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On mini-superspace limit of boundary three-point function in Liouville field theory

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Abstract: We study the mini-superspace semiclassical limit of the boundary three-point function in the Liouville field theory. We compute also matrix elements for the Morse potential quantum mechanics. An exact agreement between the former and the latter is found. We show that both of them are given by the generalized hypergeometric functions.

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Contents

1 Introduction

Recently the various semiclassical limits of the Liouville correlation functions appeared in the different instances. For example we can mention study of conformal blocks in AdS/CFT correspondence, see e.g. [1–3], semiclassical limits of the Nekrasov partition functions, see e.g. [4–9], mini-superspace limit of correlation functions in $\mathrm{AdS}_3/\mathrm{H}_3^+$ [10, 11], semiclassical limit of correlation functions in the presence of defects and boundaries [12, 13] and the most recently found application of the semiclassical limit of the Liouville field theory to the SYK problem [14].

This paper is devoted to study of the mini-superspace limit in the boundary Liouville field theory (BLFT). In the mini-superspace limit one considers a limit where only the zero mode dynamics survives and the theory is described by the corresponding quantum mechanical problem. The mini-superspace limit of the Liouville field theory was considered in [15, 16]. In these papers the matrix elements of the Liouville quantum mechanics with the exponential potential were computed. It was shown later in [17] that the DOZZ structure constants [18, 19] in this limit coincide with the matrix elements computed in [15, 16]. It was also demonstrated in [19] that the Liouville two-point function in the mini-superspace limit is in agreement with the reflection amplitude of the Liouville quantum mechanics eigenfunctions, given by the modified Bessel functions. In papers [20, 21], the mini-superspace limit of the boundary two-point function in BLFT was studied. It was found that BLFT in this limit comes down to the Morse potential quantum mechanics. It was shown that in the mini-superspace limit the boundary two-point function, computed in [22], coincides with the reflection amplitude of the eigenfunctions of the Morse potential Hamiltonian given by the Whittaker functions.

In this paper we study the mini-superspace limit of the boundary three-point function in the BLFT. The boundary three-point function in the BLFT was computed in [23] and expressed by means of the double Gamma and double Sine functions [24, 25]. Using the asymptotic properties of the double Gamma and Sine functions [10], we have shown that in the mini-superspace limit the boundary three-point function can be expressed via the Meijer functions $G^{3,2}_{3,3}$ with the unit argument or equivalently via the generalized hypergeometric functions ${}_3F_2$ with the unit argument. We have computed also the matrix elements for the Morse potential quantum mechanics and have shown that they can be expressed through the generalized hypergeometric functions ${}_3F_2$ with the unit argument as well. Using the identities, relating different generalized hypergeometric functions with the unit argument [26–28], and matching quantum and classical parameters, we have established exact agreement between the mini-superspace limit of the boundary three-point function on one side and the matrix elements for the Morse potential quantum mechanics on the other side. It is important to note, that in the BLFT, the relation of the boundary cosmological constant to the corresponding quantum parameter, appearing in the one-point function, is twofold due to a sign ambiguity in the choice of the square root branch. We found, that to match the minisuperspace limit of the boundary three-point function with the corresponding quantum mechanical matrix element, we should use the branch with the negative sign. We also found, that passing from one branch to another brings to the additional factor in the normalization of the wave functions corresponding to the boundary condition changing operators. We would like also to mention that various consequences of the branching of the BLFT parameters were considered earlier in [29].

The paper is organized as follows. In section 2 we review the BLFT and compute the mini-superspace limit of the boundary three-point function. In section 3 we compute matrix elements for the Morse potential quantum mechanics and establish precise agreement with the boundary three-point function in the mini-superspace limit found in the previous section. In appendices A, B and C we review various properties of the special functions used in the paper.

2 Boundary Liouville field theory

Let us consider the Liouville field theory on a strip $\mathbb{R} \times [0, \pi]$, parameterized by the time τ and space σ coordinates, $0 \leq \sigma \leq \pi$. The conformal invariant action has the form:

$$S = \int_{-\infty}^{\infty} d\tau \int_0^{\pi} d\sigma \left(\frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) + \int_{-\infty}^{\infty} d\tau M_1 e^{b\phi}|_{\sigma=0} + \int_{-\infty}^{\infty} d\tau M_2 e^{b\phi}|_{\sigma=\pi} , \quad (2.1)$$

where M_1 and M_2 are the corresponding boundary cosmological constants.

