PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

Physical and Mathematical Sciences 2009, № 3, p. 10–21

Mathematics

DIRICHLET WEIGHT INTEGRAL ESTIMATION TO DIRICHLET PROBLEM SOLUTION FOR THE GENERAL SECOND ORDER ELLIPTIC EQUATIONS

V. Zh. DUMANIAN*

Chair of Numerical Analysis and Mathematical Modeling, YSU

We consider the Dirichlet problem in a bounded domain $Q \subset R_n$, $\partial Q \in C^1$, for the second order linear elliptic equation

$$-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j}+\sum_{i=1}^{n}b_i(x)u_{x_i}-\sum_{i=1}^{n}(c_i(x)u)_{x_i}+d(x)u=f(x)-divF(x),\ x\in Q,$$

$$u|_{\partial Q}=u_0.$$

For the solution we prove boundedness of the Dirichlet integral with the weight $r(x)$, i.e. the function $r(x)\left|\nabla u(x)\right|^2$ is integrable over Q, where $r(x)$ is the distance from a point $x \in Q$ to the boundary ∂Q.

***Keywords*:** Dirichlet problem, elliptic equation, Dirichlet's integral.

In this paper we consider the Dirichlet problem in a bounded domain $Q \subset R_n$, $n \ge 2$, with smooth boundary $\partial Q \in C^1$ for general linear elliptic second-order equation

$$-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j}+\sum_{i=1}^{n}b_i(x)u_{x_i}-\sum_{i=1}^{n}(c_i(x)u)_{x_i}+d(x)u=f(x)-divF(x),\ x\in Q, \tag{1}$$

$$u|_{\partial Q}=u_0, \tag{2}$$

where $u_0 \in L_2(\partial Q)$, the functions f and $F=(f_1,...,f_n)$ belong to $L_{2,loc}(Q)$, $A(x)=(a_{ij}(x))$ is a symmetric matrix, whose elements are real measurable functions, and for all $\xi=(\xi_1,...,\xi_n)\in R_n$ and $x \in Q$ satisfy the condition

$$\gamma_1|\xi|^2 \le \sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j=(\xi,A(x)\xi)\le\gamma_2|\xi|^2 \tag{3}$$

with positive constants γ_1 and γ_2; the real coefficients $b(x)=(b_1(x),...,b_n(x))$, $c(x)=(c_1(x),...,c_n(x))$ and $d(x)$ are measurable and bounded functions on each strong inner subdomain of the domain Q.

The autor was partially supported by DAAD A/06/33040.

* E-mail: duman@ysu.am

As in [1–6], we assume that the unit inner normal $\overline{\nu}$ to the boundary ∂Q for all x and y in ∂Q satisfies Dini's condition

$$|\overline{\nu}(x)-\overline{\nu}(y)|\le\omega(|x-y|), \tag{4}$$

where ω is a monotone function such that $\int_0 \frac{\omega(t)}{t}dt<\infty$, and the coefficients $a_{ij}(x)$ are continuous on the boundary in the sense of Dini

$$|a_{ij}(x)-a_{ij}(y)|\le\omega(|x-y|) \tag{5}$$

for all $x\in\partial Q,\ y\in Q$ and $i,j=1,\dots,n$; without loss of generality assume that the function ω is the same in (4) and (5).

The aim of this paper is to establish the boundedness of the Dirichlet's integral with the weight $r(x)$ for the solution $u(x)$ of the problem (1), (2), i.e. the integrability of the function $r(x)|\nabla u(x)|^2$ over Q, where $r(x)$ is the distance of a point $x\in Q$ from the boundary ∂Q. This result is well known in the case of equation with smooth coefficients and Lyapunov domain (see [7–14]). In [1] this result was established for an equation without lower-order terms $(b_i=0,\ c_i=0,\ d=0)$ with $f\in W_2^{-1}$ $(F=0)$ and under the assumption that the conditions (3)–(5) are satisfied. In [2] this result was generalized for a wider class of right-hand sides. Here we show that the theorem holds for right-hand sides with

$$r^{1/2}(x)\left(1+|\ln r(x)|\right)^{3/4}|F(x)|\in L_2(Q), \tag{6}$$

$$r^{3/2}(x)\left(1+|\ln r(x)|\right)^{3/4}|f(x)|\in L_2(Q). \tag{7}$$

Moreover, the integrability of the function $r(x)|\nabla u(x)|^2$ over Q is not only necessary, but also sufficient for any solution of the equation (1) to be a solution of the Dirichlet problem with some boundary function $u_0\in L_2(\partial Q)$ (see [2, 7]).

The integrability of the function $r(x)|\nabla u(x)|^2$ over Q for the case of the equation with lower-order terms (with non-zero coefficients b and d, $c=0$) was established in [4] (see also [3]) under the assumption that the coefficients b and d satisfy the conditions: there exists a constant $K>0$ such that

$$|b(x)|\le\frac{K}{r(x)(1+|\ln r(x)|)^{3/4}},\ x\in Q, \tag{8}$$

$$\int_0 t^3(1+|\ln t|)^{3/2}D^2(t)dt<\infty,\ \text{where } D(t)\equiv\sup_{r(x)\ge t}|d(x)|. \tag{9}$$

In this article we consider the general equation and assume that the coefficient $c(x)$ satisfies the following condition (see [5, 6]):

$$\int_0 t(1+|\ln t|)^{3/2}C^2(t)dt<\infty,\ \text{where } C(t)\equiv\sup_{r(x)\ge t}|c(x)|. \tag{10}$$

By a solution of problem (1), (2) we understand a function u in $W_{2,loc}^1$, satisfying the equation (1) in the sense of generalized functions, i.e. for all $\eta\in C_0^\infty(Q)$ the integral identity

$$\int_Q (A(x)\nabla u + c(x)u, \nabla\eta)dx + \int_Q ((b(x),\nabla u) + d(x)u)\eta dx = \int_Q (f\eta + (F,\nabla\eta))dx \quad (11)$$

is satisfied, and satisfying condition (2) in the following sense: each point $x^0 \in \partial Q$ has a neighbourhood $V_{x^0} \subset \partial Q$ such that

$$\int_{V_{x^0}} (u(x+\delta\bar{\nu}(x^0)) - u_0(x))^2 ds \longrightarrow 0 \text{ as } \delta \longrightarrow +0. \quad (12)$$

Now we'll present the main result of the article.

