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Algebras with Parastrophically Uncancellable Quasigroup Equations

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Abstract. We consider 48 parastrophically uncancellable quadratic functional equations with four object variables and two quasigroup operations in two classes: balanced non–Belousov (consists of 16 equations) and non–balanced non–gemini (consists of 32 equations). A linear representation of a group (Abelian group) for a pair of quasigroup operations satisfying one of these parastrophically uncancellable quadratic equations is obtained. As a consequence of these results, a linear representation for every operation of a binary algebra satisfying one of these hyperidentities is obtained.

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Dedicated to V. D. Belousov
and G. B. Belyavskaya

1 Introduction

A binary quasigroup is usually defined to be a groupoid $(B; f)$ such that for any $a, b \in B$ there are unique solutions x and y to the following equations:

$$f(a, x) = b \quad \text{and} \quad f(y, a) = b.$$

The basic properties of quasigroups were given in books [3, 8, 9, 24]. We remind the reader of those properties we shall use in the paper.

If $(B; f)$ is quasigroup we say that f is a quasigroup operation. A loop is a quasigroup with unit (e) such that

$$f(e, x) = f(x, e) = x.$$

Groups are associative quasigroups, i. e. they satisfy:

$$f(f(x, y), z) = f(x, f(y, z))$$

and they necessarily contain a unit. A quasigroup is commutative if

$$f(x, y) = f(y, x). \tag{1.1}$$

Commutative groups are also known as Abelian groups.

A triple (α, β, γ) of bijections from a set B onto a set C is called an isotopy of a groupoid $(B; f)$ onto a groupoid $(C; g)$ provided

$$\gamma f(x, y) = g(\alpha x, \beta y)$$

for all $x, y \in B$. $(C; g)$ is then called an isotope of $(B; f)$, and groupoids $(B; f)$ and $(C; g)$ are said to be isotopic to each other. An isotopy of $(B; f)$ onto $(B; f)$ is called an autotopy of $(B; f)$. Let α and β be permutations of B and let ι denote the identity map on B. Then (α, β, ι) is a principal isotopy of a groupoid $(B; f)$ onto a groupoid $(B; g)$ means that (α, β, ι) is an isotopy of $(B; f)$ onto $(B; g)$. Isotopy is a generalization of isomorphism. Isotopic image of a quasigroup is again a quasigroup. A loop isotopic to a group is isomorphic to it. Every quasigroup is isotopic to some loop, i. e., it is a loop isotope.

If $(B; +)$ is a group, then the bijection $\alpha : B \to B$ is called a *holomorphism* of $(B; +)$ if

$$\alpha(x + y^{-1} + z) = \alpha x + (\alpha y)^{-1} + \alpha z. \tag{1.2}$$

The set of all holomorphisms of $(B; +)$ is denoted by $Hol(B; +)$. It is a group under the composition of mappings: $(\alpha \cdot \beta)x = \beta(\alpha x)$, for every $x \in B$. Note that this concept is equivalent to the concept of quasiautomorphism of groups, by [3].

A binary quasigroup $(B; f)$ is linear over a group (Abelian group) if

$$f(x, y) = \varphi x + a + \psi y,$$

where $(B; +)$ is a group (Abelian group), φ and ψ are automorphisms of $(B; +)$ and $a \in B$ is a fixed element. A quasigroup linear over an Abelian group is also called a T-quasigroup.

Quasigroups are important algebraic (combinatorial, geometric) structures which arise in various areas of mathematics and other disciplines. We mention just a few of their applications: in combinatorics (as latin squares, see [9]), in geometry (as nets/webs, see [4]), in statistics (see [11]), in special theory of relativity (see [27]), in coding theory and cryptography [25].

2 Preliminaries

We use (object) variables x, y, z, u, v, w (perhaps with indices) and operation symbols (i. e. functional variables) f, g, h (also with indices). We assume that all operation symbols represent quasigroup operations.

The set of all variables which appear in a term t is called the *content* of t and is denoted by $var(t)$. A variable x is *linear variable* in a term t when it occurs just once in t. A variable x is *quadratic variable* in a term t when it occurs twice in t. The sets of all linear and quadratic variables of term t are denoted by $var_1(t)$ and $var_2(t)$, respectively.

A *functional equation* is an equality $s = t$, where s and t are terms with symbols of unknown operations occurring in at least one of them.

Definition 1. A functional equation $s = t$ is *quadratic* if every (object) variable occurs exactly twice in $s = t$. It is *balanced* if every (object) variable appears exactly once in s and once in t.

Definition 2. A variable x from a quadratic equation $s = t$ is *linear* if x occurrs once in s and once in t; it is *left (right) quadratic* if it occurrs twice in $s\,(t)$ and *quadratic* if it is either left or right quadratic.

Definition 3. A balanced equation $s = t$ is *Belousov* if for every subterm p of s (t) there is a subterm q of t (s) such that p and q have exactly the same variables.

Definition 4. A quadratic quasigroup equation is *gemini* iff it is a theorem of TS-loops (= Steiner loops), i. e., consequence of the identities of the variety of TS-loops.

Definition 5. Functional equation $s = t$ is *generalized* if every operation symbol from $s = t$ occurrs there just once.

Definition 6. Let x be a variable occurring in a quadratic equation $s = t$. The function Lh (Rh) of the *left (right) height of the variable x in the equation $s = t$ is given by:*

- *If* $x \notin var(t)$, *then* $\mathrm{Lh}(x,t)$ $(\mathrm{Rh}(x,t))$ *is not defined,*
- $Lh(x,x) = 0$ $(Rh(x,x) = 0)$,
- *If* $t = f(t_1,t_2)$ *and both occurrences of* x *are in* t_1 *then* $\mathrm{Lh}(x,t) = 1 + \mathrm{Lh}(x,t_1)$ $(\mathrm{Rh}(x,t) = 1 + \mathrm{Rh}(x,t_1))$,
- *If* $t = f(t_1,t_2)$ *and both occurrences of* x *are in* t_2 *then* $\mathrm{Lh}(x,t) = 1 + \mathrm{Lh}(x,t_2)$ $(\mathrm{Rh}(x,t) = 1 + \mathrm{Rh}(x,t_2))$,
- *If* $t = f(t_1,t_2)$ *and* x *occurrs in both* t_1 *and* t_2 *then* $\mathrm{Lh}(x,t) = 1+\mathrm{Lh}(x,t_1)$ $(\mathrm{Rh}(x,t) = 1 + \mathrm{Rh}(x,t_2))$,
- $\mathrm{Lh}(x, s=t) = \begin{cases} \mathrm{Lh}(x,s) & \textit{if } x \in var(s), \\ \mathrm{Lh}(x,t) & \textit{otherwise,} \end{cases}$
- $\mathrm{Rh}(x, s=t) = \begin{cases} \mathrm{Rh}(x,t) & \textit{if } x \in var(t), \\ \mathrm{Rh}(x,s) & \textit{otherwise.} \end{cases}$

Definition 7. Let $s = t$ be a quadratic equation. It is a *level equation iff* $\mathrm{Lh}(x, s = t) = \mathrm{Rh}(y, s = t)$ *for all variables* x, y *of* $s = t$.