Let us review some facts on the boundary Liouville field theory [22, 23, 30]. The primary fields of the Liouville field theory are V_α, associated with the vertex operators $e^{2\alpha\phi}$. They have conformal dimension

$$\Delta_\alpha = \alpha(Q - \alpha), \qquad Q = b + \frac{1}{b} . \quad (2.2)$$

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In the presence of the boundary with the cosmological constant M the primary fields V_α have the one-point functions:

$$\langle 0|V_\alpha(z,\bar{z})|0\rangle = \frac{U_\sigma(\alpha)}{|z-\bar{z}|^{2\Delta_\alpha}}\,, \tag{2.3}$$

where

$$U_\sigma(\alpha) = \frac{2}{b}(\pi\mu\gamma(b^2))^{(Q-2\alpha)/2b}\Gamma(1-b(Q-2\alpha))\Gamma(-b^{-1}(Q-2\alpha))\cos(\pi(2\sigma-Q)(2\alpha-Q)) \tag{2.4}$$

and the parameter σ is related to the boundary cosmological constant M by the relation:

$$M = \sqrt{\frac{\mu}{\sin(\pi b^2)}}\cos\pi b\,(2\sigma - Q)\ . \tag{2.5}$$

Besides the bulk primary fields, in boundary conformal field theory exist also boundary condition changing operators, parameterized by the types of the switched boundary conditions and conformal weights. In the case of the BLFT they are given by the fields $\Psi_\beta^{\sigma_1\sigma_2}$ with the conformal weight $\Delta_\beta = \beta(Q-\beta)$. They have the two-point function:

$$\langle 0|\Psi_{\beta_1}^{\sigma_1\sigma_2}(x)\Psi_{\beta_2}^{\sigma_2\sigma_1}(0)|0\rangle = \frac{\delta(\beta_2+\beta_1-Q)+S(\beta_1,\sigma_2,\sigma_1)\delta(\beta_2-\beta_1)}{|x|^{2\Delta_{\beta_1}}}\,, \tag{2.6}$$

where

$$S(\beta,\sigma_2,\sigma_1) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{Q-2\beta}{2b}} \times \frac{\Gamma_b(2\beta-Q)}{\Gamma_b(Q-2\beta)}\frac{S_b(\sigma_2+\sigma_1-\beta)S_b(2Q-\sigma_2-\sigma_1-\beta)}{S_b(\sigma_2-\sigma_1+\beta)S_b(\sigma_1-\sigma_2+\beta)} \tag{2.7}$$

and the three-point function:

$$\langle 0|\Psi_{\beta_3}^{\sigma_1\sigma_3}(x_3)\Psi_{\beta_2}^{\sigma_3\sigma_2}(x_3)\Psi_{\beta_1}^{\sigma_2\sigma_1}(x_3)|0\rangle = \frac{C^{\sigma_3\sigma_2\sigma_1}_{\beta_3\beta_2\beta_1}}{|x_{21}|^{\Delta_1+\Delta_2-\Delta_3}|x_{32}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}\,, \tag{2.8}$$

$$C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1} \equiv C^{\sigma_3\sigma_2\sigma_1}_{Q-\beta_3,\beta_2,\beta_1}\,, \tag{2.9}$$

$$C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1} = R_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\int_{-i\infty}^{i\infty}\frac{d\tau}{i}J_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\,, \tag{2.10}$$

where

$$\begin{aligned}R_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} &= (\pi\mu\gamma(b^2)b^{2-2b^2})^{\frac{1}{2b}(\beta_3-\beta_2-\beta_1)} \\ &\times\frac{\Gamma_b(2Q-\beta_1-\beta_2-\beta_3)\Gamma_b(\beta_2+\beta_3-\beta_1)\Gamma_b(Q+\beta_2-\beta_1-\beta_3)\Gamma_b(Q+\beta_3-\beta_2-\beta_1)}{\Gamma_b(2\beta_3-Q)\Gamma_b(Q-2\beta_2)\Gamma_b(Q-2\beta_1)\Gamma_b(Q)} \\ &\times\frac{S_b(\beta_3+\sigma_1-\sigma_3)S_b(Q+\beta_3-\sigma_3-\sigma_1)}{S_b(\beta_2+\sigma_2-\sigma_3)S_b(Q+\beta_2-\sigma_3-\sigma_2)}\end{aligned} \tag{2.11}$$

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and

$$J_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} = \frac{S_b(U_1+\tau)S_b(U_2+\tau)}{S_b(V_1+\tau)S_b(V_2+\tau)}\frac{S_b(U_3+\tau)S_b(U_4+\tau)}{S_b(V_3+\tau)S_b(V_4+\tau)}\,, \tag{2.12}$$

$$\begin{aligned} U_1 &= \sigma_2+\sigma_1-\beta_1, & V_1 &= Q+\sigma_2+\beta_3-\beta_1-\sigma_3\,, \\ U_2 &= Q+\sigma_2-\beta_1-\sigma_1, & V_2 &= 2Q+\sigma_2-\beta_3-\sigma_3-\beta_1\,, \\ U_3 &= \sigma_2+\beta_2-\sigma_3, & V_3 &= 2\sigma_2\,, \\ U_4 &= Q+\sigma_2-\beta_2-\sigma_3, & V_4 &= Q\,. \end{aligned} \tag{2.13}$$

The $\Gamma_b(x)$ and $S_b(x)$ in the formulae above denote the double Gamma and Sine functions reviewed in appendix A.