Theorem. Assume that the conditions (3)–(10) are satisfied, and let u be a solution of the Dirichlet problem (1), (2) in $W^1_{2,loc}$. Then the function $r(x)|\nabla u(x)|^2$ is integrable over Q, i.e. $\int_Q r(x)|\nabla u(x)|^2 dx < \infty$.

Proof of the Theorem. We'll follow the scheme of the proof of Lemma 1 in [1]. Let $x^0 \in \partial Q$ be an arbitrary point of the boundary ∂Q of the domain Q, and (x', x_n) be a local coordinate system with the origin x^0, and x_n-axis be directed along the inner normal $\bar{\nu}(x^0)$ to ∂Q at the point x^0. Since ∂Q belongs to the class C^1, there exists positive number $r_{x^0} > 0$ and a function $\varphi_{x^0} \in C^1(R_{n-1})$ with $\varphi_{x^0}(0) = 0, \nabla\varphi_{x^0}(0) = 0$ and $|\nabla\varphi_{x^0}(x')| \le \frac{1}{2}$ for all $x' \in R_{n-1}$, such that the intersection of the domain Q with the sphere $U_{x^0}^{(r_{x^0})} = \{x : |x - x^0| < r_{x^0}\}$ of radius r_{x^0} about x^0 has the form $Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n > \varphi_{x^0}(x')\}$.

Then, of course, $\partial Q \cap U_{x^0}^{(r_{x^0})} = U_{x^0}^{(r_{x^0})} \cap \{(x', x_n) : x_n = \varphi_{x^0}(x')\}$.

We assume that r_{x^0} is such that $\partial Q \cap U_{x^0}^{(r_{x^0})}$ belongs to the neighbourhood V_{x^0} in condition (12) (this can be achieved by decrease of r_{x^0}). Then

$$\int_{\{x' \in R_{n-1} : |x'| < \frac{2}{\sqrt{5}} r_{x^0}\}} (u(x', \varphi_{x^0}(x') + \delta) - u_0(x', \varphi_{x^0}(x')))^2 dx' \to 0 \text{ as } \delta \to +0.$$

Let $\ell_{x^0} = \frac{r_{x^0}}{\sqrt{2}}$. Following [1], let us select a finite subcovering $U_{x^m}^{(l_{x^m})}$, $m = 1, \ldots, p$, from the covering $\left\{U_{x^0}^{(\ell_{x^0})}, x^0 \in \partial Q\right\}$ of the boundary ∂Q; for brevity denote the balls $U_{x^m}^{(r_{x^m})}$, $m = 1, \ldots, p$, by U_m, r_{x^m} by r_m, ℓ_{x^m} by ℓ_m and φ_{x^m} by φ_m. Set $h = \frac{1}{3}\left(\frac{2}{\sqrt{5}} - \frac{\sqrt{2}}{2}\right)\min(1, r_1, \ldots, r_p)$. Then, each of the curvilinear cylinders $\Pi_m^{\ell_m+h,h} = \{(x', x_n) : |x'| < \ell_m + h, \varphi_m(x') < x_n < \varphi_m(x') + h\}$ lies in the corresponding ball U_m, as well as in $U_m \cap Q$ (recall that here (x', x_n) are the coordinates of a point in a local system of coordinates with origin at x^m). Let $\ell_0 \in (0, h/4)$ be such that the complement of the domain

$Q_{3\ell_0} = \{x \in Q : r(x) = \mathrm{dist}\,(x, \partial Q) > 3\ell_0\}$ in Q lies in the union of the "cylinders" $\Pi_m^{\ell_m, h} = \{(x', x_n) : |x'| < \ell_m,\ \varphi_m(x') < x_n < \varphi_m(x') + h\},\ m = 1, \ldots, p$:

$$Q^{3\ell_0} = \{x \in Q : r(x) = \mathrm{dist}\,(x, \partial Q) \le 3\ell_0\} \subset \bigcup_{m=1}^{p} \Pi_m^{\ell_m, h}.$$

Put $\Pi_m^h = \Pi_m^{\ell_m + l_0, h} \subset \Pi_m^{\ell_m + h, h} \subset U_m \cap Q,\ Q_m = (Q \setminus Q^{2\ell_0}) \cup \Pi_m^h,\ Q_m' = (Q \setminus Q^{3l_0}) \cup \Pi_m^{\ell_m, h}$.

It is easy to see that for all $x = (x', x_n) \in \Pi_m^h,\ m = 1, \ldots, p$,

$$r(x) \le x_n - \varphi_m(x') \le \frac{\sqrt{5}}{2} r(x) < \frac{4}{3} r(x). \tag{13}$$

We fix an index m, $1 \le m \le p$, and take a local coordinate system with origin at x^m; further the dependence of the function φ_m on the index m will not be indicated: $\varphi = \varphi_m$. We define a mapping L of the space R_n onto itself by relation $L(x) = (x', x_n - \varphi(x'))$, where $x = (x', x_n);\ L_{-1}(y) = (y', y_n + \varphi(y'))$. The image of a set under the mapping L will be denoted by the same letter with ~ on top; in particular, $L(Q) = \tilde{Q},\ L(Q_m) = \tilde{Q}_m,\ L(\Pi_m^h) = \tilde{\Pi}_m^h,\ L(\Pi_m^{\ell_m, h}) = \tilde{\Pi}_m^{\ell_m, h}$

Let $u(x)$ be a solution in $W_{2,loc}^1$ of the problem (1), (2). We take an arbitrary function $\tilde{\eta}$ in $W_2^1(\tilde{Q})$ with support in $\tilde{Q}$. Then, the function $\eta(x) = \tilde{\eta}(x', x_n - \varphi(x')),\quad x = (x', x_n) \in Q$, belongs to $W_2^1(Q)$ and its support is contained in Q.