Example 1. The following are various functional equations:

(commutativity)	$f(x,y) = f(y,x),$	(2.1)
(associativity)	$f(f(x,y),z) = f(x,f(y,z)),$	(2.2)
(mediality)	$f(f(x,y),f(u,v)) = f(f(x,u),f(y,v)),$	(2.3)
(paramediality)	$f(f(x,y),f(u,v)) = f(f(v,y),f(u,x)),$	(2.4)
(distributivity)	$f(x,f(y,z)) = f(f(x,y),f(x,z)),$	(2.5)
(transitivity)	$f(f(x,y),f(y,z)) = f(x,z),$	(2.6)
(intermediality)	$f(f(x,y),f(y,u)) = f(f(x,v),f(v,u)),$	(2.7)
(extramediality)	$f(f(x,y),f(u,x)) = f(f(v,y),f(u,v)),$	(2.8)
(4-palindromic identity)	$f(f(x,y),f(u,v)) = f(f(v,u),f(y,x)),$	(2.9)
(idempotency)	$f(x,x) = x,$	(2.10)
(trivial)	$f(x,y) = f(x,y),$	(2.11)
	$f(x,f(y,z)) = f(f(z,y),x).$	(2.12)

Associativity, (para)mediality, 4-palindromic, trivial identity and (2.12) are balanced, transitivity, intermediality and extramediality are quadratic but not balanced and idempotency and (left) distributivity are not even quadratic. Commutativity, trivial, 4-palindromic and (2.12) are gemini functional equations and since they are balanced, they are Belousov equations as well. The equations (2.2) − (2.8) are non-gemini and non-Belousov. Commutativity, mediality, paramediality, intermediality, extramediality, 4–palindromic and trivial identity are level equations.

Every quasigroup satisfying (para)medial identity is called *(para)medial quasigroup*. Every quasigroup satisfying 4-palindromic identity is called 4-*palindromic quasigroup*.

Theorem 1 (Toyoda [26])**.** *If $(B;f)$ is a medial quasigroup then there exists an Abelian group $(B;+)$ such that $f(x,y) = \varphi(x) + c + \psi(y)$, where $\varphi,\psi \in Aut(B;+)$, $\varphi\psi = \psi\varphi$ and $c \in B$.*

Theorem 2 (Němec, Kepka [23])**.** *If $(B;f)$ is a paramedial quasigroup then there exists an Abelian group $(B;+)$ such that $f(x,y) = \varphi(x) + c + \psi(y)$, where $\varphi,\psi \in Aut(B;+)$, $\varphi\varphi = \psi\psi$ and $c \in B$.*

More generaly, considering the following equations with two functional variables, we can define the notion of (para)medial pair of operations:

$$f_1(f_2(x,y),f_2(u,v)) = f_2(f_1(x,u),f_1(y,v)), \tag{2.13}$$

$$f_1(f_2(x,y),f_2(u,v)) = f_2(f_1(v,y),f_1(u,x)). \tag{2.14}$$

Definition 8. A pair (f_1,f_2) of binary operations is called *(para)medial pair of operations* if the algebra $(B;f_1,f_2)$ satisfies the equation (2.13) ((2.14)).

Definition 9. A binary algebra $\mathbf{B} = (B; F)$ is called *(para)medial algebra* if every pair of operations of the algebra $\mathbf{B}$ is (para)medial (or, the algebra $\mathbf{B}$ satisfies (para)medial hyperidentity).

The following theorem generalizes above results by Toyoda and Němec, Kepka:

Theorem 3 (Nazari, Movsisyan [22], Ehsani, Movsisyan [10])**.** *Let the set B form a quasigroup under the binary operations f_1 and f_2. If the pair of binary operations (f_1, f_2) is (para)medial, then there exists a binary operation $'+'$ under which B forms an Abelian group and for arbitrary elements $x, y \in B$ we have:*

$$f_i(x, y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where c_is are fixed elements of B, and $\varphi_i, \psi_i \in Aut(B; +)$ for $i = 1, 2$, such that:

$\varphi_1\psi_2 = \psi_2\varphi_1$, $\varphi_2\psi_1 = \psi_1\varphi_2$, $\psi_1\psi_2 = \psi_2\psi_1$ *and* $\varphi_1\varphi_2 = \varphi_2\varphi_1$ *should be satisfied by the medial pair of operations,*

$\varphi_1\varphi_2 = \psi_2\psi_1$, $\varphi_2\varphi_1 = \psi_1\psi_2$, $\varphi_1\psi_2 = \varphi_2\psi_1$ *and* $\psi_1\varphi_2 = \psi_2\varphi_1$ *should be satisfied by the paramedial pair of operations.*

The group $(B; +)$, is unique up to isomorphisms.

The following results will be frequently utilized.

Theorem 4 (Aczél, Belousov, Hosszú [1], see also [2])**.** *Let the set B form a quasigroup under six operations $A_i(x, y)$ (for $i = 1, \ldots, 6$). If these operations satisfy the following equation:*

$$A_1(A_2(x, y), A_3(u, v)) = A_4(A_5(x, u), A_6(y, v)), \tag{2.15}$$

for all elements x, y, u and v of the set B then there exists an operation $'+'$ under which B forms an abelian group isotopic to all these six quasigroups. And there exist eight permutations α, β, γ, δ, ϵ, ψ, φ, χ of B such that:

$$\begin{aligned} A_1(x, y) &= \delta x + \varphi y, \\ A_2(x, y) &= \delta^{-1}(\alpha x + \beta y), \\ A_3(x, y) &= \varphi^{-1}(\chi x + \gamma y), \\ A_4(x, y) &= \psi x + \epsilon y, \\ A_5(x, y) &= \psi^{-1}(\alpha x + \chi y), \\ A_6(x, y) &= \epsilon^{-1}(\beta x + \gamma y). \end{aligned}$$

Theorem 5 (Krapež [14])**.** *If the set B forms a quasigroup under four operations $A_i(x, y)$ (for $i = 1, \ldots, 4$) and if these operations satisfy the equation of generalized transitivity:*

$$A_1(A_2(x, y), A_3(y, z)) = A_4(x, z),$$

for all elements $x, y, z \in B$, then there exists an operation $'+'$ under which B forms a group isotopic to all these quasigroups and there exist permutaions α, β, γ, δ, ϵ, ψ, φ, χ of B such that

$$\begin{aligned} A_1(x,y) &= \alpha x + \beta y, \\ A_2(x,y) &= \alpha^{-1}(\alpha\gamma x + \alpha\delta y), \\ A_3(x,y) &= \beta^{-1}(\beta\epsilon x + \beta\psi y), \\ A_4(x,y) &= \varphi x + \chi y. \end{aligned}$$

Theorem 6 (Krapež [13], Belousov [5]). *A quasigroup satisfying a balanced but not Belousov equation is isotopic to a group.*

Theorem 7 (Krapež, Taylor [16]). *A quasigroup satisfying a quadratic but not gemini equation is isotopic to a group.*

3 Parastrophically uncancellable quadratic equations with two function variables

We consider parastrophically uncancellable quadratic quasigroup equations of the form:

$$f_1(f_2(x_1,x_2), f_2(x_3,x_4)) = f_2(f_1(x_5,x_6), f_1(x_7,x_8)) \tag{Eq}$$

where $x_i \in \{x, y, u, v\}$, for $i = 1, \ldots, 8$. Therefore, the equation (Eq) is quadratic level quasigroup equation with four (object) variables each appearing twice in the equation and with two function variables each appearing three times in the equation. There are 48 such equations and we attempt to solve them all.

