional formula of k-valued logic, $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of φ and P' be some subset of P.

Following [9] we give the generalizations of determinative conjuncts and determinative disjunctive normal forms notions.

Definition 2.1 Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in E_{\mathrm{k}}^m$, the conjunct $K^\sigma = \left\{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\right\}$ is called $\varphi - \frac{i}{k-1}$-determinative $(0 \le i \le k-1)$, if assigning $\sigma_j\ (1 \le j \le m)$ to each p_{i_j} and successively using replacement-rules and, if it is necessary, the auxiliary relations for replacement also, we obtain the value $\frac{i}{k-1}$ of φ independently of the values of the remaining variables.

Every $\varphi - \frac{i}{k-1}-$ determinative conjunct is called also φ-determinative or determinative for φ.

Example It is not difficult to see that the conjuncts $\{\mathrm{p}_1\}$, $\{\neg\neg\mathrm{p}_3\}$, $\{\mathrm{p}_2\}$, $\{\neg\mathrm{p}_1, \neg\mathrm{p}_2\}$are determinative for formula $(\mathrm{p}_1 \supset \mathrm{p}_2) \supset (\mathrm{p}_3 \supset (\neg\mathrm{p}_2 \supset \mathrm{p}_1))$ in 3-valued Łukasiewicz's system based on (1) conjunction, (1) disjunction, (1) implication, (1) negation and (1) exponent. Note that correctness of this statement for conjunct $\{\neg\mathrm{p}_1, \neg\mathrm{p}_2\}$must be proved using the auxiliary relations for replacement as well.

Definition 2.2 A DNF $D = \{K_1, K_2, \ldots, K_j\}$ is called determinative DNF (DDNF) for φ if $\varphi = D$ and if "1" (every of values $\frac{1}{2} \le \frac{i}{\mathrm{k-1}} \le 1$) is (are) fixed as designated value, then every conjunct $K_i\ (1 \le i \le j)$ is 1-determinative ($\frac{i}{\mathrm{k-1}}$ − determinative from indicated intervale) for φ.

Remark 2.1 As in [3] it is also easily proved, that

1) if for some $k-$tautology φ, the minimal number of literals, containing in φ-determinative conjunct, is m,then φ-determinative DNF has at least k^m conjuncts;
2) if for some $k-$tautology φ there is such $m$that every conjunct with m literals is φ-determinative, then there is φ-determinative DNF with no more than k^m conjuncts.

3 Definitions of universal systems for MVL and some properties of them

3.1 Sequent type system US for all versions of MVL

Sequent system uses the denotation of sequent $\Gamma \vdash \Delta$ where Γ (antecedent) and Δ (succedent) are finite (may be empty) sequences (or sets) of propositional formulas.

For every literal C and for any set of literals Γ the axiom scheme of propositional system **US** is $\Gamma, C \to C$.

For every formulas A, B, for any sets of literals Γ, each $\sigma_1, \sigma_2, \sigma$ from the set E_k and for $* \in \{\&, \vee, \supset\}$ the logical rules of **US** are:

$$\vdash * \frac{\Gamma \vdash A^{\sigma_1} \text{and } \Gamma \vdash B^{\sigma_2}}{\Gamma \vdash (A*B)^{\varphi_*(A,B,\sigma_1,\sigma_2)})}$$
$$\vdash \exp \frac{\Gamma \vdash A^{\sigma_1} \text{and } \Gamma \vdash B^{\sigma_2}}{\Gamma \vdash \left(A^B\right)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)}}$$
$$\vdash \neg \frac{\Gamma \vdash A^{\sigma}}{\Gamma \vdash (\neg A)^{\varphi_\neg(A,\sigma)}}$$

literals elimination $\vdash$

$$\frac{\Gamma, p^0 \vdash A,\ \Gamma, p^{\frac{1}{k-1}} \vdash A, \ldots, \Gamma, p^{\frac{k-2}{k-1}} \vdash A, \Gamma,\ p^1 \vdash A}{\Gamma \vdash A},$$

where many-valued functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_\neg(A, \sigma)$, must be defined individually for each version of MVL such, that