Denoting $u(y', y_n + \varphi(y'))$ by $\tilde{u}(y),\ f(y', y_n + \varphi(y'))$ by $\tilde{f}(y)$ and $d(y', y_n + \varphi(y'))$ by $\tilde{d}(y),$ from the integral identity (11) we get

$$\int_{\tilde{Q}} \left(\sum_{i,j=1}^{n} \tilde{a}_{ij}(y) \tilde{u}_{y_i}(y) \tilde{\eta}_{y_j}(y) + \sum_{i=1}^{n} \tilde{c}_i(y) \tilde{u}(y) \tilde{\eta}_{y_i}(y) \right) dy + \int_{\tilde{Q}} \left(\sum_{i=1}^{n} \tilde{b}_i(y) \tilde{u}_{y_i}(y) + \tilde{d}(y) \tilde{u}(y) \right) \tilde{\eta}(y) dy =$$

$$= \int_{\tilde{Q}} \tilde{f}(y) \tilde{\eta}(y) dy + \int_{\tilde{Q}} \sum_{i=1}^{n} \tilde{f}_i(y) \tilde{\eta}_{y_i}(y) dy, \tag{$\widetilde{11}$}$$

where the matrix $\tilde{A}(y) = (\tilde{a}_{ij}(y))$ and the vectors $\tilde{b}(y) = (\tilde{b}_1(y), \ldots, \tilde{b}_n(y)),$ $\tilde{c}(y) = (\tilde{c}_1(y), \ldots, \tilde{c}_n(y)),\ \tilde{F}(y) = (\tilde{f}_1(y), \ldots, \tilde{f}_n(y))$ have the form:

$\tilde{a}_{ij}(y) = a_{ij}(y', y_n + \varphi(y'))$ for $i < n,\ j < n$,

$$\tilde{a}_{ni}(y) = \tilde{a}_{in}(y) = a_{ni}(y', y_n + \varphi(y')) - \sum_{k=1}^{n-1} a_{ki}(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k} \text{ for } i < n,$$

$$\tilde{a}_{nn}(y) = \sum_{k,m=1}^{n-1} \frac{\partial \varphi(y')}{\partial y_k} a_{km}(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_m} - 2 \sum_{k=1}^{n-1} a_{nk}(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k} +$$

$$+ a_{nn}(y', y_n + \varphi(y')),$$

$\tilde{b}_i(y) = b_i(y', y_n + \varphi(y'))$ for $i < n$,

$$\tilde{b}_n(y) = b_n(y', y_n + \varphi(y')) - \sum_{k=1}^{n-1} b_k(y', y_n + \varphi(y')) \frac{\partial \varphi(y')}{\partial y_k},$$

$\tilde{c}_i(y) = c_i(y', y_n + \varphi(y'))$ for $i < n$,

$$\tilde{c}_n(y)=c_n(y',y_n+\varphi(y'))-\sum_{k=1}^{n-1}c_k(y',y_n+\varphi(y'))\frac{\partial\varphi(y')}{\partial y_k},$$

$\tilde{f}_i(y)=f_i(y',y_n+\varphi(y'))$ for $i<n$,

$$\tilde{f}_n(y)=f_n(y',y_n+\varphi(y'))-\sum_{k=1}^{n-1}f_k(y',y_n+\varphi(y'))\frac{\partial\varphi(y')}{\partial y_k}.$$

This means that the function $\tilde{u}(y)$ (in $W^1_{2,loc}(\tilde{Q})$) is a solution of the equation

$$-\text{div}\,(\tilde{A}(y),\nabla\tilde{u}(y))+(\tilde{b}(y),\nabla\tilde{u}(y))-\text{div}(\tilde{c}(y)\tilde{u}(y))+\tilde{d}(y)\tilde{u}(y)=\tilde{f}(y)-\text{div}\,\tilde{F}(y).\quad(\tilde{1})$$

The matrix $\tilde{A}(y)$ is positively defined uniformly with respect to $y\in\tilde{Q}$, and the coefficient $\tilde{a}_{nn}(y)$ satisfies the inequalities

$$\gamma_1\le\gamma_1(1+|\nabla\varphi(y')|^2)\le\tilde{a}_{nn}(y)\le\gamma_2(1+|\nabla\varphi(y')|^2)\le\frac{5}{4}\gamma_2.$$

Denote by $A_0(y)=(a^0_{ij}(y))$ the matrix, the elements of which are defined on $\tilde{\Pi}^h_m$ and have the following form: $a^0_{ij}(y)=\tilde{a}_{ij}(y)$ for $i<n,\ j<n$,

$$a^0_{ni}(y)=a^0_{in}(y)=a^0_{in}(y',y_n)=\frac{1}{\text{mes}_{n-1}\{\xi\in R_{n-1}:|\xi|<y_n\}}\int_{\{\xi\in R_{n-1}:|\xi-y'|<y_n\}}\tilde{a}_{in}(\xi,0)d\xi\quad\text{for } i<n,$$

$$a^0_{nn}(y)=\tilde{a}_{nn}(y',0).$$

In [1] it was established that in $\tilde{\Pi}^h_m$

$$\left(\sum_{i=1}^{n}\left|a^0_{in}(y)-\tilde{a}_{in}(y)\right|^2\right)^{1/2}\le\tilde{\omega}(y_n)\tag{14}$$

and

$$\left|\frac{\partial a^0_{in}(y)}{\partial y_i}\right|\le\frac{\tilde{\omega}(y_n)}{y_n},\quad i=1,\dots,n-1,\tag{15}$$

where $\tilde{\omega}(t)=C\tilde{\omega}(2\sqrt{2}t)$ ($\omega(t)$ follows from the conditions (4) and (5)); the constant C depends only on n and γ_2.