There is a correspondence between generalized quadratic quasigroup equations and connected cubic graphs, namely Krstić graphs. Two such equations are parastrophically equivalent iff they have the same (i.e. isomorphic) Krstić graphs. Furthermore, an equation is parastrophically uncancellable iff the corresponding Krstić graph is 3–connected. For more detailed account of this correspondence see [16, 17] and [18].

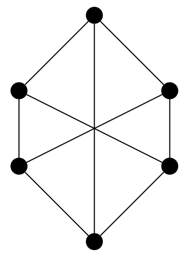
For everyone of the 48 equations (Eq) there is a corresponding generalized equation:

$$f_1(f_3(x_1,x_2), f_4(x_3,x_4)) = f_2(f_5(x_5,x_6), f_6(x_7,x_8)) \tag{GEq}$$

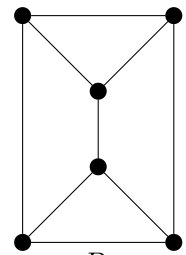
(where $x_i \in \{x, y, u, v\}$, for $i = 1, \ldots, 8$) with the appropriate Krstić graph. This Krstić graph *will be assumed* to be the Krstić graph of (Eq) as well. All these equations can be partitioned into two classes, depending on their Krstić graphs, as follows:

– 16 balanced (and non-Belousov) equations with the Krstić graph $K_{3,3}$,

– 32 non–balanced non-gemini equations with the Krstić graph P_3.



$K_{3,3}$



P_3

To characterize a pair of quasigroup operations which satisfies a non–Belousov balanced functional equation, we need the notion of Lbranch (Rbranch) and the following properties of holomorphisms which were proved for Muofang loops in [19].

Definition 10. Let t be a term and x a variable. We define:

- If $x \notin var(t)$, then $\text{Lbranch}(x,t)$ $(\text{Rbranch}(x,t))$ is not defined,
- $\text{Lbranch}(x,x) = \Lambda$ $(\text{Rbranch}(x,x) = \Lambda)$ (Λ is the empty word),
- If $t = f_i(t_1,t_2)$ and both occurrences of x are in t_1, then $\text{Lbranch}(x,t) = \alpha_i \,\text{Lbranch}(x,t_1)$ $(\text{Rbranch}(x,t) = \alpha_i \,\text{Rbranch}(x,t_1))$,
- If $t = f_i(t_1,t_2)$ and both occurrences of x are in t_2, then $\text{Lbranch}(x,t) = \beta_i \,\text{Lbranch}(x,t_2)$ $(\text{Rbranch}(x,t) = \beta_i \,\text{Rbranch}(x,t_2))$,
- If $t = f_i(t_1,t_2)$ and x occurrs in both t_1 and t_2, then $\text{Lbranch}(x,t) = \alpha_i \,\text{Lbranch}(x,t_1)$ $(\text{Rbranch}(x,t) = \beta_i \,\text{Rbranch}(x,t_2))$,
- $\text{Lbranch}(x, s = t) = \begin{cases} \text{Lbranch}(x,s) & \text{if } \ x \in var(s), \\ \text{Lbranch}(x,t) & \text{otherwise} \end{cases}$
- $\text{Rbranch}(x, s = t) = \begin{cases} \text{Rbranch}(x,t) & \text{if } \ x \in var(t), \\ \text{Rbranch}(x,s) & \text{otherwise} \end{cases}$

Lemma 1. *Let the identity:*

$$\alpha_1(x+y) = \alpha_2(x) + \alpha_3(y)$$

be satisfied for bijections $\alpha_1, \alpha_2, \alpha_3$ on the group $(B;+)$. Then $\alpha_1, \alpha_2, \alpha_3 \in Hol(B;+)$.

Lemma 2. *Every holomorphism α of the group $(B;+)$ has the following forms:*

$$\alpha x = \varphi_1 x + k_1, \qquad \alpha x = k_2 + \varphi_2 x,$$

where $\varphi_1, \varphi_2 \in Aut(B;+)$ and $k_1, k_2 \in B$.

4 Equations with Krstić graph $K_{3,3}$

The class of non–gemini balanced (and therefore non–Belousov) quadratic functional equations consists of the following 16 equations with four object variables x, y, u, v and two quasigroup operations f_1, f_2:

$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(x,u), f_1(y,v)) \tag{4.1}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(x,u), f_1(v,y)) \tag{4.2}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(x,v), f_1(y,u)) \tag{4.3}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(x,v), f_1(u,y)) \tag{4.4}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(y,u), f_1(x,v)) \tag{4.5}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(y,u), f_1(v,x)) \tag{4.6}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(y,v), f_1(x,u)) \tag{4.7}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(y,v), f_1(u,x)) \tag{4.8}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(u,x), f_1(y,v)) \tag{4.9}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(u,x), f_1(v,y)) \tag{4.10}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(u,y), f_1(x,v)) \tag{4.11}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(u,y), f_1(v,x)) \tag{4.12}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(v,x), f_1(y,u)) \tag{4.13}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(v,x), f_1(u,y)) \tag{4.14}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(v,y), f_1(x,u)) \tag{4.15}$$
$$f_1(f_2(x,y), f_2(u,v)) = f_2(f_1(v,y), f_1(u,x)) \tag{4.16}$$

The following result generalizes, on the one hand Theorem 3, and on the other, the results from and immediately after Example 7 in [12].

Theorem 8. *Let the balanced non–Belousov quasigroup equations* (4.j) (j = 1,. . . ,16) *have the Krstić graph* $K_{3,3}$*. A general solution of any of* (4.j) *is given by:*

$$f_i(x,y) = \alpha_i x + c_i + \beta_i y \quad (i = 1, 2) \tag{4.17}$$

where:

- $(B;+)$ *is an arbitrary Abelian group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2, c_2) = f_2(c_1, c_1)$*,*
- α_i, β_i $(i = 1, 2)$ *are arbitrary automorphisms of* $+$ *such that:*

$$\text{Lbranch}(z, (4.j)) = \text{Rbranch}(z, (4.j)) \tag{4.18}$$

  for all variables z *of the equation* (4.j)*.*

The group $(B;+)$ *is unique up to isomorphism.*

Proof. (1) To show that the pair (f_1, f_2) of operations is a solution of (4.j), just replace $f_i(x, y)$ in (4.j) using (4.17) and all conditions (4.18).