1. formulas $A^{\sigma_1} \supset \left(B^{\sigma_2} \supset (A*B)^{\varphi_*(A,B,\sigma_1,\sigma_2)}\right)$, $A^{\sigma_1} \supset \left(B^{\sigma_2} \supset \left(A^B\right)^{\varphi_{\exp}(A,B,\sigma_1,\sigma_2)}\right)$ and

$A^{\sigma} \supset (\neg A)^{\varphi_\neg(A,\sigma)}$ must be k-tautology in this version,

2. if for some $\sigma_1, \sigma_2, \sigma$ the value of $\sigma_1 * \sigma_2$ ($\sigma_1^{\sigma_2}$, $\neg\sigma$) is one of ***designed values*** in this version of MVL, then $(\sigma_1*\sigma_2)^{\varphi_*(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = \sigma_1*\sigma_2$, $((\sigma_1^{\sigma_2})^{\varphi_{\exp}(\sigma_1,\sigma_2,\sigma_1,\sigma_2)} = \sigma_1^{\sigma_2}$, $(\neg\sigma)^{\varphi_\neg(\sigma,\sigma)} = \neg\sigma)$.

We use the well known notion of proof in sequent systems. We say that formula ***A is derived in US iff the sequent*** $\vdash A$ ***is deduced in US.***

3.1.1 Examples of US for some versions of MVL

Here we give some systems, which are described in [9–12].

a)For the first of constructed s**ystems LN**$_k$(Łukasiewicz's negation) with fixed "1" as designated value, use conjunction, disjunction, (1) implication, (1) negation and (1) exponent, and constants $\boldsymbol{\delta} = \frac{i}{k-1}(1 \le \mathrm{i} \le \mathrm{k}-2)$ for using (1)exponent functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_\neg(A, \sigma)$ are defined as follows:

$$\varphi_*(A, B, \sigma_1, \sigma_2) = \sigma_1 * \sigma_2$$
$$\varphi_{exp}(A, B, \sigma_1, \sigma_2) = \sigma_1^{\sigma_2}$$
$$\varphi_\neg(A, \sigma) = \neg\sigma.$$

b) For the second systems **CN**$_3$ (cyclically permuting negation) with fixed "1" as designated value, use conjunction, disjunction, (2) implication, (2) negation and (2) exponent the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{exp}(A, B, \sigma_1, \sigma_2)$, $\varphi_\neg(A, \sigma)$ are defined as follows:

$$\varphi_\supset(A, B, \sigma_1, \sigma_2) = (\sigma_1 \supset \sigma_2)\,\&\, \left(\neg\left(A \vee \bar{A}\right) \vee \left(\bar{\bar{B}} \supset B\right)\right) \vee \left(\neg\left(A \vee \bar{\bar{A}}\right) \& \neg\left(B \vee \bar{B}\right)\right),$$
$$\varphi_\vee(A, B, \sigma_1, \sigma_2) = (\sigma_1 \vee \sigma_2) \vee \left(\left(A \supset \bar{A}\right) \& \neg\left(\bar{B} \vee \bar{\bar{B}}\right)\right) \vee \left(\neg\left(\bar{A} \vee \bar{\bar{A}}\right) \& \left(B \supset \bar{B}\right)\right),$$
$$\varphi_\&(A, B, \sigma_1, \sigma_2) = (\sigma_1 \& \sigma_2) \vee \left(\left(A \& \bar{\bar{A}}\right) \vee \left(B \& \bar{B}\right)\right) \vee \left(\left(A \& \bar{A}\right) \vee \left(B \& \bar{\bar{B}}\right)\right.$$
$$\varphi_{\exp}(A, B, \sigma_1, \sigma_2) = \sigma_1^{\sigma_2} \vee \left(\neg\left(\sigma_1^{\sigma_2}\right) \& \neg\left(\neg\left(A^{\sigma_1} \& \bar{B}^{\sigma_2}\right) \vee \neg\neg\left(A^{\sigma_1} \& \bar{B}^{\sigma_2}\right)\right)\right)$$
$$\varphi_\neg(\mathrm{A}, \sigma) = \neg\sigma.$$