Let $\delta_0<\dfrac{\ell_0}{2}$ be a fixed positive number; further the dependence on the chosen and fixed numbers $p,r_m,\ell_m,m=1,\dots,p,\ell_0,n,\gamma_1,\gamma_2,\delta_0$ will not be indicated in the notation. For an arbitrary $\delta\in(0,\delta_0)$ we define the function $\varrho_\delta(y)$ on the domain $\tilde{Q}_m$ by

$$\varrho_\delta(y)=\begin{cases}0 & \text{for } |y'|<\ell_m+\ell_0,\ 0<y_n<\delta,\\ y_n-\delta & \text{for } |y'|<\ell_m+\ell_0,\ \delta\le y_n\le 4\delta_0,\\ 4\delta_0-\delta & \text{for the remaining points } y \text{ in } \tilde{Q}_m.\end{cases}$$

The function ϱ_δ satisfies the inequalities

$$r_\delta(x)\le\varrho_\delta\big(L(x)\big)\le\frac{4}{3}r_{\frac{3}{4}\delta}(x)\quad\text{for all } x\in Q_m,\tag{16}$$

where $r_\delta(x)=\min\{3\delta_0,\max\{0,r(x)-\delta\}\}$. Moreover, $\left\|\left|\nabla\varrho_\delta\right|\right\|_{L_\infty(\tilde{Q}_m)}\le 1$. We fix a function $\psi\in C^1(\overline{Q})$ such that $\psi(x)=1$ for $x\in Q'_m$, $\psi=0$ for $x\in Q^{5l_0/2}\setminus\Pi_m^{\ell_m+l_0/2,h}$ and $0\le\psi(x)\le 1$ for all $x\in Q$; it will also be assumed that for $|y'|<\ell_m+\ell_0$ and $0<y_n<2\ell_0$ the function $\tilde{\psi}(y)=\psi(L_{-1}(y))$ does not depend on y_n.

Taking in the integral identity $(\widetilde{11})$ the function $\varrho_\delta(y)\tilde{\psi}(y)\tilde{u}(y)$ instead of $\tilde{\eta}(y)$, we get

$$\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{\psi}(\nabla\tilde{u},\tilde{A}\nabla\tilde{u})dy+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{u}(\nabla\tilde{\psi},\tilde{A}\nabla\tilde{u})dy+\int\limits_{\tilde{Q}_m}\tilde{\psi}\tilde{u}(\nabla\varrho_\delta,\tilde{A}\nabla\tilde{u})dy+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{\psi}\tilde{u}(\tilde{b},\nabla\tilde{u})dy+$$
$$+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{\psi}(\nabla\tilde{u},\tilde{c}\tilde{u})dy+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{u}(\nabla\tilde{\psi},\tilde{c}\tilde{u})dy+\int\limits_{\tilde{Q}_m}\tilde{\psi}\tilde{u}(\nabla\varrho_\delta,\tilde{c}\tilde{u})dy+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{\psi}\tilde{d}\tilde{u}^2dy= \quad (17)$$
$$=\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{\psi}\tilde{u}\tilde{f}dy+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{\psi}(\tilde{F},\nabla\tilde{u})dy+\int\limits_{\tilde{Q}_m}\varrho_\delta\tilde{u}(\tilde{F},\nabla\tilde{\psi})dy+\int\limits_{\tilde{Q}_m}\tilde{\psi}\tilde{u}(\tilde{F},\nabla\varrho_\delta)dy.$$

In view of (16)

$$\tilde{I}_1^{(m)}(\delta)=\int\limits_{\tilde{Q}_m}\varrho_\delta(y)\tilde{\psi}(y)(\nabla\tilde{u}(y),\tilde{A}(y)\nabla\tilde{u}(y))dy\ge\int\limits_{Q'_m}r_\delta(x)(\nabla u(x),A(x)\nabla u(x))dx\ge$$
$$\ge\gamma_1\int\limits_{Q'_m}r_\delta(x)|\nabla u(x)|^2\,dx.$$

We are going to obtain upper estimates for the remaining terms of equality (17).

Let us estimate the integral $\tilde{I}_2^{(m)}(\delta)=\int\limits_{\tilde{Q}_m}\varrho_\delta(y)\tilde{u}(y)(\nabla\tilde{\psi}(y),\tilde{A}(y)\nabla\tilde{u}(y))dy.$

Again in view of (16) we get

$$|\tilde{I}_2^{(m)}(\delta)|\le\frac{4}{3}\int\limits_{Q_m}r_{\frac{3}{4}\delta}(x)|u(x)||(\nabla\psi(x),A(x)\nabla u(x))|dx\le\frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\gamma_2\int\limits_{Q_m}r_{\frac{3}{4}\delta}(x)|u(x)||\nabla u(x)|dx\le$$
$$\le\frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\gamma_2\left\{\int\limits_{Q_m}r_{\frac{3}{4}\delta}(x)u^2(x)dx\right\}^{1/2}\left\{\int\limits_{(Q\setminus Q^{\frac{5\ell_0}{2}})\cup\Pi_m^{\ell_m+\frac{\ell_0}{2},h}}r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx\right\}^{1/2}\le$$
$$\le\frac{4}{3}\|\psi\|_{C^1(\overline{Q})}\gamma_2\left\{\int\limits_{Q}r(x)u^2(x)dx\right\}^{1/2}\left\{\frac{\delta}{2}\int\limits_{(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}})\setminus Q^{\frac{3\delta}{4}}}|\nabla u(x)|^2\,dx+2\int\limits_{Q}r_\delta(x)|\nabla u(x)|^2\,dx\right\}^{1/2}\le$$
$$\le\varepsilon\int\limits_{Q}r_\delta(x)|\nabla u(x)|^2\,dx+\frac{\varepsilon\delta}{4}\int\limits_{(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}})\setminus Q^{\frac{3\delta}{4}}}|\nabla u(x)|^2\,dx+\frac{C'_2}{\varepsilon}\int\limits_{Q}r(x)u^2(x)dx,$$

where $0<\varepsilon<1$ is to be chosen later.