(2) An equation (4.j) is an instance of the appropriate generalized equation (GEq) with the Krstić graph $K_{3,3}$. Therefore, all operations of (GEq) are isotopic to an Abelian group $+$ and the main operations f_1, f_2 can be chosen to be principally isotopic to it (see [17]):

$$f_i(x, y) = \lambda_i x + \varrho_i y \quad (i = 1, 2).$$

Replace this in (Eq) to get:

$$\lambda_1 f_2(x_1, x_2) + \varrho_1 f_2(x_3, x_4) = \lambda_2 f_1(x_5, x_6) + \varrho_2 f_1(x_7, x_8). \tag{4.19}$$

Since variables x_1, x_2 are separated on the right hand side of equation (4.19), replacing x_3 and x_4 by 0, we get:

$$\lambda_1(\lambda_2 x_1 + \varrho_2 x_2) + d = \sigma x_1 + \tau x_2$$

for $d = \varrho_1(\lambda_2 0 + \varrho_2 0)$ and appropriate σ, τ depending on n. Therefore:

$$\lambda_1(z + w) = \sigma\lambda_2^{-1} z + T\tau\varrho_2^{-1} w$$

(where $Tx = x - d$) and $\lambda_1 \in Hol(B; +)$.

Analogously we get $\varrho_1, \lambda_2, \varrho_2 \in Hol(B; +)$.

Using Lemma 2 we easily get (4.17) for $i = 1, 2$ where α_i, β_i are automorphisms of $(B; +)$.

Replace f_1 and f_2 in (4.j):

$$\alpha_1(\alpha_2 x_1 + c_2 + \beta_2 x_2) + c_1 + \beta_1(\alpha_2 x_3 + c_2 + \beta_2 x_4) =$$

$$= \alpha_2(\alpha_1 x_5 + c_1 + \beta_1 x_6) + c_2 + \beta_2(\alpha_1 x_7 + c_1 + \beta_1 x_8).$$

Replacing $x_1 = x_2 = x_3 = x_4 = 0$, we get:

$$\alpha_1 c_2 + c_1 + \beta_1 c_2 = \alpha_2 c_1 + c_2 + \beta_2 c_1,$$

i. e. $f_1(c_2, c_2) = f_2(c_1, c_1)$.

For $x_2 = x_3 = x_4 = 0$, we get:

$$\mathrm{Lbranch}(x_1, (4.\mathrm{j})) = \alpha_1\alpha_2 \mathrm{x}_1 = \gamma\delta \mathrm{x}_1 = \mathrm{Rbranch}(\mathrm{x}_1, (4.\mathrm{j}))$$

for some $\gamma, \delta \in \{\alpha_1, \beta_1, \alpha_2, \beta_2\}$ depending on j.

Analogously:

$$\mathrm{Lbranch}(x_i, (4.\mathrm{j})) = \mathrm{Rbranch}(\mathrm{x_i}, (4.\mathrm{j}))$$

for $i = 2, 3, 4$.

The uniqueness of the group $(B; +)$ follows from the Albert Theorem (see [6]): If two groups are isotopic, then they are isomorphic. □

5 Equations with Krstić graph P_3

There exist 32 parastrophically uncancellable non-gemini and non-balanced quadratic functional equations with four object variables and two operations:

$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(y,v),f_1(u,v)) \tag{5.1}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(y,v),f_1(v,u)) \tag{5.2}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(u,v),f_1(y,v)) \tag{5.3}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(u,v),f_1(v,y)) \tag{5.4}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(v,y),f_1(u,v)) \tag{5.5}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(v,y),f_1(v,u)) \tag{5.6}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(v,u),f_1(y,v)) \tag{5.7}$$
$$f_1(f_2(x,y),f_2(x,u)) = f_2(f_1(v,u),f_1(v,y)) \tag{5.8}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(x,v),f_1(u,v)) \tag{5.9}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(x,v),f_1(v,u)) \tag{5.10}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(u,v),f_1(x,v)) \tag{5.11}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(u,v),f_1(v,x)) \tag{5.12}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(v,x),f_1(u,v)) \tag{5.13}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(v,x),f_1(v,u)) \tag{5.14}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(v,u),f_1(x,v)) \tag{5.15}$$
$$f_1(f_2(x,y),f_2(y,u)) = f_2(f_1(v,u),f_1(v,x)) \tag{5.16}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(y,v),f_1(u,v)) \tag{5.17}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(y,v),f_1(v,u)) \tag{5.18}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(u,v),f_1(y,v)) \tag{5.19}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(u,v),f_1(v,y)) \tag{5.20}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(v,y),f_1(u,v)) \tag{5.21}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(v,y),f_1(v,u)) \tag{5.22}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(v,u),f_1(y,v)) \tag{5.23}$$
$$f_1(f_2(x,y),f_2(u,x)) = f_2(f_1(v,u),f_1(v,y)) \tag{5.24}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(x,v),f_1(u,v)) \tag{5.25}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(x,v),f_1(v,u)) \tag{5.26}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(u,v),f_1(x,v)) \tag{5.27}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(u,v),f_1(v,x)) \tag{5.28}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(v,x),f_1(u,v)) \tag{5.29}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(v,x),f_1(v,u)) \tag{5.30}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(v,u),f_1(x,v)) \tag{5.31}$$
$$f_1(f_2(x,y),f_2(u,y)) = f_2(f_1(v,u),f_1(v,x)) \tag{5.32}$$

The next theorem gives a general solution of the equation (5.10) which generalizes the *intermedial equation* (see equation (4.36) and Theorem 8.4 of [15] for the original definition of intermedial equation).

Lemma 3. *A general solution of the equation* (5.10) *is given by:*

$$f_i(x, y) = \alpha_i x + c_i + \beta_i y \quad (i = 1, 2) \tag{5.33}$$

where:

- $(B; +)$ *is an arbitrary group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2, c_2) = f_2(c_1, c_1)$,
- α_i, β_i $(i = 1, 2)$ *are arbitrary automorphisms of* $+$ *such that:*

$$\text{Lbranch}(z, (5.10)) = \text{Rbranch}(z, (5.10)) \tag{5.34}$$

for $z \in \{x, u\}$ *and*

$$\text{Lbranch}(w_i, (5.10))w_i + c_i + \text{Rbranch}(w_i, (5.10))w_i = c_i \tag{5.35}$$

for $i \in \{1, 2\}, w_1 = y$ *and* $w_2 = v$.

The group $(B; +)$ *is unique up to isomorphism.*

Proof. (1) To show that the pair (f_1, f_2) of operations is a solution of (5.10), just replace $f_i(x, y)$ in (5.10) using (5.33) and all conditions (5.34), (5.35).