The three-point function has the property, that setting one of the field to the vacuum, one recovers the two-point function. For example it was checked in [23] that

$$\lim_{\beta_1\to 0} C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1} = \delta(\beta_3-\beta_2)+S(\beta_2,\sigma_3,\sigma_2)\delta(\beta_3+\beta_2-Q)\,. \tag{2.14}$$

Let us now consider the minisuperspace limit of the boundary three-point function.

As the warm-up exercise we review the minisuperspace limit of the boundary two-point function (2.7), computed in [20]. It is argued in [20] that one should take the limit $b\to 0$ and scale the parameters β and σ in the following way:

$$\beta = \frac{Q}{2}+ikb \tag{2.15}$$

and

$$\begin{aligned} \sigma_1 &= \frac{1}{4b}+\rho_1 b\,, \\ \sigma_2 &= \frac{1}{4b}+\rho_2 b\,. \end{aligned} \tag{2.16}$$

Using the formulae (A.7), (A.8) and (A.10) in appendix A one can obtain easily:

$$S(\beta,\sigma_2,\sigma_1)\to\left(\frac{4\pi\mu}{b^2}\right)^{-ik}\frac{\Gamma(2ik)}{\Gamma(-2ik)}\frac{\Gamma\left(\rho_1+\rho_2-\frac{1}{2}-ik\right)}{\Gamma\left(\rho_1+\rho_2-\frac{1}{2}+ik\right)}\,. \tag{2.17}$$

To compute the mini-superspace limit of the boundary three-point function we will use the ansatz (2.16) for all the three boundary condition parameters:

$$\begin{aligned} \sigma_1 &= \frac{1}{4b}+\rho_1 b\,, \\ \sigma_2 &= \frac{1}{4b}+\rho_2 b\,, \\ \sigma_3 &= \frac{1}{4b}+\rho_3 b\,. \end{aligned} \tag{2.18}$$

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For the primary fields parameters we will use the ansatz suggested in [17] for calculation of the mini-superspace limit of the bulk three-point function:

$$\beta_1 = \frac{Q}{2} + ik_1 b\,, \tag{2.19}$$
$$\beta_2 = \eta b\,,$$
$$\beta_3 = \frac{Q}{2} + ik_2 b\,.$$

It is convenient to denote

$$\rho_1 + \rho_2 = 1 - \lambda\,, \tag{2.20}$$
$$\rho_2 - \rho_3 = \xi\,, \tag{2.21}$$

implying also

$$\rho_1 + \rho_3 = 1 - \lambda - \xi\,. \tag{2.22}$$

Inserting (2.18) and (2.19) in (2.12), using the formulas (A.7), (A.8), (A.9) in appendix A, and rescaling the integration variable $\tau \to b\tau$, one obtains in the limit $b \to 0$

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i} J_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} \to \tag{2.23}$$
$$2^{-7/2}(\pi b^2)^{-\lambda+ik_1} b^{-1}\pi^{-2}$$
$$\times \int_{-i\infty}^{i\infty} \frac{d\tau}{i} \frac{\Gamma(-\tau)\Gamma(\tau-ik_1+1/2-\lambda)\Gamma(\eta+\xi+\tau)\Gamma(ik_1-ik_2-\xi-\tau)\Gamma(ik_2+ik_1-\xi-\tau)}{\Gamma(\eta-\xi-\tau)}\,.$$

Using the definition of the Meijer G-functions, reviewed in appendix B, one can write

$$\int_{-i\infty}^{i\infty} \frac{d\tau}{i} J_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix} \to 2^{-5/2}(\pi b^2)^{-\lambda+ik_1} b^{-1}\pi^{-1} G^{3,2}_{3,3}\left(1 \middle| \begin{matrix}\frac{1}{2}+\lambda+ik_1, 1-\eta-\xi, \eta-\xi\\ 0, ik_1-ik_2-\xi, ik_1+ik_2-\xi\end{matrix}\right)$$
$$= 2^{-5/2}(\pi b^2)^{-\lambda+ik_1} b^{-1}\pi^{-1} G^{3,2}_{3,3}\left(1 \middle| \begin{matrix}\frac{1}{2}+\lambda+\xi+ik_1, 1-\eta, \eta\\ \xi, ik_1-ik_2, ik_1+ik_2\end{matrix}\right)\,. \tag{2.24}$$

In the second line we used the identity (B.4) in appendix B.