c) For ***LN***$_{3,2}$**-** *Łukasiewicz's logic* with *fixed "1/2" and "1" as designated values, and with (1)* conjunction, (1) disjunction, (1) implication, (1) negation *and* (1) exponent, *and constants 0, 1/2 and 1 for using* (1)exponent we have

$$\varphi_*(A, B, \sigma_1, \sigma_2) = \left(\left(A^{\sigma_1} \& B^{\sigma_2}\right) \& \neg(A * B)\right) \supset \neg\left(\left(A^{\sigma_1} \& B^{\sigma_2}\right) \& \neg(A * B)\right)$$
$$\varphi_{exp}(A, B, \sigma_1, \sigma_2) = \left(\left(A^{\sigma_1} \& B^{\sigma_2}\right) \& \neg\left(A^B\right)\right) \supset \neg\left(\left(A^{\sigma_1} \& B^{\sigma_2}\right) \& \neg\left(A^B\right)\right)$$
$$\varphi_\neg(\mathrm{A}, \sigma) = (A \& \sigma) \supset \neg(A \& \sigma)$$

The work with other version of MVL is in progress.

3.1.2 Completeness of US

Here we give at first for the system **US** some generalization of Kalmar's proof of deducibility for two-valued tautologies in the classical propositional logic [4].

Lemma 3.1 *Let* $P = \{p_1, p_2, \ldots, p_n\}$ *be the set of all variables of any formula* A*, then for every* $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ *the sequent* $p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A^{A(\delta_1,\delta_2,\ldots,\delta_n)}$ *is proved in* **US**.

Proof is given by induction on number n of logical connectives in the formula A. For $n = 1$ we have:

$p^{\delta} \vdash p^{\delta}$ for every δ from $E_{k.}$

Suppose that statement is valid for number of logical connectives is less than n. If the number of logical connectives is n, then formula A can be in one of the following forms:

1. $A = A_1 * A_2$, where $* \in \{\&, \vee, \supset\}$,
2. $A = A_1^{A_2}$,
3. $A = \neg A_1$.

For case 1. $A_1\left(\tilde{\delta}\right) = \sigma_1$, $A_2\left(\tilde{\delta}\right) = \sigma_2 => A\left(\tilde{\delta}\right) = \sigma_1 * \sigma_2$

By induction hypothesis in **US** are proved the sequents

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A_1^{\sigma_1}$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A_2^{\sigma_2}$$

Use the inference rule $\vdash *$ we have that in **US** is proved the sequent

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash (A_1 * A_2)^{\varphi_*(A_1,A_2,\sigma_1,\sigma_2)})$$

For case 2 .$A_1\left(\tilde{\delta}\right) = \sigma_1$, $A_2\left(\tilde{\delta}\right) = \sigma_2 => A\left(\tilde{\delta}\right) = \left(A_1^{A_2}\right)^{\varphi_{\exp}(A_1,A_2,\sigma_1,\sigma_2)}$ and we must use the rule $\vdash$ exp.

For case 3. $A_1\left(\tilde{\delta}\right) = \sigma => A\left(\tilde{\delta}\right) = (\neg A_1)^{\varphi_\neg(A_1,\sigma)}$ we must use the rule $\vdash \neg$. □

Corollary *If Ais k-tautology, then for every* $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ *in* **US** *is proved the sequent*

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A.$$

Really we must use the properties 2) of the functions $\varphi_*(A, B, \sigma_1, \sigma_2)$, $\varphi_{\exp}(A, B, \sigma_1, \sigma_2)$ *and* $\varphi_\neg(A, \sigma)$.

Theorem 3.1 *Any formula is derived in* **US** *iff it is k-tautology.*

Proof It is obvious that every formula, which is derived in **US** is k-tautology.

Let $P = \{p_1, p_2, \ldots, p_n\}$ $(n \geq 1)$ be the set of all variables of any tautology A. For every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ by above corollary we have $p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A$.