(2) The equation (5.10) is an instance of the generalized intermedial equation:

$$f_1(h_1(x, y), h_2(y, u)) = f_2(h_3(x, v), h_4(v, u)). \tag{GI}$$

Choose $v = a$ for some $a \in B$ and define $\gamma x = h_1(x, a), \delta u = h_2(a, u)$ and $g(x, u) = f_2(\gamma x, \delta u)$. We get:

$$f_1(h_1(x, y), h_2(y, u)) = g(x, u) \tag{GT}$$

which is the generalized transitivity equation. By Theorem 5 all operations of this equation are isotopic to a group $+$ and the main operations f_1, g can be chosen to be principally isotopic to it:

$$f_1(x, y) = \lambda_1 x + \varrho_1 y, \qquad g(x, y) = \lambda_3 x + \varrho_3 y.$$

It follows that $f_2(x, y) = \lambda_3\gamma^{-1}x + \varrho_3\delta^{-1}y = \lambda_2 x + \varrho_2 y$ for appropriate λ_2, ϱ_2. Replacing this in (5.10) we get:

$$\lambda_1(\lambda_2 x + \varrho_2 y) + \varrho_1(\lambda_2 y + \varrho_2 u) = \lambda_2(\lambda_1 x + \varrho_1 v) + \varrho_2(\lambda_1 v + \varrho_1 u). \tag{5.36}$$

If we choose $\varrho_2 u = \varrho_1 v = 0$ and define $d = \varrho_2(\lambda_1\varrho_1^{-1}0 + \varrho_1\varrho_2^{-1}0)$ we get:

$$\lambda_1(\lambda_2 x + \varrho_2 y) + \varrho_1\lambda_2 y = \lambda_2\lambda_1 x + d$$

which implies that $\lambda_1 \in Hol(B;+)$.

Analogously we get $\varrho_1, \lambda_2, \varrho_2 \in Hol(B;+)$.

Using Lemma 2 we easily get (5.33) for $i = 1, 2$ where α_i, β_i are automorphisms of $(B;+)$.

Replace f_1 and f_2 in (5.10):

$$\alpha_1(\alpha_2 x + c_2 + \beta_2 y) + c_1 + \beta_1(\alpha_2 y + c_2 + \beta_2 u) =$$

$$= \alpha_2(\alpha_1 x + c_1 + \beta_1 v) + c_2 + \beta_2(\alpha_1 v + c_1 + \beta_1 u).$$

Putting $x = y = u = v = 0$, we get:

$$\alpha_1 c_2 + c_1 + \beta_1 c_2 = \alpha_2 c_1 + c_2 + \beta_2 c_1,$$

i. e. $f_1(c_2, c_2) = f_2(c_1, c_1)$.

For $y = u = v = 0$ we get:

$$\text{Lbranch}(x, (5.10)) = \alpha_1\alpha_2 = \alpha_2\alpha_1 = \text{Rbranch}(x, (5.10)).$$

Analogously:

$$\text{Lbranch}(u, (5.10)) = \text{Rbranch}(u, (5.10)),$$

$$\text{Lbranch}(y, (5.10))y + c_1 + \text{Rbranch}(u, (5.10))y = \alpha_1\beta_2 y + c_1 + \beta_1\alpha_2 y = c_1,$$

$$\text{Lbranch}(v, (5.10))v + c_2 + \text{Rbranch}(v, (5.10))v = \alpha_2\beta_1 v + c_2 + \beta_2\alpha_1 v = c_2.$$

The uniqueness of the group $(B;+)$ follows from the Albert Theorem. □

Lemma 4. *A general solution of the equation* (5.j) (j = 1, 2, 5, 6, 9, 13, 14, 17, 18, 21, 22, 25, 26, 29, 30) *is given by:*

$$f_i(x, y) = \alpha_i x + c_i + \beta_i y \quad (i = 1, 2) \tag{5.37}$$

where:

- $(B;+)$ *is an arbitrary Abelian group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2, c_2) = f_2(c_1, c_1)$,
- α_i, β_i $(i = 1, 2)$ *are arbitrary automorphisms of* $+$ *such that:*

$$\text{Lbranch}(z, (5.\text{j})) = \text{Rbranch}(z, (5,\text{j})) \tag{5.38}$$

for all linear variables z *of* (5.j) *and*

$$\text{Lbranch}(w, (5.\text{j}))w + \text{Rbranch}(w, (5.\text{j}))w = 0 \tag{5.39}$$

for all quadratic variables w *from the equation.*

The group $(B;+)$ *is unique up to isomorphism.*

Proof. (1) To show that the pair (f_1, f_2) of operations is a solution of (5.j), just replace $f_i(x, y)$ in (5.j) using (5.37) and all conditions (5.38), (5.39).

(2) The crucial property of all 15 equations (5.j) is that, by applying duality to some of non–main operations of the generalized version of (5.j), they may be transformed into equation (GI):

$$f_1(h_1(x, y), h_2(y, u)) = f_2(h_3(x, v), h_4(v, u))$$

which, by the proof of Lemma 3, has a solution:

$$f_i(x, y) = \alpha_i x + c_i + \beta_i y \quad (i = 1, 2)$$

where $(B; +)$ is a group and α_i, β_i are automorphisms of $+$.

Replacing f_1, f_2 in (5.j), we get:

$$\begin{aligned} &\alpha_1(\alpha_2 x_1 + c_2 + \beta_2 x_2) + c_1 + \beta_1(\alpha_2 x_3 + c_2 + \beta_2 x_4) = \\ =\ &\alpha_2(\alpha_1 x_5 + c_1 + \beta_1 x_6) + c_2 + \beta_2(\alpha_1 x_7 + c_1 + \beta_1 x_8). \end{aligned} \tag{5.40}$$

Just as in the proof of Lemma 3, we conclude that $f_1(c_2, c_2) = f_2(c_1, c_1)$. Let us define $c = f_1(c_2, c_2)$.

To prove the properties from the statement of the lemma, we need to discuss the arrangement $x_1 \dots x_4 = x_5 \dots x_8$ of variables in the equation (5.40). It is easy to see:

- The order of first (i. e. left) appearances of variables is always $xyuv$.
- $x_1 = x$.
- Since P_3 has no loops, $x_2 = y$.
- Either x or y is quadratic, but not both.
- Variable u is always linear.
- Variable v is always quadratic.
- Arrangement $xyyu = xvvu$ is not allowed.