Since the estimate is valid for solutions of the elliptic equation (1) (see [15])

$$\int\limits_{\Omega'}|\nabla u|^2\,dx\le C_0(n,\gamma_1,\gamma_2)\left(\frac{1}{\sigma^2}\int\limits_{\Omega}u^2dx+\int\limits_{\Omega}(|b|^2+|c|^2+|d|)u^2dx+\sigma^2\int\limits_{\Omega}f^2dx+\int\limits_{\Omega}|F|^2\,dx\right),\quad(18)$$

where $\Omega'\subset\subset\Omega$ and $\sigma=\operatorname{dist}(\Omega',\partial\Omega)$, then in view of (8), (9) and (10) it follows that

$$\delta \int\limits_{(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}})\setminus Q^{\frac{3\delta}{4}}} |\nabla u(x)|^2\,dx \le C_0\delta\Bigg(\bigg(\frac{16}{\delta^2}+\big\||b|\big\|^2_{L_\infty(Q_{\frac{\delta}{2}}\setminus \overline{Q}_{\frac{3}{2}\delta})}\bigg)\int\limits_{(\Pi_m^{\ell_m+\ell_0,h}\cap Q^{\frac{3\delta}{2}})\setminus Q^{\frac{\delta}{2}}} u^2(x)dx+$$

$$+\int\limits_{(\Pi_m^{\ell_m+\ell_0,h}\cap Q^{\frac{3\delta}{2}})\setminus Q^{\frac{\delta}{2}}}\big(|c|^2+|d|\big)u^2(x)dx+\frac{\delta^2}{16}\int\limits_{(\Pi_m^{\ell_m+\ell_0,h}\cap Q^{\frac{3\delta}{2}})\setminus Q^{\frac{\delta}{2}}} f^2(x)dx+\int\limits_{(\Pi_m^{\ell_m+\ell_0,h}\cap Q^{\frac{3\delta}{2}})\setminus Q^{\frac{\delta}{2}}}|F(x)|^2\,dx\Bigg)\le$$

$$\le C_2''\Bigg(\bigg(1+\frac{1}{(1+|\ln\frac{3}{2}\delta|)^{3/2}}+\delta\int\limits_{3\delta/8}^{3\delta/2}(C^2(t)+D(t))\,dt\bigg)\max_{\frac{\delta}{2}\le y_n\le 2\delta}\int\limits_{|y'|<\ell_m+\ell_0}\tilde u^2(y',y_n)dy'+$$

$$+\frac{1}{(1+|\ln\frac{3}{2}\delta|)^{3/2}}\int\limits_{(\Pi_m^{\ell_m+\ell_0,h}\cap Q^{\frac{3\delta}{2}})\setminus Q^{\frac{\delta}{2}}} r^3(x)(1+|\ln r(x)|)^{3/2}f^2(x)dx+$$

$$+\frac{1}{(1+|\ln\frac{3}{2}\delta|)^{3/2}}\int\limits_{(\Pi^{\ell_m+\ell_0,h}\cap Q^{\frac{3\delta}{2}})\setminus Q^{\frac{\delta}{2}}} r(x)(1+|\ln r(x)|)^{3/2}|F(x)|^2\,dx).$$

We introduce the notations

$$M=\max_{0\le y_n\le h}\int\limits_{|y'|<\ell_m+\ell_0}\tilde u^2(y',y_n)dy',\quad \|f\|^2=\int\limits_Q r^3(x)(1+|\ln r(x)|)^{3/2}f^2(x)dx,$$

$$\|F\|^2=\int\limits_Q r(x)(1+|\ln r(x)|)^{3/2}|F(x)|^2\,dx.$$

Since by (9), (10) $\delta\int\limits_{\frac{3}{8}\delta}^{\frac{3}{2}\delta}(C^2(t)+D(t))dt\le\frac{8}{3}\int\limits_{\frac{3}{8}\delta}^{\frac{3}{2}\delta}t(C^2(t)+D(t))dt\le\frac{8}{3}\Bigg(\int\limits_{\frac{3}{8}\delta}^{\frac{3}{2}\delta}t(1+|\ln t|)^{3/2}C^2(t)dt+$

$+\Bigg(\int\limits_{\frac{3}{8}\delta}^{\frac{3}{2}\delta}\frac{1}{t(1+|\ln t|)^{3/2}}dt\Bigg)^{1/2}\Bigg(\int\limits_{\frac{3}{8}\delta}^{\frac{3}{2}\delta}t^3(1+|\ln t|)^{3/2}D^2(t)dt\Bigg)^{1/2}\Bigg)\le C_2'''$, then we have

$$\delta\int\limits_{(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}})\setminus Q^{\frac{3\delta}{4}}}|\nabla u(x)|^2\,dx\le\tilde C_0(M+\|f\|^2+\|F\|^2).\qquad(19)$$

Therefore, the estimation is valid: $|\tilde I_2^{(m)}(\delta)|\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+I_2^{(m)}(\varepsilon)$,

where $I_2^{(m)}(\varepsilon)=C_2\left(\frac{1}{\varepsilon}\int\limits_Q r(x)u^2(x)dx+\varepsilon(M+\|f\|^2+\|F\|^2)\right)$.

Let us estimate the integral $\tilde I_3^{(m)}(\delta)=\int\limits_{\tilde Q_m}\tilde\psi(y)\tilde u(y)(\nabla\varrho_\delta(y),\tilde A(y)\nabla\tilde u(y))dy$.

$$\tilde I_3^{(m)}(\delta)=\int\limits_\delta^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde\psi(y')\tilde u(y',y_n)(\nabla\varrho_\delta(y_n),(\tilde A(y',y_n)-A_0(y',y_n))\nabla\tilde u(y',y_n))dy'dy_n-$$

$$-\frac{1}{2}\int\limits_\delta^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde\psi(y')\tilde u^2(y',y_n)\sum_{i=1}^{n-1}\frac{\partial a_{in}^0(y',y_n)}{\partial y_i}dy'dy_n-$$

$$-\frac{1}{2}\int\limits_\delta^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde u^2(y',y_n)\sum_{i=1}^{n-1}a_{in}^0(y',y_n)\frac{\partial\tilde\psi(y')}{\partial y_i}dy'dy_n+\frac{1}{2}\int\limits_{|y'|<\ell_m+\ell_0}\tilde a_{nn}(y',0)\tilde\psi(y')\tilde u^2(y',4\delta_0)dy'-$$

$$-\frac{1}{2}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{a}_{nn}(y',0)\tilde{\psi}(y')\tilde{u}^2(y',\delta)dy'=\tilde{I}_{31}^{(m)}(\delta)+\tilde{I}_{32}^{(m)}(\delta)+\tilde{I}_{33}^{(m)}(\delta)+\tilde{I}_{34}^{(m)}(\delta_0)+\tilde{I}_{35}^{(m)}(\delta).$$