For every $\delta_1, \delta_2, \ldots, \delta_{n-1}$ we take into consideration the following k numbers of truth values

$$\delta_1, \delta_2, \ldots, \delta_{n-1}, 0$$
$$\delta_1, \delta_2, \ldots, \delta_{n-1}, \frac{1}{k-1}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\delta_1, \delta_2, \ldots, \delta_{n-1}, \frac{k-2}{k-1}$$
$$\delta_1, \delta_2, \ldots, \delta_{n-1}, 1,$$

for which we have by Corollary of Main Lemma the derivations of the following sequents in **US**

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^0 \vdash A$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^{1/\mathrm{k}-1} \vdash A$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^{\mathrm{k}-2/\mathrm{k}-1} \vdash A$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^1 \vdash A$$

Using the rule ″literals elimination $\vdash$″ we have the derivation in **US** of sequent

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash A.$$

Therefore, the number of premises is $n-1$ now. Repeating the above steps, we will obtain finally the derivation of tautology A in **US.** □

Note that this proof is the analogous to proof of the corresponding theorem for the 2-valued logic [4].

3.2 The universal elimination system UE for all versions of MVL

The axioms of Elimination systems **UE** are not fixed, but for every $\boldsymbol{k-valued\varphi}$ formula each conjunct from some DDNF of $\boldsymbol{\varphi}$ can be considered as an axiom.

For k-valued logic the inference rule is *elimination rule* (ε-rule)

$$\frac{K_0 \cup \{p^0\}, K_1 \cup \left\{p^{\frac{1}{k-1}}\right\}, \ldots, K_{k-2} \cup \left\{p^{\frac{k-2}{k-1}}\right\}, K_{k-1} \cup \{p^1\}}{K_0 \cup K_1 \cup \ldots \cup K_{k-2} \cup K_{k-1}},$$

where mutual supplementary literals (variables with corresponding (1) or (2) exponents) are eliminated.

Following [3], a finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of **UE** or is inferred from earlier conjuncts in the sequence by ε-rule is called a proof in **UE**. A DNF $D = \{K_1, K_2, \ldots, K_l\}$ is $k-$tautological if using ε-rule can be proved the empty conjunct (Ø) from the axioms $\{K_1, K_2, \ldots, K_l\}$.

The completeness of these systems is obvious.

4 Bounds of proof complexity measures in *UE*

4.1 Definitions of main proof complexities

Four main characteristics of the proof are considered in the theory of proof complexity. Following [12] we give the formal definitions of all proof complexity measures.

If a proof in the system $\boldsymbol{\Phi}$ is a sequence of lines, where each line is an axiom, or is derived from previous lines by one of a finite set of allowed inference rules, then a $\boldsymbol{\Phi}$-configuration is a set of such lines. A sequence of $\boldsymbol{\Phi}$-configurations $\{\boldsymbol{D_0}, \boldsymbol{D_1}, \ldots, \boldsymbol{D_r}\}$ is said to be $\boldsymbol{\Phi}$-derivation if $\boldsymbol{D_0}$ is empty set and for all $\boldsymbol{t}$ $(\mathbf{1} \leq \boldsymbol{t} \leq \boldsymbol{r})$ the set $\boldsymbol{D_t}$ is obtained from $\boldsymbol{D_{t-1}}$ by one of the following derivation steps:

Axiom Download: $\boldsymbol{D_t} = \boldsymbol{D_{t-1}} \cup \{\boldsymbol{L_A}\}$, where $\boldsymbol{L_A}$ is an axiom of $\boldsymbol{\Phi}$.

Inference: $\boldsymbol{D_t} = \boldsymbol{D_{t-1}} \cup \{\boldsymbol{L}\}$, for some $\boldsymbol{L}$ inferred by one of the inference rules for $\boldsymbol{\Phi}$from a set of assumptions, belonging to $\boldsymbol{D_{t-1}}$.

Erasure: $\boldsymbol{D_t} \subset \boldsymbol{D_{t-1}}$.