For further purposes, it is convenient to present the Meijer $G^{3,2}_{3,3}$-function (2.24) in a special way, use of which becomes clear in the next section. Namely, first we decompose the $G^{3,2}_{3,3}$-function as a sum of ${}_3F_2$ hypergeometric functions with the unit argument according to eq. (B.2) in appendix B. Afterwards we transform obtained in this way ${}_3F_2$ hypergeometric functions with the unit argument successively applying the identities (C.1) and (C.2) in appendix C. We end up with

$$G^{3,2}_{3,3}\left(1\left|\begin{matrix}\frac{1}{2}+\lambda+ik_1+\xi, 1-\eta, \eta\\ \xi, ik_1-ik_2, ik_1+ik_2\end{matrix}\right.\right)=$$
$$\frac{\Gamma(\xi+\eta)\Gamma\left(\frac{1}{2}+\lambda-ik_1\right)}{\sin\pi\left(ik_1+\frac{1}{2}+\lambda\right)}$$
$$\times\left[\frac{\Gamma(2ik_2)\Gamma(ik_1-ik_2+\eta)\Gamma\left(\frac{1}{2}-ik_2-\lambda-\xi\right)}{\Gamma(-ik_1+ik_2+\eta)\Gamma\left(-ik_2+\frac{1}{2}+\lambda+\eta\right)\Gamma\left(-ik_2+\frac{1}{2}-\lambda+\eta\right)\Gamma\left(ik_2+\frac{1}{2}+\lambda-\eta\right)}\right.$$
$$\times{}_3F_2\left(\begin{matrix}-ik_1-ik_2+\eta, ik_1-ik_2+\eta, \frac{1}{2}+\lambda+\xi-ik_2;\\ 1-2ik_2, \frac{1}{2}+\lambda-ik_2+\eta : 1\end{matrix}\right)$$
$$+\frac{\Gamma(-2ik_2)\Gamma(ik_1+ik_2+\eta)\Gamma\left(\frac{1}{2}+ik_2-\lambda-\xi\right)}{\Gamma(-ik_1-ik_2+\eta)\Gamma\left(ik_2+\frac{1}{2}+\lambda+\eta\right)\Gamma\left(ik_2+\frac{1}{2}-\lambda+\eta\right)\Gamma\left(-ik_2+\frac{1}{2}+\lambda-\eta\right)}$$
$$\left.\times{}_3F_2\left(\begin{matrix}ik_1+ik_2+\eta, -ik_1+ik_2+\eta, \frac{1}{2}+\lambda+\xi+ik_2;\\ 1+2ik_2, \frac{1}{2}+\lambda+ik_2+\eta : 1\end{matrix}\right)\right]. \tag{2.25}$$

Now inserting (2.18) and (2.19) in (2.11), and using the formulae (A.3)–(A.11) in appendix A, we obtain for the prefactor (2.11) in the limit $b \to 0$

$$R_{\sigma_2,\beta_3}\begin{bmatrix}\beta_2 & \beta_1\\ \sigma_3 & \sigma_1\end{bmatrix}\to\left(\frac{4\pi\mu}{b^2}\right)^{(ik_2-ik_1-\eta)/2}4(\pi b^2)^{-ik_1+\lambda}b\pi^{3/2} \tag{2.26}$$
$$\times\frac{\Gamma(-ik_1+ik_2+\eta)\Gamma(-ik_1-ik_2+\eta)}{\Gamma(2ik_2)\Gamma(-2ik_1)\Gamma\left(\frac{1}{2}-ik_2-\lambda-\xi\right)\Gamma(\eta+\xi)}.$$

Combining (2.26) and (2.25) finally we obtain:[1]

$$C^{\sigma_3\sigma_2\sigma_1}_{\beta_3|\beta_2\beta_1}\to C^{\lambda\xi}_{k_2|\eta k_1}=\left(\frac{4\pi\mu}{b^2}\right)^{(ik_2-ik_1-\eta)/2}\frac{\Gamma(\frac{1}{2}+\lambda-ik_1)}{\sin\pi\left(ik_1+\frac{1}{2}+\lambda\right)\Gamma(2ik_2)\Gamma(-2ik_1)\Gamma\left(\frac{1}{2}-ik_2-\lambda-\xi\right)}$$
$$\times\left[\frac{\Gamma(2ik_2)\Gamma(ik_1-ik_2+\eta)\Gamma(-ik_1-ik_2+\eta)\Gamma\left(\frac{1}{2}-ik_2-\lambda-\xi\right)}{\Gamma\left(-ik_2+\frac{1}{2}+\lambda+\eta\right)\Gamma\left(-ik_2+\frac{1}{2}-\lambda+\eta\right)\Gamma\left(ik_2+\frac{1}{2}+\lambda-\eta\right)}\right.$$
$$\times{}_3F_2\left(\begin{matrix}-ik_1-ik_2+\eta, ik_1-ik_2+\eta, \frac{1}{2}+\lambda+\xi-ik_2;\\ 1-2ik_2, \frac{1}{2}+\lambda-ik_2+\eta : 1\end{matrix}\right)$$
$$+\frac{\Gamma(-2ik_2)\Gamma(ik_1+ik_2+\eta)\Gamma(-ik_1+ik_2+\eta)\Gamma\left(\frac{1}{2}+ik_2-\lambda-\xi\right)}{\Gamma\left(ik_2+\frac{1}{2}+\lambda+\eta\right)\Gamma\left(ik_2+\frac{1}{2}-\lambda+\eta\right)\Gamma\left(-ik_2+\frac{1}{2}+\lambda-\eta\right)}$$
$$\left.\times{}_3F_2\left(\begin{matrix}ik_1+ik_2+\eta, -ik_1+ik_2+\eta, \frac{1}{2}+\lambda+\xi+ik_2;\\ 1+2ik_2, \frac{1}{2}+\lambda+ik_2+\eta : 1\end{matrix}\right)\right]. \tag{2.27}$$

As we will see in the next section, especially important role plays the case when $\xi = -\eta$.