In view of (14) and (13)

$$|\tilde{I}_{31}^{(m)}(\delta)|\le\int\limits_{\delta}^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{\psi}(y')|\tilde{u}(y)||\nabla\tilde{u}(y)|\tilde{\omega}(y_n)dy'dy_n\le I_{31}^{(m)'}(\delta)+$$

$$+\tilde{\omega}(4\delta_0)\int\limits_{\delta_0}^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{\psi}(y')|\tilde{u}(y)||\nabla\tilde{u}(y)|dy'dy_n,$$

where $I_{31}^{(m)'}(\delta)=\left(\int\limits_{\delta}^{\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{\psi}(y')y_n|\nabla\tilde{u}(y)|^2\,dy'dy_n\right)^{1/2}\left(M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n\right)^{1/2}\le$

$$\le\left(\frac{\sqrt{5}}{2}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q_{\frac{2\delta}{\sqrt{5}}}}r(x)|\nabla u(x)|^2\,dx\right)^{1/2}\left(M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n\right)^{1/2}\le$$

$$\le\left(4\sqrt{5}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}}r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx\right)^{1/2}\left(M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n\right)^{1/2}\le\frac{\varepsilon}{2}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}}r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx+$$

$$+\frac{8\sqrt{5}}{\varepsilon}M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n.$$

Further, in view of (19)

$$\frac{\varepsilon}{2}\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}}r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx\le\frac{\varepsilon}{2}\left(\frac{\delta}{2}\int\limits_{(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}})\setminus Q^{\frac{3\delta}{4}}}|\nabla u(x)|^2\,dx+2\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx\right)\le$$

$$\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+\frac{\varepsilon}{4}\tilde{C}_0(M+\|f\|^2+\|F\|^2).$$

Thus, $|\tilde{I}_{31}^{(m)}(\delta)|\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+I_{31}^{(m)}(\delta_0,\varepsilon)$, where

$$I_{31}^{(m)}(\delta_0,\varepsilon)=\frac{\varepsilon}{4}\tilde{C}_0(M+\|f\|^2+\|F\|^2)+\frac{8\sqrt{5}}{\varepsilon}M\int\limits_0^{\delta_0}\frac{\tilde{\omega}^2(y_n)}{y_n}dy_n+$$

$$+\tilde{\omega}(4\delta_0)\int\limits_{\delta_0}^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{\psi}(y')|\tilde{u}(y)||\nabla\tilde{u}(y)|dy'dy_n.$$

In view of (15)

$$|\tilde{I}_{32}^{(m)}(\delta)|\le\frac{n-1}{2}\int\limits_0^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{\psi}(y')\tilde{u}^2(y',y_n)\frac{\tilde{\omega}(y_n)}{y_n}dy'dy_n\le M\frac{n-1}{2}\int\limits_0^{4\delta_0}\frac{\tilde{\omega}(y_n)}{y_n}dy_n=I_{32}^{(m)}(\delta_0).$$

$$|\tilde{I}_{33}^{(m)}(\delta)|\le\frac{1}{2}\int\limits_0^{4\delta_0}\int\limits_{|y'|<\ell_m+\ell_0}\tilde{u}^2(y',y_n)\left|\sum_{i=1}^{n-1}a_{in}^0(y',y_n)\frac{\partial\tilde{\psi}(y')}{\partial y_i}\right|dy'dy_n=I_{33}^{(m)}(\delta_0).$$

$|\tilde I_{35}^{(m)}(\delta)|\le\frac{5}{8}\gamma_2 M=I_{35}^{(m)}$. Thus, we get $|\tilde I_3^{(m)}(\delta)|\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+I_3^{(m)}(\varepsilon)$, where $I_3^{(m)}(\varepsilon)=I_{31}^{(m)}(\delta_0,\varepsilon)+I_{32}^{(m)}(\delta_0)+I_{33}^{(m)}(\delta_0)+\tilde I_{34}^{(m)}(\delta_0)+I_{35}^{(m)}$.

Let us estimate the integral $\tilde I_4^{(m)}(\delta)=\int\limits_{\tilde Q_m}\varrho_\delta(y)\tilde\psi(y)\tilde u(y)(\tilde b(y),\nabla\tilde u(y))dy$. In view of (16)

$$|\tilde I_4^{(m)}(\delta)|\le\frac{4}{3}\int\limits_{Q_m}r_{\frac{3}{4}\delta}(x)\psi(x)|u(x)||b(x)||\nabla u(x)|dx\le\left(2\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}}r(x)\psi(x)u^2(x)|b(x)|^2\,dx\right)^{1/2}\times$$

$$\times\left(\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}}\psi(x)r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx\right)^{1/2}\le\frac{\varepsilon}{2}\int\limits_{(Q\setminus Q^{\frac{5\ell_0}{2}})\cup\Pi_m^{\ell_m+\frac{\ell_0}{2},h}}r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx+$$

$$+\frac{1}{\varepsilon}\int\limits_{Q_m\cap Q_{\frac{3}{4}\delta}}\frac{K^2u^2(x)}{r(x)\left(1+|\ln r(x)|\right)^{3/2}}dx\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+\frac{\varepsilon}{4}\tilde C_0(M+\|f\|^2+\|F\|^2)+$$

$$+\frac{K^2}{\varepsilon}\left(\int\limits_{Q\setminus Q^{2\ell_0}}\frac{u^2(x)}{r(x)(1+|\ln r(x)|)^{3/2}}dx+\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q_{\frac{3\delta}{4}}}\frac{u^2(x)}{r(x)(1+|\ln r(x)|)^{3/2}}dx\right)\le$$

$$\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+\frac{\varepsilon}{4}\tilde C_0(M+\|f\|^2+\|F\|^2)+\frac{K^2}{2\varepsilon\ell_0}\|u\|^2_{L_2(Q)}+$$

$$+\frac{\sqrt5K^2}{2\varepsilon}\int\limits_{\frac{3\delta}{4}}^{h}\int\limits_{|y'|<\ell_m+\ell_0}\frac{\tilde u^2(y)}{y_n(1+|\ln y_n|)^{3/2}}dy'dy_n\le\varepsilon\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx+I_4^{(m)}(\varepsilon),$$

where $I_4^{(m)}(\varepsilon)=\frac{\varepsilon}{4}\tilde C_0(M+\|f\|^2+\|F\|^2)+\frac{K^2}{2\varepsilon\ell_0}\|u\|^2_{L_2(Q)}+\frac{\sqrt5K^2}{2\varepsilon}M\int\limits_0^h\frac{dy_n}{y_n(1+|\ln y_n|)^{3/2}}$.