There are two possibilities: x is either linear or quadratic.

a) Variable x is linear (and y is quadratic).
   Again, there are two possibilities: Either $x_3 = y$ or $x_3 = u$.

   a1) $x_3 = y$ (and $x_4 = u$).
       Yet again, there are two possibilities: Either $x_5 = x$ or $x_5 = v$.

a11) The arrangement of variables is $xyyu = xvuv$.
We have equation (5.9). Replacing $x = y = 0$ in (5.40), we get:

$$c + \beta_1\beta_2 u = \alpha_2 c_1 + \alpha_2\beta_1 v + c_2 + \beta_2\alpha_1 u + \beta_2 c_1 + \beta_2\beta_1 v. \tag{5.41}$$

For $v = 0$ we get:

$$\beta_2 c_1 + \beta_1\beta_2 u = \beta_2\alpha_1 u + \beta_2 c_1 \tag{5.42}$$

and for $u = 0$:

$$c - \beta_2\beta_1 v = \alpha_2 c_1 + \alpha_2\beta_1 v + c_2 + \beta_2 c_1. \tag{5.43}$$

Applying (5.42) and (5.43) to (5.41), we conclude:

$$c + \beta_1\beta_2 u - \beta_2\beta_1 v = c - \beta_2\beta_1 v + \beta_1\beta_2 u$$

which is, after cancellation from the left, equivalent to commutativity of $+$. Therefore $(B; +)$ is an Abelian group.

a12) The arrangement of variables is $xyyu = vx(uv \text{ or } vu)$.
Replacement $y = u = 0$ leads to:

$$\alpha_1\alpha_2 x + c = \alpha_2\alpha_1 v + \alpha_2 c_1 + \alpha_2\beta_1 x + c_2 + t(v) \tag{5.44}$$

where

$$t(v) = \begin{cases} \beta_2\alpha_1 v + \beta_2 c_1 & \text{if } x_7 = v, \\ \beta_2 c_1 + \beta_2\beta_1 v & \text{if } x_7 = u. \end{cases}$$

Note that in both cases $t(0) = \beta_2 c_1$. Putting $x = 0$, we get:

$$t(v) = -c_2 - \alpha_2 c_1 - \alpha_2\alpha_1 v + c \tag{5.45}$$

while replacement $v = 0$ leads to:

$$\alpha_1\alpha_2 x + \alpha_2 c_1 = \alpha_2 c_1 + \alpha_2\beta_1 x. \tag{5.46}$$

Using (5.45) and (5.46) in (5.44), we conclude:

$$\alpha_1\alpha_2 x + c = \alpha_2\alpha_1 v + \alpha_1\alpha_2 x - \alpha_2\alpha_1 v + c$$

which implies that the group $(B; +)$ is Abelian.

a2) $x_3 = u$ (and $x_4 = y$).
The arrangement of variables is $xyuy = (xv \text{ or } vx)(uv \text{ or } vu)$. Replacement $x = v = 0$ in (5.j) yields:

$$\alpha_1 c_2 + \alpha_1\beta_2 y + c_1 + \beta_1\alpha_2 u + \beta_1 c_2 + \beta_1\beta_2 y = t(u) \tag{5.47}$$

where

$$t(u) = \begin{cases} \alpha_2 c_1 + c_2 + \beta_2\alpha_1 u + \beta_2 c_1 & \text{if } x_7 = u, \\ c + \beta_2\beta_1 u & \text{if } x_7 = v. \end{cases}$$

Note that in both cases $t(0) = c$. Putting $y = 0$ in (5.47), we get:

$$\alpha_1 c_2 + c_1 + \beta_1\alpha_2 u + \beta_1 c_2 = t(u) \tag{5.48}$$

while replacement $u = 0$ yields:

$$\alpha_1 c_2 + \alpha_1\beta_2 y + c_1 = c - \beta_1\beta_2 y - \beta_1 c_2. \tag{5.49}$$

Feeding (5.48) and (5.49) in (5.47), we get:

$$c - \beta_1\beta_2 y - \beta_1 c_2 + \beta_1\alpha_2 u + \beta_1 c_2 = \alpha_1 c_2 + c_1 + \beta_1\alpha_2 u + \beta_1 c_2 - \beta_1\beta_2 y$$

which implies commutativity of $+$.

b) Variable x is quadratic (and y is linear).
The arrangement of variables is $xy(xu \text{ or } ux) = (yv \text{ or } vy)(uv \text{ or } vu)$. Let $u = v = 0$. We have:

$$\alpha_1\alpha_2 x + \alpha_1 c_2 + \alpha_1\beta_2 y + c_1 + s(x) = t(y) \tag{5.50}$$

where:

$$s(x) = \begin{cases} \beta_1\alpha_2 x + \beta_1 c_2 & \text{if } x_3 = x, \\ \beta_1 c_2 + \beta_1\beta_2 x & \text{if } x_3 = u, \end{cases}$$

$$t(y) = \begin{cases} \alpha_2\alpha_1 y + c & \text{if } x_5 = y, \\ \alpha_2 c_1 + \alpha_2\beta_1 y + c_2 + \beta_2 c_1 & \text{if } x_5 = v. \end{cases}$$

Note that $s(0) = \beta_1 c_2$ and $t(0) = c$. Specifying $x = 0$, we get:

$$\alpha_1 c_2 + \alpha_1\beta_2 y + c_1 + \beta_1 c_2 = t(y) \tag{5.51}$$

while $y = 0$ yields:

$$c_1 + s(x) = -\alpha_1 c_2 - \alpha_1\alpha_2 x + c. \tag{5.52}$$

Feeding (5.51) and (5.52) into (5.50), we get:

$$\alpha_1\alpha_2 x + \alpha_1 c_2 + \alpha_1\beta_2 y - \alpha_1 c_2 - \alpha_1\alpha_2 x + \alpha_1 c_2 = \alpha_1 c_2 + \alpha_1\beta_2 y$$

which implies that the group $(B;+)$ is Abelian.

Because of commutativity of $+$ and the condition for c, the equation (5.j) reduces to:

$$\begin{aligned} &\alpha_1\alpha_2 x_1 + \alpha_1\beta_2 x_2 + \beta_1\alpha_2 x_3 + \beta_1\beta_2 x_4 = \\ &= \alpha_2\alpha_1 x_5 + \alpha_2\beta_1 x_6 + \beta_2\alpha_1 x_7 + \beta_2\beta_1 x_8), \end{aligned}$$

which is equivalent to the system:

$$\text{Lbranch}(z, (5.\text{j})) = \text{Rbranch}(\text{z}, (5,\text{j}))$$

$$\text{Lbranch}(w, (5.\text{j}))\text{w} + \text{Rbranch}(\text{w}, (5.\text{j}))\text{w} = 0$$

for all linear variables z and all quadratic variables w.

The uniqueness of the group $(B;+)$ follows from the Albert Theorem. □

Lemma 5. *A general solution of the equation* (5.23) *is given by:*

$$\begin{cases} f_1(x,y) = & \alpha_1 x + c_1 + \beta_1 y \\ f_2(x,y) = & \beta_2 y + c_2 + \alpha_2 x \end{cases} \tag{23}$$

where:

- $(B;+)$ *is an arbitrary group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2,c_2) = f_2(c_1,c_1)$*,*
- α_i, β_i $(i = 1,2)$ *are arbitrary automorphisms of* $+$ *such that:*

$$\mathrm{Lbranch}(z,(5.23)) = \mathrm{Rbranch}(z,(5.23)) \tag{5.53}$$

for $z \in \{y,u\}$*,*

$$\mathrm{Lbranch}(x,(5.23))x + c_1 + \mathrm{Rbranch}(x,(5.23))x = c_1 \tag{5.54}$$

,

$$\mathrm{Rbranch}(v,(5.23))v + c_2 + \mathrm{Lbranch}(v,(5.23))v = c_2. \tag{5.55}$$

The group $(B;+)$ *is unique up to isomorphism.*

Proof. (1) To show that the pair (f_1, f_2) of operations is a solution of (5.23), just replace $f_i(x,y)$ in (5.23) using (23) and all conditions (5.53)–(5.55).