A $\boldsymbol{\Phi}$-proof of a tautology $\boldsymbol{\varphi}$ is a $\boldsymbol{\Phi}$-derivation $\{\boldsymbol{D_0}, \boldsymbol{D_1}, \ldots, \boldsymbol{D_r}\}$ such that $\tilde{\boldsymbol{\varphi}} \in \boldsymbol{D_r}$, where $\tilde{\boldsymbol{\varphi}}$ is empty conjunct in $\boldsymbol{ECN_k}$and $\tilde{\boldsymbol{\varphi}}$ is $\boldsymbol{\varphi}$ in $\boldsymbol{CN_k}$-cut-free.

By $|\boldsymbol{\varphi}|$ we denote the size of a formula $\boldsymbol{\varphi}$, defined as the number of all logical signs entries. It is obvious that the full size of a formula, which is understood to be the number of all symbols, is bounded by some linear function in$|\boldsymbol{\varphi}|$.

The ***size*** (**l**) of a $\boldsymbol{\Phi}$-proof is a sum of the sizes of all lines in a proof, where lines that are derived multiple times are counted without repetitions. The ***steps*** (***t***) of a $\boldsymbol{\Phi}$-proof is the number of axioms downloads and inference steps in it. The ***space*** (s)of a $\boldsymbol{\Phi}$-proof is the maximal space of a configuration in a proof, where the space of a configuration is the total number of logical signs in a configuration, counted with repetitions. The ***width*** (***w***) of a $\boldsymbol{\Phi}$-proof is the size of the widest line in a proof.

Let $\boldsymbol{\Phi}$ be a proof system and $\boldsymbol{\varphi}$ be a tautology. As known the minimal possible value of ***t-complexity***(***l-complexity,s-complexity,w-complexity***)for all proofs of tautology $\boldsymbol{\varphi}$ in $\boldsymbol{\Phi}$ is denoted by $\boldsymbol{t^{\Phi}_{\varphi}}\left(\mathbf{l^{\Phi}_{\varphi}}, \boldsymbol{s^{\Phi}_{\varphi}}, \boldsymbol{w^{\Phi}_{\varphi}}\right)$.

Let Φ_1 and Φ_2 be two different proof systems.

Definition 4.1 The system Φ_1 *p-simulates* the system Φ_2 if there exist the polynomial $p()$ such that for each formula φ provable both in the systems $\Phi_1$and Φ_2, we have $l^{\Phi 1}_{\varphi} \leq p\left(l^{\Phi 2}_{\varphi}\right)$.

Definition 4.2 The systems $\Phi_1$and Φ_2 are *p-equivalent,* if systems $\Phi_1$and Φ_2*p-l-simulate* each other.

4.2 Bounds of proof complexity measures of some classes of k-tautologies in some modifications of *UE*

In some papers in area of propositional proof complexity for 2-valued classical logic [3,5] the following tautologies (Topsy-Turvy Matrix) play key role

$$\boldsymbol{TTM_{n,m}} = \bigvee_{(\sigma_1,\sigma_2,\ldots,\sigma_n)\in E^n} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}^{\sigma_j} \left(\boldsymbol{n}\geq\mathbf{1}, \mathbf{1}\leq\boldsymbol{m}\leq\mathbf{2}^{\boldsymbol{n}}-1\right).$$

For all fixed $\boldsymbol{n} \geq \mathbf{1}$ and $\boldsymbol{m}$ in above indicated intervals every formula of this kind expresses the following true statement: given a 0,1-matrix of order $\boldsymbol{n} \times \boldsymbol{m}$ we can "topsy-turvy" some strings (writing 0 instead of 1 and 1 instead of 0) so that each column will contain at least one 1.

In [3] is proved that for formulas $\boldsymbol{\varphi_n} = \boldsymbol{TTM_{n,m}}$ for every $\boldsymbol{n} \geq \mathbf{1}$ and $\boldsymbol{m} = \mathbf{2^n} - \mathbf{1}$ the following bounds are true:

$\log_2 (|\varphi_n|) = \theta (n)$;

$\log_2\log_2 (t (\varphi_n)) = \theta (n)$;

$\log_2\log_2 (l (\varphi_n)) = \theta (n)$;

$\log_2 (s (\varphi_n)) = \theta (n)$;

$\log_2 (w (\varphi_n)) = \theta (n)$.