Let us estimate the integral $\tilde I_5^{(m)}(\delta)=\int\limits_{\tilde Q_m}\varrho_\delta(y)\tilde\psi(y)(\tilde c(y)\tilde u(y),\nabla\tilde u(y))dy$.

In view of (10), (16) we have

$$|\tilde I_5^{(m)}(\delta)|\le\frac{4}{3}\int\limits_{Q_m}r_{\frac{3}{4}\delta}(x)\psi(x)|c(x)||u(x)||\nabla u(x)|dx\le\frac{\varepsilon}{2}\int\limits_{Q_m}\psi(x)r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx+$$

$$+\frac{16}{9\varepsilon}\int\limits_{Q_m}\psi(x)r_{\frac{3}{4}\delta}(x)|c(x)|^2u^2(x)dx\le\frac{\varepsilon}{2}\int\limits_{(Q\setminus Q^{\frac{5\ell_0}{2}})\cup\Pi_m^{\ell_m+\frac{\ell_0}{2},h}}r_{\frac{3}{4}\delta}(x)|\nabla u(x)|^2\,dx+$$

$$+\frac{16}{9\varepsilon}\int\limits_{Q_m\cap Q_{\frac{3\delta}{4}}}\psi(x)r(x)C^2(r(x))u^2(x)dx\le\frac{\varepsilon}{2}\left(\frac{\delta}{2}\int\limits_{(\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q^{\frac{5\delta}{4}})\setminus Q^{\frac{3\delta}{4}}}|\nabla u(x)|^2\,dx+2\int\limits_Q r_\delta(x)|\nabla u(x)|^2\,dx\right)+$$

$$+\frac{16}{9\varepsilon}\left(\int\limits_{Q\setminus Q^{2\ell_0}}\psi(x)r(x)C^2(r(x))u^2(x)dx+\int\limits_{\Pi_m^{\ell_m+\frac{\ell_0}{2},h}\cap Q_{3\delta/4}}\psi(x)r(x)C^2(r(x))u^2(x)dx\right)\le$$

$$\le \varepsilon \int_Q r_\delta(x)|\nabla u(x)|^2\,dx + \frac{\varepsilon}{4}\tilde{C}_0(M+\|f\|^2+\|F\|^2)+\frac{16}{9\varepsilon}C^2(2\ell_0)\int_Q r(x)u^2(x)dx+$$

$$+\frac{16}{9\varepsilon}\int_{\frac{3\delta}{4}}^{h}\int_{|y'|<\ell_m+\frac{\ell_0}{2}}\tilde{u}^2(y',y_n)y_nC^2\left(\frac{2}{\sqrt{5}}y_n\right)dy'dy_n \le \varepsilon\int_Q r_\delta(x)|\nabla u(x)|^2\,dx+\frac{\varepsilon}{4}\tilde{C}_0(M+\|f\|^2+\|F\|^2)+$$

$$+\frac{16}{9\varepsilon}C^2(2\ell_0)\int_Q r(x)u^2(x)dx+\frac{16}{9\varepsilon}M\int_0^h y_nC^2\left(\frac{2}{\sqrt{5}}y_n\right)dy_n.$$

Thus, we get $|\tilde{I}_5^{(m)}(\delta)|\le \varepsilon\int_Q r_\delta(x)|\nabla u(x)|^2\,dx + I_5^{(m)}(\varepsilon)$, where

$$I_5^{(m)}(\varepsilon)=\frac{\varepsilon}{4}\tilde{C}_0(M+\|f\|^2+\|F\|^2)+\frac{16}{9\varepsilon}C^2(2\ell_0)\int_Q r(x)u^2(x)dx+\frac{16}{9\varepsilon}M\int_0^h y_nC^2\left(\frac{2}{\sqrt{5}}y_n\right)dy_n.$$

Let us estimate the integral $\tilde{I}_6^{(m)}(\delta)=\int_{\tilde{Q}_m}\varrho_\delta(y)\tilde{u}(y)(\tilde{c}(y)\tilde{u}(y),\nabla\tilde{\psi}(y))dy$.

Again in view of (10), (16)

$$|\tilde{I}_6^{(m)}(\delta)|\le\frac{4}{3}\int_{Q_m} r_{\frac{3}{4}\delta}(x)C(r(x))u^2(x)|\nabla\psi(x)|\,dx\le\frac{4}{3}\|\psi\|_{C^1(\bar{Q})}\left(\int_{Q\setminus Q^{2\ell_0}} r_{\frac{3}{4}\delta}(x)C(r(x))u^2(x)dx+\right.$$

$$\left.+\int_{\Pi_m^{\ell_m+\ell_0,h}\cap Q_{3\delta/4}} r_{\frac{3}{4}\delta}(x)C(r(x))u^2(x)dx\right)\le\frac{4}{3}\|\psi\|_{C^1(\bar{Q})}\left(C(2\ell_0)\int_Q r(x)u^2(x)dx+\right.$$

$$\left.+\int_{3\delta/4}^{h}\int_{|y'|<\ell_m+\ell_0}\tilde{u}^2(y',y_n)y_nC\left(\frac{2}{\sqrt{5}}y_n\right)dy'dy_n\right)\le$$

$$\le\frac{4}{3}\|\psi\|_{C^1(\bar{Q})}\left(C(2\ell_0)\int_Q r(x)u^2(x)dx+M\int_0^h y_nC\left(\frac{2}{\sqrt{5}}y_n\right)dy_n\right)\le$$

$$\le\frac{4}{3}\|\psi\|_{C^1(\bar{Q})}\left(C(2\ell_0)\int_Q r(x)u^2(x)dx+M\int_0^h y_ndy_n+M\int_0^h y_nC^2\left(\frac{2}{\sqrt{5}}y_n\right)dy_n\right)=\tilde{I}_6^{(m)}.$$

Let us estimate the integral $\tilde{I}_7^{(m)}(\delta)=\int_{\tilde{Q}_m}\tilde{\psi}(y)\tilde{u}(y)(\tilde{c}(y)\tilde{u}(y),\nabla\varrho_\delta(y))dy$.