(2) Define new quasigroup f_3 to be the dual quasigroup of f_2, i. e. $f_3(x,y) = f_2(y,x)$. The equation (5.23) transforms into equation (5.10) with a general solution given by Lemma 3:

$$\begin{cases} f_1(x,y) = & \alpha_1 x + c_1 + \beta_1 y \\ f_3(x,y) = & \alpha_3 x + c_3 + \beta_3 y \end{cases} \tag{23*}$$

where:

- $(B;+)$ is an arbitrary group,
- c_1, c_3 are arbitrary elements of B such that $f_1(c_3,c_3) = f_3(c_1,c_1)$,
- α_i, β_i $(i = 1,3)$ are arbitrary automorphisms of $+$ such that:

$$\alpha_1\alpha_3 = \alpha_3\alpha_1$$

$$\beta_1\beta_3 = \beta_3\beta_1$$

$$\alpha_1\beta_3 x + c_1 + \beta_1\beta_3 x = c_1$$

$$\alpha_3\beta_1 v + c_3 + \beta_3\alpha_1 v = c_3.$$

Define: $\alpha_2 = \beta_3, \beta_2 = \alpha_3$ and $c_2 = c_3$ and replace in (23*) to get:
$f_2(x, y) = f_3(y, x) = \alpha_3 y + c_2 + \beta_3 x = \beta_2 y + c_2 + \alpha_2 x$, and

$$\alpha_1\beta_2 = \beta_2\alpha_1$$

$$\beta_1\alpha_2 = \alpha_2\beta_1$$

$$\alpha_1\alpha_2 x + c_1 + \beta_1\alpha_2 x = c_1$$

$$\beta_2\beta_1 v + c_2 + \alpha_2\alpha_1 v = c_2,$$

which is:

$$\text{Lbranch}(z, (5.23)) = \text{Rbranch}(z, (5.23))$$

for $z \in \{y, u\}$, and

$$\text{Lbranch}(x, (5.23))x + c_1 + \text{Rbranch}(x, (5.23))x = c_1,$$

$$\text{Rbranch}(v, (5.23))v + c_2 + \text{Lbranch}(v, (5.23))v = c_2.$$

Trivially, $f_1(c_2, c_2) = f_2(c_1, c_1)$.
The uniqueness of the group $(B; +)$ follows from the Albert Theorem. □

Lemma 6. *A general solution of the equation* (5.k) (k = 3, 4, 7, 8, 11, 12, 15, 16, 19, 20, 24, 27, 28, 31, 32) *is given by:*

$$f_i(x, y) = \alpha_i x + c_i + \beta_i y \quad (i = 1, 2) \tag{5.56}$$

where:

- $(B; +)$ *is an arbitrary Abelian group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2, c_2) = f_2(c_1, c_1)$,
- α_i, β_i $(i = 1, 2)$ *are arbitrary automorphisms of* $+$ *such that:*

$$\text{Lbranch}(z, (5.\text{k})) = \text{Rbranch}(z, (5, \text{k})) \tag{5.57}$$

for all linear variables z *of* (5.j) *and*

$$\text{Lbranch}(w, (5.\text{k}))w + \text{Rbranch}(w, (5.\text{k}))w = 0 \tag{5.58}$$

for all quadratic variables w *from the equation.*

The group $(B; +)$ *is unique up to isomorphism.*

Proof. (1) To show that the pair (f_1, f_2) of operations is a solution of (5.k), just replace $f_i(x,y)$ in (5.k) using (5.56) and all conditions (5.57), (5.58).

(2) Let us prove that the solution given in the lemma is general in the case $k = 3$.

The equation (5.3) has arrangement of variables equal to $xyxu = uvyv$. Let us replace the operation f_2 in (5.3) by the dual operation $f_3(x,y) = f_2^*(x,y) = f_2(y,x)$. We get the equation

$$f_1(f_3(y,x), f_3(u,x)) = f_3(f_1(y,v), f_1(u,v))$$

with the arrangement of variables equal to $yxux = yvuv$. Normalizing (i.e. applying the permutation (xy) to variables) we get the equation (5.25) with a general solution given in Lemma 4:

$$f_i(x,y) = \alpha_i x + c_i + \beta_i y \quad (i = 1,3) \tag{5.59}$$

where:

- $(B;+)$ is an arbitrary group,
- c_1, c_3 are arbitrary elements of B such that $f_1(c_3, c_3) = f_3(c_1, c_1)$,
- α_i, β_i $(i = 1,3)$ are arbitrary automorphisms of $+$ such that:

$$\text{Lbranch}(z, (5.25)) = \text{Rbranch}(z, (5.25)) \tag{5.60}$$

  for all linear variables z of (5.25) and

$$\text{Lbranch}(w, (5.25))w + \text{Rbranch}(w, (5.25))w = 0 \tag{5.61}$$

  for all quadratic variables w from the equation.

Conditions (5.60) and (5.61) evaluate to:

$$\alpha_1\alpha_3 = \alpha_3\alpha_1$$

$$\beta_1\alpha_3 = \beta_3\alpha_1$$

$$\alpha_1\beta_3 x + \beta_1\beta_3 x = 0$$

$$\alpha_3\beta_1 v + \beta_3\beta_1 v = 0.$$

Define: $\alpha_2 = \beta_3, \beta_2 = \alpha_3, c_2 = c_3$ and replace in (5.59) to get:
$f_2(x,y) = f_3(y,x) = \alpha_3 y + c_3 + \beta_3 x = \beta_2 y + c_2 + \alpha_2 x = \alpha_2 x + c_2 + \beta_2 y$, and

$$\alpha_1\beta_2 = \beta_2\alpha_1$$

$$\beta_1\beta_2 = \alpha_2\alpha_1$$

$$\alpha_1\alpha_2 x + \beta_1\alpha_2 x = 0$$

$$\beta_2\beta_1 v + \alpha_2\beta_1 v = 0,$$

which is:

$$\text{Lbranch}(z,(5,3)) = \text{Rbranch}(z,(5,3))$$

for $z \in \{y,u\}$, and

$$\text{Lbranch}(w,(5.3))w + \text{Rbranch}(x,(5.3))w = 0,$$

for $w \in \{x,v\}$.

Trivially, $f_1(c_2,c_2) = f_2(c_1,c_1)$.

Analogously, we can transform (5.4) into (5.29), (5.7) into (5.26), (5.8) into (5.30), (5.11) into (5.17), (5.12) into (5.21), (5.15) into (5.18), (5.16) into (5.22), (5.19) into (5.9), (5.20) into (5.13), (5.24) into (5.14), (5.27) into (5.1), (5.28) into (5.5), (5.31) into (5.2), (5.32) into (5.6) and prove appropriate relationships between $\alpha_i, \beta_i, c_i\,(i = 1,2)$ for these equations, using results given in Lemma 4. □

Definition 11. Let $\partial : B \longrightarrow B$ be the natural antiautomorphism of the group $(B;+)$ with itself so that $\partial(x+y) = y+x$.