[1]Probably up to some inessential numerical factors.

For $\xi = -\eta$ eq. (2.27) simplifies and takes the form:

$$\begin{aligned} C^{\lambda(-\eta)}_{k_2|\eta k_1} = & \left(\frac{4\pi\mu}{b^2}\right)^{(ik_2-ik_1-\eta)/2} \frac{\Gamma\left(\frac{1}{2}+\lambda-ik_1\right)}{\sin\pi\left(ik_1+\frac{1}{2}+\lambda\right)\Gamma(2ik_2)\Gamma(-2ik_1)\Gamma\left(\frac{1}{2}-ik_2-\lambda+\eta\right)} \\ & \times\left[\frac{\Gamma(2ik_2)\Gamma(ik_1-ik_2+\eta)\Gamma(-ik_1-ik_2+\eta)}{\Gamma\left(-ik_2+\frac{1}{2}+\lambda+\eta\right)\Gamma\left(ik_2+\frac{1}{2}+\lambda-\eta\right)}\right. \\ & \times {}_3F_2\left(\begin{array}{c} -ik_1-ik_2+\eta, ik_1-ik_2+\eta, \frac{1}{2}+\lambda-\eta-ik_2; \\ 1-2ik_2, \frac{1}{2}+\lambda-ik_2+\eta : 1 \end{array}\right) \\ & +\frac{\Gamma(-2ik_2)\Gamma(ik_1+ik_2+\eta)\Gamma(-ik_1+ik_2+\eta)}{\Gamma\left(ik_2+\frac{1}{2}+\lambda+\eta\right)\Gamma\left(-ik_2+\frac{1}{2}+\lambda-\eta\right)} \\ & \left.\times\, {}_3F_2\left(\begin{array}{c} ik_1+ik_2+\eta, -ik_1+ik_2+\eta, \frac{1}{2}+\lambda-\eta+ik_2; \\ 1+2ik_2, \frac{1}{2}+\lambda+ik_2+\eta : 1 \end{array}\right)\right]. \end{aligned} \tag{2.28}$$

Let us consider the limit $\beta_2 \to 0$ and correspondingly $\eta \to 0$.

Using, that as we explained in appendix C, in this limit ${}_3F_2$ becomes ${}_2F_1$, which for the unit argument is given by the eq. (C.3), it is straightforward to show that:

$$\lim_{\eta\to 0} C^{\lambda(-\eta)}_{k_2|\eta k_1} = \delta(k_1-k_2) + \left(\frac{4\pi\mu}{b^2}\right)^{-ik_1} \frac{\Gamma(2ik_1)}{\Gamma(-2ik_1)} \frac{\Gamma\left(\frac{1}{2}-\lambda-ik_1\right)}{\Gamma\left(\frac{1}{2}-\lambda+ik_1\right)} \delta(k_1+k_2) \tag{2.29}$$

in agreement with (2.17).

3 Matrix elements for the Morse potential quantum mechanics

In the mini-superspace limit the boundary Liouville field theory is described by the Hamiltonian with the Morse potential [20, 21]. The corresponding eigenfuntions satisfy the Schrödinger equation:

$$-\frac{\partial^2\psi}{\partial\phi_0^2} + \pi\mu e^{2b\phi_0}\psi + (M_1+M_2)e^{b\phi_0}\psi = k^2b^2\psi\,. \tag{3.1}$$

The relation between parameters M_i, appearing in the Schrödinger equation, and parameters ρ_i, used in the previous section, can be found using (2.18) and (2.5) and taking the limit $b \to 0$:

$$M_i = \sqrt{\frac{\mu}{\sin(\pi b^2)}}\sin\pi b^2(2\rho_i-1) \to \pm(\mu\pi)^{1/2}b(2\rho_i-1)\,. \tag{3.2}$$