We have generalized the notion "topsy-turvy" as follow:

Definition 4.3 Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in E^m_k$ and $\delta = \frac{i}{k-1}$ $(0\leq i\leq k-1)$ we call δ-(1)-topsy-turvy-result (δ-(2)-topsy-turvy-result) the cortege $\tilde{\sigma\delta}$, which contains every σ_j $(1 \leq j \leq m)$ **with (1) exponent δ for (1) negation** (**with (2) exponentδ for*(2) negation***).

The results of proof complexities measures investigations for generalizations of above tautologies in k-valued logics with (1) conjunction, (1) disjunction, (1) implication, (1) negation ((1) conjunction, (1) disjunction, (2) implication, (2) negation) and only **1** as designated value are given in [8,10,14]:

For 1-k-tautologies $(k \geq 3)$ $\boldsymbol{\varphi_n} = \mathbf{TTM_{n,m}}$ for every $\mathbf{n} \geq \mathbf{1}$ and $\mathbf{m} = \mathbf{k^{[n/k]}}$, where $\mathbf{TTM_{n,m}} = \bigvee_{(\sigma_1,\sigma_2,\ldots,\sigma_n)\in E^n_k} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} \mathbf{p_{i_j}^{\sigma_j}}$ the following bounds are true

$\log_k (|\varphi_n|) = \theta (n)$;

$\log_k\log_k (t (\varphi_n)) = \theta (n)$;

$\log_k\log_k (l (\varphi_n)) = \theta (n)$;

$\log_k (s (\varphi_n)) = \theta (n)$;

$\log_k (w (\varphi_n)) = \theta (n)$.

It is not difficult to prove the same bounds for (2) disjunction as well.

In this paper we will generalize, in particular, these results for ≥1/2-k-tautologies in various logics.

Given $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in E^m_k$ and $\delta = \frac{i}{k-1}$ $(0 \leq i \leq k-1)$ we denote by $|\tilde{\sigma}(\boldsymbol{\delta})|$ the number of $\boldsymbol{\delta}$ occurrences in $\tilde{\sigma}$.

Lemma 4.1 *In given k-valued* $\mathbf{0}, \frac{\mathbf{1}}{\mathbf{k-1}}, \ldots, \frac{\mathbf{k-2}}{\mathbf{k-1}}, \mathbf{1}$*-matrix of order* $\mathbf{n} \times \mathbf{m}$ *can do 0-(1)-"topsy-turvy" some strings such that each column will contain at least one of value* $\frac{\mathbf{i}}{\mathbf{k-1}}$ *from interval [1/2,1], iff* $m \leq 2^n - 1$.

Proof is given by induction on number $\mathbf{n}$ of matrix strings.For $\mathbf{n} = \mathbf{1}, \mathbf{m} = 1$ proof is obvious. Suppose that statement is valid for $\mathbf{n}$ strings. If the number of strings is $\mathbf{n} + 1$ and the number of columns is $m \leq 2^{n+1}$ -1, we consider the last string $\tilde{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_m)$. If sum of $|\tilde{\sigma}(\delta)|$ for $\delta = \frac{\mathbf{i}}{\mathbf{k-1}} < 1/2$ is more or equal than m/2then after 0-(1)-topsy-turvy we obtain in the last string at least 2^n numbers $\delta = \frac{\mathbf{i}}{\mathbf{k-1}} \geq 1/2$, therefore, we will have at least 2^n columns, which contain at least one $\delta = \frac{\mathbf{i}}{\mathbf{k-1}} \geq 1/2$. If sum of $|\tilde{\sigma}(\delta)|$ for $\delta = \frac{\mathbf{i}}{\mathbf{k-1}} < 1/2$ is less than m/2 then we can do nothing (1-(1)-topsy-turvy). In this case also we will have at least 2^n columns, which contain at least one $\delta = \frac{\mathbf{i}}{\mathbf{k-1}} \geq 1/2$. For submatrix from the other columns and first $\mathbf{n}$ strings the statement is valid by induction supposition. □

Corollary 1 *For every* $\mathbf{n} \geq \mathbf{1}$ *and* $\mathbf{m} \leq \mathbf{2^n} - \mathbf{1}$ *the following formulas*

$$\mathrm{L}\boldsymbol{TTM}_{n,m} = \bigvee_{(\sigma_1,\sigma_2,\ldots,\sigma_n)\in E^n} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}^{\sigma_j}, \text{ where } E=\{0,1\},$$

are ≥ 1/2-k-tautologies in k-valued logics with (1) conjunction, (1) or (2) disjunction, (1) implication, (1) negation and (1) exponent.