It is easy to see that for all natural numbers n, $\partial(x_1 + x_2 + \cdots + x_n) = x_n + x_{n-1} + \cdots x_1$. In particular $\partial(x+y+z) = z+y+x$. Also, for all even (odd) j and all terms t: $\partial^j(t) = t$ ($\partial^j(t) = \partial(t)$).

We may now combine Lemmas 3 and 5 into:

Theorem 9. *A general solution of the equation* (5.j) (j = 10,23) *is given by:*

$$\begin{cases} f_1(x,y) = \alpha_1 x + c_1 + \beta_1 y \\ f_2(x,y) = \partial^{\mathrm{j}}(\alpha_2 x + c_2 + \beta_2 y) \end{cases}$$

where:

- $(B;+)$ *is an arbitrary group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2,c_2) = f_2(c_1,c_1)$,
- α_i, β_i $(i = 1,2)$ *are arbitrary automorphisms of* $+$ *such that:*

  $$\text{Lbranch}(z,(5.\mathrm{j})) = \text{Rbranch}(z,(5.\mathrm{j}))$$

  for all linear variables z *of the equation* (5.j) *and*

  $$\text{Lbranch}(w_i,(5.\mathrm{j}))w_i + c_i + \text{Rbranch}(w_i,(5.\mathrm{j}))w_i = c_i$$

  for $i \in \{1,2\}$, *where* w_1 *is the left quadratic variable while* w_2 *is the right quadratic variable of* (5.j).

The group $(B;+)$ *is unique up to isomorphism.*

Likewise, Theorem 8 and Lemmas 4 and 6 can be combined into:

Theorem 10. *A general solution of the equation* $(m.\mathrm{j_m})$ $(m = 4,5;\ 1 \leq \mathrm{j}_4 \leq 16;\ 1 \leq \mathrm{j}_5 \leq 32;\ \mathrm{j}_5 \neq 10, 23)$ *is given by:*

$$f_i(x,y) = \alpha_i x + c_i + \beta_i y \quad (i = 1,2)$$

where:

- $(B;+)$ *is an arbitrary Abelian group,*
- c_1, c_2 *are arbitrary elements of* B *such that* $f_1(c_2,c_2) = f_2(c_1,c_1)$*,*
- α_i, β_i $(i = 1,2)$ *are arbitrary automorphisms of* $+$ *such that:*

  $$\mathrm{Lbranch}(z,(m.\mathrm{j_m})) = \mathrm{Rbranch(z,(m.j_m))}$$

  for all linear variables z *of* $(m.\mathrm{j_m})$ *and*

  $$\mathrm{Lbranch}(w,(m.\mathrm{j_m}))\mathrm{w} + \mathrm{Rbranch(w,(m.j_m))w} = 0$$

  for all quadratic variables w *from the equation.*

The group $(B;+)$ *is unique up to isomorphism.*

6 Algebras with Parastrophically Uncancellable Quadratic Hyperidentities

By [20, 21], a hyperidentity (or $\forall(\forall)$-identity) is a second-order formula of the following form:

$$\forall f_1, \ldots, f_k \forall x_1, \ldots, x_n \qquad (w_1 = w_2),$$

where w_1, w_2 are words (terms) in the alphabet of function variables $f_1, \ldots, f_k$ and object variables $x_1, \ldots, x_n$. However hyperidentities are usually presented without universal quantifiers: $w_1 = w_2$. The hyperidentity $w_1 = w_2$ is said to be satisfied in the algebra $(B;F)$ if this equality holds whenever every function variable f_i is replaced by an arbitrary operation of the corresponding arity from F and every object variable x_i is replaced by an arbitrary element of B.

Now, as a consequence of the results of the previous section, we can establish the following representation of a binary algebra satisfying one of the non-gemini hyperidentities.

Theorem 11. *Let* $(B;F)$ *be a binary algebra with quasigroup operations which satisfy one of the non–gemini hyperidentities* $(m.\mathrm{j_m})$ $(m = 4,5;\ 1 \leq \mathrm{j}_4 \leq 16;\ 1 \leq \mathrm{j}_5 \leq 32)$*. Then there exists an Abelian group* $(B;+)$ *such that every operation* $f_i \in F$ *is represented by:*

$$f_i(x,y) = \alpha_i(x) + c_i + \beta_i(y),$$

where:

- c_i $(i = 1, \dots, |F|)$ *are arbitrary elements of* B *such that* $f_l(c_k, c_k) = f_k(c_l, c_l)$ *for* $1 \leqslant l, k \leqslant |F|$,
- α_i, β_i $(i = 1, \dots, |F|)$ *are arbitrary automorphisms of* $+$ *such that:*

$$\mathrm{Lbranch}(z, (m.\mathrm{j_m})) = \mathrm{Rbranch}(\mathrm{z}, (\mathrm{m.j_m}))$$

for all linear variables z *of* $(m.\mathrm{j_m})$ *and*

$$\mathrm{Lbranch}(w, (m.\mathrm{j_m}))\mathrm{w} + \mathrm{Rbranch}(\mathrm{w}, (\mathrm{m.j_m}))\mathrm{w} = 0$$

for all quadratic variables w *from the equation.*

Proof. Let us consider the pair (f_1, f_1) of operations satisfying equation $(m.\mathrm{j_m})$ (for $m = 4$ or 5; j_4 is some of $1, 2, \dots, 16$ while j_5 is some of $1, 2, \dots, 32$). Then

$$f_1(x, y) = \alpha_1(x) + c_1 + \beta_1(y)$$

where $+$ is a group and α_1, β_1 its automorphisms. In the case of equation (5.10) ((5.23)) the group $+$ is commutative by Theorem 1 (Theorem 2). In all other cases $+$ is commutative by Theorem 10.

For any $i \in F$, $i \neq 1$, the pair (f_1, f_i) also satisfies $(m.\mathrm{j_m})$, hence both are principally isotopic to a group (perhaps other than $+$). Anyway, f_i is also principally isotopic to $+$ and by Theorem 9 or 10

$$f_i(x, y) = \alpha_i(x) + c_i + \beta_i(y)$$

where $c_i \in B$ and $\alpha_i, \beta_i \in Aut(B; +)$ such that

$$\mathrm{Lbranch}(z, (m.\mathrm{j_m})) = \mathrm{Rbranch}(\mathrm{z}, (\mathrm{m.j_m}))$$

for all linear variables z of $(m.j_m)$ and

$$\mathrm{Lbranch}(w, (m.\mathrm{j_m}))\mathrm{w} + \mathrm{Rbranch}(\mathrm{w}, (\mathrm{m.j_m}))\mathrm{w} = 0$$

for all quadratic variables w from the equation.

The rest of the proof is easy. □

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