The solution of the eq. (3.1) is given by the Whittaker functions $W_{\mu,\nu}(y)$ [31, 32]:

$$\begin{aligned} \psi = & \mathcal{N}\left[e^{-y/2}y^{ik}\frac{\Gamma(-2ik)}{\Gamma\left(\frac{1}{2}-ik+\frac{M_1+M_2}{2b\sqrt{\pi\mu}}\right)}{}_1F_1\left(\frac{1}{2}+ik+\frac{M_1+M_2}{2b\sqrt{\pi\mu}}, 1+2ik, y\right)\right. \\ & \left.+e^{-y/2}y^{-ik}\frac{\Gamma(2ik)}{\Gamma\left(\frac{1}{2}+ik+\frac{M_1+M_2}{2b\sqrt{\pi\mu}}\right)}{}_1F_1\left(\frac{1}{2}-ik+\frac{M_1+M_2}{2b\sqrt{\pi\mu}}, 1-2ik, y\right)\right] \\ \equiv & \mathcal{N}\, W_{-\frac{M_1+M_2}{2b\sqrt{\pi\mu}}, ik}(y)y^{-\frac{1}{2}}\,, \end{aligned}$$

where

$$y = \frac{2\sqrt{\pi\mu}}{b} e^{b\phi_0} \,, \tag{3.3}$$

$\mathcal{N}$ is the normalization and ${}_1F_1(a,c,z)$ is the confluent hypergeometric function:

$${}_1F_1(a,c,z) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{z^n}{n!} \,. \tag{3.4}$$

Now we wish to compute matrix elements of the "vertex operator" $e^{\eta b\phi_0}$, between the wave functions corresponding to the boundary condition changing operators. According to this solution to the operator $\Psi_{\beta_1}^{\sigma_2\sigma_1}$ corresponds the wave function $\mathcal{N}_1 W_{\chi_1,ik_1}(y) y^{-\frac{1}{2}}$ with

$$\chi_1 = -\frac{M_1 + M_2}{2b\sqrt{\pi\mu}} = \pm\lambda \tag{3.5}$$

and to the operator $\Psi_{\beta_3}^{\sigma_1\sigma_3}$ corresponds the wave function $\mathcal{N}_2 W_{\chi_2,ik_2}(y) y^{-\frac{1}{2}}$ with

$$\chi_2 = -\frac{M_1 + M_3}{2b\sqrt{\pi\mu}} = \pm(\lambda+\xi) \,. \tag{3.6}$$

The corresponding integral can be found in [31, 32]:

$$\begin{aligned}
\mathcal{M}_{\eta k_1 k_2}^{\chi_1\chi_2} &= \mathcal{N}_1 \mathcal{N}_2^* \int_{-\infty}^{\infty} W_{\chi_1,ik_1}(y) y^{-\frac{1}{2}} W_{\chi_2,-ik_2}(y) y^{-\frac{1}{2}} e^{\eta b\phi_0} d\phi_0 \\
&= \frac{\mathcal{N}_1 \mathcal{N}_2^*}{b} \left(\frac{4\pi\mu}{b^2}\right)^{-\eta/2} \int_0^{\infty} W_{\chi_1,ik_1}(y) W_{\chi_2,-ik_2}(y) y^{\eta-2} dy \\
&= \mathcal{N}_1 \mathcal{N}_2^* (4\pi\mu b^{-2})^{-\eta/2} b^{-1} \\
&\quad \times \left[\frac{\Gamma(ik_1 - ik_2 + \eta)\Gamma(-ik_1 - ik_2 + \eta)\Gamma(2ik_2)}{\Gamma\left(\frac{1}{2} - \chi_2 + ik_2\right)\Gamma\left(\frac{1}{2} - \chi_1 - ik_2 + \eta\right)} \right. \\
&\quad \times {}_3F_2\left(\begin{matrix} -ik_1 - ik_2 + \eta, ik_1 - ik_2 + \eta, \frac{1}{2} - \chi_2 - ik_2; \\ 1 - 2ik_2, \frac{1}{2} - \chi_1 - ik_2 + \eta : 1 \end{matrix} \right) \\
&\quad + \frac{\Gamma(ik_1 + ik_2 + \eta)\Gamma(-ik_1 + ik_2 + \eta)\Gamma(-2ik_2)}{\Gamma\left(\frac{1}{2} - \chi_2 - ik_2\right)\Gamma\left(\frac{1}{2} - \chi_1 + ik_2 + \eta\right)} \\
&\quad \left. \times {}_3F_2\left(\begin{matrix} ik_1 + ik_2 + \eta, -ik_1 + ik_2 + \eta, \frac{1}{2} - \chi_2 + ik_2; \\ 1 + 2ik_2, \frac{1}{2} - \chi_1 + ik_2 + \eta : 1 \end{matrix} \right) \right] .
\end{aligned} \tag{3.7}$$

Comparing (3.7) with (2.28) we see that they coincide if we set:

$$\chi_1 = -\lambda \,, \tag{3.8}$$

$$\chi_2 = -\lambda + \eta \,, \tag{3.9}$$

$$\mathcal{N}_1 = \frac{(4\pi\mu b^{-2})^{-ik_1/2} b^{1/2}}{\sin\pi\left(\frac{1}{2} + ik_1 + \lambda\right)} \frac{\Gamma\left(\frac{1}{2} + \lambda - ik_1\right)}{\Gamma(-2ik_1)} \,, \tag{3.10}$$

$$\mathcal{N}_2 = \frac{1}{\pi} (4\pi\mu b^{-2})^{-ik_2/2} b^{1/2} \sin\pi\left(\frac{1}{2} + ik_2 - \lambda + \eta\right) \frac{\Gamma\left(\frac{1}{2} + \lambda - \eta - ik_2\right)}{\Gamma(-2ik_2)} \,. \tag{3.11}$$