Lemma 4.2 *In given k-valued* $\mathbf{0}, \frac{\mathbf{1}}{\mathbf{k-1}}, \ldots, \frac{\mathbf{k-2}}{\mathbf{k-1}}, \mathbf{1}$*-matrix of order* $\boldsymbol{n} \times \boldsymbol{m}$ *can do (2)-"topsy-turvy" some strings such,* *that each column will contain at least one of value* $\frac{i}{\mathbf{k-1}}$ *from interval [1/2,*$\mathbf{1}$*], iff* $m \le 2^n - 1$.

Proof is given by induction on number n of matrix strings. For $n = 1$,$m = 1$ *proof is obvious*. Suppose that statement is valid for n strings. If the number of strings is $n + 1$ and the number of columns is $m \le 2^{n+1}$ -1, we consider the last string $\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m)$. If sum of $|\ \tilde{\sigma}(\delta)|$ for $\delta = \frac{i}{k-1} < 1/2$ is more or equal than $m/2$ then after [k/2]/(k-1)-(2)-topsy-turvy we obtain in the last string at least 2^n numbers $\delta = \frac{i}{k-1} \ge 1/2$, therefore, we will have at least 2^n columns, which contain at least one $\delta = \frac{i}{k-1} \ge 1/2$. If sum of $|\ \tilde{\sigma}(\delta)|$ for $\delta = \frac{i}{k-1} > 1/2$ is less than $m/2$ then we can do nothing (1-(1)-topsy-turvy). In this case also we will have at least 2^n columns, which contain at least one $\delta = \frac{i}{k-1} \ge 1/2$. For submatrix from the other columns and first n strings the statement is valid by induction supposition. □

Corollary 2 *For every* $\boldsymbol{n} \ge \mathbf{1}$ *and* $\boldsymbol{m} \le \mathbf{2}^{\boldsymbol{n}} - \mathbf{1}$ *the following formulas*

$$\boldsymbol{CTTM_{n,m}} = \bigvee_{(\sigma_1,\sigma_2,\ldots,\sigma_n)\in E^n} \bigwedge_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}^{\sigma_j}, \text{ where } E=\{0,1\},$$

are $\ge$***1/2-k*-tautologies** *in k-valued logics with (1) conjunction, (1) or (2) disjunction, (2) implication, (2) negation and (2) exponent.*

Lemma 4.3 *The bounds of minimal possible value of* ***s-complexity*** *for all proofs of k-tautology* $\boldsymbol{\varphi}$ *with* $\boldsymbol{n}$ *variables in* ***UE*** *are:* $s_{\varphi} = O\left(n^2\right)$ *and* $s_{\varphi} = \Omega\left(n\right)$.

Proof is given by analogy with the proof of Lemma 3. from [14].

Theorem 4.1 *For every* $k \ge 3$ *there are the sequences of* $\ge$*1/2-k− tautologies* $\varphi_{\mathbf{n}}$*, for the proof complexity measures of which in some variants of the system* $\boldsymbol{UE}_k$ *are valid in the following equations:*

$$log_2\left(|\varphi_n|\right) = \theta\left(n\right);$$

$$log_2 log_k\left(t\left(\varphi_n\right)\right) = \theta\left(n\right);$$

$$log_2 log_k\left(l\left(\varphi_n\right)\right) = \theta\left(n\right);$$

$$log_2\left(s\left(\varphi_n\right)\right) = \theta\left(n\right);$$

$$log_2\left(w\left(\varphi_n\right)\right) = \theta\left(n\right).$$

Proof As $\boldsymbol{\varphi_n}$ we take the formulas $\boldsymbol{LTTM_{n,m}}$ ($\boldsymbol{CTTM_{n,m}}$) for every $\boldsymbol{n} \ge \mathbf{1}$and $\boldsymbol{m} = \mathbf{2}^{\boldsymbol{n}} - \mathbf{1}$. For upper bounds we use the perfect DNF of $\boldsymbol{\varphi_n}$, and for lower bounds—the properties of $\boldsymbol{\varphi_n}$–determinative conjuncts.