This result leads us to the following conclusion on a role of the exponential operator $e^{\eta b\phi_0}$. Combining (3.5) and (3.6) with lower signes, as indicating in (3.8) and (3.9), and also remembering (2.18) and (2.21) one has

$$\frac{M_3 - M_2}{2\sqrt{\pi\mu}} = b\xi = -b\eta = \sigma_2 - \sigma_3\,. \tag{3.12}$$

Therefore recalling also that the exponential operator $e^{\eta b\phi_0}$ should correspond to a boundary condition changing operator $\Psi^{\sigma_3\sigma_2}_{\beta_2}$, one concludes that the operator $e^{\eta b\phi_0}$ in the semiclassical limit produces change of the boundary condition given by (3.12).

It is instructive to compare the normalizations of the wave functions found here with those used in [20]. Let us compute for this purpose the matrix element (3.7) for $\eta \to 0$ and $\chi_1 = \chi_2$. In this limit we obtain:

$$\begin{aligned}\mathcal{M}^{\chi_1\chi_1}_{0k_1k_2} &= \frac{\mathcal{N}_1\mathcal{N}_2^* b^{-1}\Gamma(2ik_1)\Gamma(-2ik_1)}{\Gamma\left(\frac{1}{2}-\chi_1+ik_1\right)\Gamma\left(\frac{1}{2}-\chi_1-ik_1\right)}\delta(k_1-k_2)\\ &\quad+\frac{\mathcal{N}_1\mathcal{N}_2^* b^{-1}\Gamma(2ik_1)\Gamma(-2ik_1)}{\Gamma\left(\frac{1}{2}-\chi_1-ik_1\right)\Gamma\left(\frac{1}{2}-\chi_1+ik_1\right)}\delta(k_1+k_2)\,.\end{aligned} \tag{3.13}$$

For χ_1, χ_2, $\mathcal{N}_1$, $\mathcal{N}_2$, chosen as in (3.8)–(3.11), with $\eta = 0$, the expression (3.13) surely coincides with the two-point function (2.29). But note that for

$$\chi_1 = \lambda\,, \tag{3.14}$$

$$\chi_2 = \lambda\,, \tag{3.15}$$

$$\mathcal{N}_1 = (4\pi\mu b^{-2})^{-ik_1/2} b^{1/2}\frac{\Gamma\left(\frac{1}{2}-\lambda-ik_1\right)}{\Gamma(-2ik_1)}\,, \tag{3.16}$$

$$\mathcal{N}_2 = (4\pi\mu b^{-2})^{-ik_2/2} b^{1/2}\frac{\Gamma\left(\frac{1}{2}-\lambda-ik_2\right)}{\Gamma(-2ik_2)}\,, \tag{3.17}$$

the expression (3.13) again coincides with the two-point function (2.29). This was established in [20].

This shows that passing from one branch of the square root to another introduces additional sine factors in the normalization of the wave functions in a way to keep unchanged the two-point functions.

4 Conclusion

We discussed in this paper semiclassical properties of the boundary three-point functions. We found perfect agreement with the corresponding quantum mechanical calculations. The matching of the calculations required to consider the negative branch in the branched correspondence of the classical and quantum parameters. We show that passing from one branch to another leads to the change in the normalization of the wave functions. We also found the flip of the boundary conditions induced by the exponential operators in the minisuperspace limit.

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A Double Gamma and double Sine functions

Here we review double Gamma $\Gamma_b(x)$ and double Sine $S_b(x)$ functions [24, 25].

$\Gamma_b(x)$ can be defined by means of the integral representation

$$\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\right]. \tag{A.1}$$

It has the property:

$$\Gamma_b(x+b) = \sqrt{2\pi}b^{bx-\frac{1}{2}}\Gamma^{-1}(bx)\Gamma_b(x)\,. \tag{A.2}$$