It is not difficult to see, that number of variables of φ_n is $\boldsymbol{n}\mathbf{2}^{\boldsymbol{n}}\left(\mathbf{2}^{\boldsymbol{n}} - \mathbf{1}\right)$, the minimal number of variables in every $\boldsymbol{\varphi_n}$–determinative conjuncts is $\mathbf{2}^{\boldsymbol{n}} - \mathbf{1}$, therefore, by Remark 3.1.2. the minimal number of $\boldsymbol{\varphi_n}$–determinative conjuncts is $\mathbf{k}^{\mathbf{2}^{n}-\mathbf{1}}$, hence the number of axioms, using in the each variant of system $\boldsymbol{UE_k}$, must be at least $\mathbf{k}^{\mathbf{2}^{n}-\mathbf{1}}$. Therefore, using these statements and Lemma 4.3.,we can obtain all upper and lower bounds. □

Corollary *If we take the analogous formulas for k-valued logics for every* $\mathbf{n} \ge \mathbf{1}$ *and corresponding* $\boldsymbol{m}$*, then the analogous results can be obtained for more variants of* $\boldsymbol{UE_k}$ *.*

It is important to notice that all above bounds are valid in cut-free sequent and cut-free Frege systems, which are defined in [11] for each above variants of MVL and which are polynomially equivalent to corresponding Elimination system. It is not difficult to prove that the systems **UE** for every version of MVL are polynomially equivalent to some simple generalization of corresponding systems **US**, where in the rules with two or more premises we did not take the same set Γ but may be the different sets $\Gamma_1, \Gamma_2, \ldots, \Gamma_k (k \ge 2)$ and the set $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_k$ in the result of inference rules, therefore, the same proof complexities bounds for the same classes of formulas are valid in generalization of the corresponding **US** as well.

5 Discussion

Our idea can be realized for other variants of MVL, which can be constructed with other combination of main logical functions, defined in point 2.1., may be with other definitions of main logical functions as well. In particular, we did not investigate some combinations with (2) conjunction. Our idea can be realized for Fuzzy logic, where the set of truth values of variables is [0, 1]. As literals we can take simple literals p^{α} from some number α from [0,1] and interval-literals $p^{I}$from some interval I from [0,1]; as contrary literals we can consider $p^{[0,\alpha]}$ and $p^{[1-\alpha,1]}$ for some α, which is more or equal 0,5. Replacement rules must correspond to definitions of main operations in corresponding Fuzzy system. Determinative conjunct can be defined directly as above for 2-valued classical logic, but we must prove also some relation between the determinative conjuncts in classical and in Fuzzy logic. Determinative DNF and Elimination rules can be defined as above. It is necessary to note that suggested here Elimination Systems can be easy transformed into corresponding cut-free Frege systems, some of which are described in [11]. Investigations of these problems and analogous investigations for other interesting logics are in progress.

6 Conclusion

In this paper, we developed some idea for construction of Universal Sequent proof system for all variants of MVL and give realization of this system for some versions of MVL. The suggested method of mentioned proof system definition can be easily realized for many other variants of MVL.

The completeness the constructed systems is proved directly by generalization of the corresponding proofs method for 2-valued classical logic. The second Universal Elimination system is based on the notion of determinative disjunctive form. The completeness of such system obviously follows from its construction. The preference of such systems is also the simple strategy of proof steps choice and the possibility of the automatic receipt of exponential lower bounds for tautologies with specific properties: minimal numbers of literals in determinative conjunct must be by order nearly equal to the size of formula.

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