The double Sine function $S_b(x)$ may be defined in term of $\Gamma_b(x)$ as

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}\,. \tag{A.3}$$

It has an integral representation:

$$\log S_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{\sinh t(Q-2x)}{2\sinh bt \sinh b^{-1}t} - \frac{Q-2x}{2t}\right) \tag{A.4}$$

and the properties:

$$S_b(x+b) = 2\sin(\pi bx)S_b(x)\,, \tag{A.5}$$

$$S_b(x+1/b) = 2\sin(\pi x/b)S_b(x)\,. \tag{A.6}$$

For $b \to 0$ the double Gamma $\Gamma_b(x)$ and double Sine $S_b(x)$ functions have the asymptotic behaviour [10]:

$$S_b(bx) \to (2\pi b^2)^{x-\frac{1}{2}}\Gamma(x)\,, \tag{A.7}$$

$$S_b\left(\frac{1}{2b}+bx\right) \to 2^{x-\frac{1}{2}}\,, \tag{A.8}$$

$$S_b\left(\frac{1}{b}+bx\right) \to \frac{2\pi(2\pi b^2)^{x-\frac{1}{2}}}{\Gamma(1-x)}\,, \tag{A.9}$$

$$\Gamma_b(bx) \to (2\pi b^3)^{\frac{1}{2}(x-\frac{1}{2})}\Gamma(x)\,, \tag{A.10}$$

$$\Gamma_b(Q-bx) \to \sqrt{2\pi}(2\pi b)^{\frac{1}{2}(\frac{1}{2}-x)}\,. \tag{A.11}$$

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B Meijer G-functions

The Meijer G-functions can be defined via the integral [31]:

$$G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}x^s ds\,. \tag{B.1}$$

In this paper we will consider the $G^{3,2}_{3,3}$ function. It admits the decomposition [31]:

$$\begin{aligned}G^{3,2}_{3,3}\left(x\,\middle|\,\begin{matrix}a_1,a_2,a_3\\ b_1,b_2,b_3\end{matrix}\right)&=\frac{\Gamma(a_1-a_2)\Gamma(1+b_1-a_1)\Gamma(1+b_2-a_1)\Gamma(1+b_3-a_1)}{\Gamma(1+a_3-a_1)}x^{a_1-1}\\&\times{}_3F_2\left(\begin{matrix}1+b_1-a_1,1+b_2-a_1,1+b_3-a_1\\ 1+a_2-a_1,1+a_3-a_1;\,x^{-1}\end{matrix}\right)\\&+\frac{\Gamma(a_2-a_1)\Gamma(1+b_1-a_2)\Gamma(1+b_2-a_2)\Gamma(1+b_3-a_2)}{\Gamma(1+a_3-a_2)}x^{a_2-1}\\&\times{}_3F_2\left(\begin{matrix}1+b_1-a_2,1+b_2-a_2,1+b_3-a_2\\ 1+a_1-a_2,1+a_3-a_2;\,x^{-1}\end{matrix}\right).\end{aligned}\tag{B.2}$$

Here ${}_3F_2$ is the generalized hypergeometric function:

$${}_3F_2\left(\begin{matrix}a,b,c;\\ d,e:x\end{matrix}\right)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{x^n}{n!}\,,$$

where

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}\tag{B.3}$$

is the Pochhammer symbol. We will need also the following property of the Meijer G-functions:

$$x^{\xi}G^{3,2}_{3,3}\left(x\,\middle|\,\begin{matrix}a_1,a_2,a_3\\ b_1,b_2,b_3\end{matrix}\right)=G^{3,2}_{3,3}\left(x\,\middle|\,\begin{matrix}a_1+\xi,a_2+\xi,a_3+\xi\\ b_1+\xi,b_2+\xi,b_3+\xi\end{matrix}\right).\tag{B.4}$$

C ${}_3F_2$ and ${}_2F_1$ hypergeometric functions with unit argument

The ${}_3F_2$ function with the unit argument satisfies the identities [26–28]:

$$\begin{aligned}{}_3F_2\left(\begin{matrix}a,b,c;\\ d,e:1\end{matrix}\right)&=\frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(c-b)}{\Gamma(e-b)\Gamma(d-b)\Gamma(1+b-a)\Gamma(c)}{}_3F_2\left(\begin{matrix}b,1+b-d,1+b-e;\\ 1+b-c,1+b-a:1\end{matrix}\right)\\&+\frac{\Gamma(1-a)\Gamma(d)\Gamma(e)\Gamma(b-c)}{\Gamma(e-c)\Gamma(d-c)\Gamma(1+c-a)\Gamma(b)}{}_3F_2\left(\begin{matrix}c,1+c-e,1+c-d;\\ 1+c-b,1+c-a:1\end{matrix}\right),\end{aligned}\tag{C.1}$$

$${}_3F_2\left(\begin{matrix}a,b,c;\\ d,e:1\end{matrix}\right)=\frac{\Gamma(d)\Gamma(d+e-a-b-c)}{\Gamma(d-a)\Gamma(d+e-b-c)}{}_3F_2\left(\begin{matrix}e-c,e-b,a;\\ d+e-b-c,e:1\end{matrix}\right).\tag{C.2}$$

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Note that if one of the "upper" arguments of the ${}_3F_2$ function coincides with one of the "lower" argument it becomes ${}_2F_1$ function:

$$ {}_2F_1\left(\begin{matrix} a,b; \\ c:x \end{matrix}\right) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\,. $$

The ${}_2F_1$ function with the unit argument is equal to:

$$ {}_2F_1\left(\begin{matrix} a,b; \\ c:1 \end{matrix}\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,. \tag{C.3} $$

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