In view of (10)

$$|\tilde{I}_7^{(m)}(\delta)|\le\int_{\tilde{Q}_m}|\tilde{c}(y)|\tilde{u}^2(y)dy\le\int_{Q\setminus Q^{2\ell_0}}|c(x)|u^2(x)dx+\int_{\Pi_m^{\ell_m+\ell_0,h}}|c(x)|u^2(x)dx\le$$

$$\le C(2\ell_0)\int_Q u^2(x)dx+M\int_0^h C\left(\frac{2}{\sqrt{5}}y_n\right)dy_n\le C(2\ell_0)\int_Q u^2(x)dx+$$

$$+M\int_0^h\frac{1}{y_n\left(1+|\ln y_n|\right)^{3/2}}dy_n+M\int_0^h y_n\left(1+|\ln y_n|\right)^{3/2}C^2\left(\frac{2}{\sqrt{5}}\right)dy_n=I_7^{(m)}.$$

Let us estimate the integral $\tilde{I}_8^{(m)}(\delta)=\int_{\tilde{Q}_m}\varrho_\delta(y)\tilde{\psi}(y)\tilde{d}(y)\tilde{u}^2(y)dy.$

In view of (9)

$$|\tilde{I}_8^{(m)}(\delta)|\le\frac{4}{3}\int_{Q_m} r_{\frac{3}{4}\delta}(x)\psi(x)\left|d(x)\right|u^2(x)dx\le\frac{4}{3}\int_{Q_m\cap Q_{\frac{3}{4}\delta}} r(x)D(r(x))\psi(x)u^2(x)dx\le$$

$$\le\frac{4}{3}\left(D(2\ell_0)\int_{Q\setminus Q^{2\ell_0}} r(x)u^2(x)dx+\int_{\Pi_m^{\ell_m+\ell_0,h}\cap Q_{3\delta/4}} r(x)D(r(x))u^2(x)dx\right)\le$$

$$\le\frac{4}{3}\left(D(2\ell_0)\int_Q r(x)u^2(x)dx+\int_{\frac{3\delta}{4}}^{h}\int_{|y'|<\ell_m+\ell_0} y_n D\left(\frac{2}{\sqrt{5}}y_n\right)\tilde{u}^2(y',y_n)dy'dy_n\right)\le$$

$$\le C_8\left(\int_Q r(x)u^2(x)dx+M\int_0^h y_n D\left(\frac{2}{\sqrt{5}}y_n\right)dy_n\right)=I_8^{(m)}.$$

Let us estimate the integral $\tilde{I}_9^{(m)}(\delta)=\int_{\tilde{Q}_m}\varrho_\delta(y)\tilde{\psi}(y)\tilde{u}(y)\tilde{f}(y)dy.$

$$|\tilde{I}_9^{(m)}(\delta)|\le\frac{4}{3}\int_{Q_m} r(x)\left|u(x)\right|\left|f(x)\right|dx\le C_9\left(\|u\|_{L_2(Q)}^2+\|f\|^2+M\int_0^h\frac{dy_n}{y_n(1+|\ln y_n|)^{3/2}}\right)=I_9^{(m)}.$$

Let us estimate the integral $\tilde{I}_{10}^{(m)}(\delta)=\int_{\tilde{Q}_m}\varrho_\delta(y)\tilde{\psi}(y)(\tilde{F}(y),\nabla\tilde{u}(y))dy.$

Analogously to the estimations of $\tilde{I}_2^{(m)}(\delta)$ and $\tilde{I}_4^{(m)}(\delta)$ we have:

$$|\tilde{I}_{10}^{(m)}(\delta)|\le\frac{4}{3}\int_{Q_m} r_{\frac{3}{4}\delta}(x)\psi(x)\,|F(x)|\,|\nabla u(x)|\,dx\le\frac{\varepsilon}{2}\int_{Q_m}\psi(x)r_{\frac{3}{4}\delta}(x)\,|\nabla u(x)|^2\,dx+$$

$$+\frac{16}{9\varepsilon}\|F\|^2\le\varepsilon\int_Q r_\delta(x)\,|\nabla u(x)|^2\,dx+\frac{\varepsilon}{4}\tilde{C}_0(M+\|f\|^2+\|F\|^2)+\frac{16}{9\varepsilon}\|F\|^2=$$

$$=\varepsilon\int_Q r_\delta(x)\,|\nabla u(x)|^2\,dx+I_{10}^{(m)}(\varepsilon).$$

Let us estimate the integral $\tilde{I}_{11}^{(m)}(\delta)=\int_{\tilde{Q}_m}\varrho_\delta(y)\tilde{u}(y)(\tilde{F}(y),\nabla\tilde{\psi}(y))dy.$

$$|\tilde{I}_{11}^{(m)}(\delta)|\le\frac{4}{3}\int_{Q_m} r_{\frac{3}{4}\delta}(x)\,|u(x)|\,|F(x)|\,|\nabla\psi(x)|\,dx\le\frac{4}{3}\|\psi\|_{C^1(\bar{Q})}\left(\int_Q r(x)u^2(x)dx+\|F\|^2\right)=I_{11}^{(m)}.$$

And finally, let us estimate the integral $\tilde{I}_{12}^{(m)}(\delta)=\int_{\tilde{Q}_m}\tilde{\psi}(y)\tilde{u}(y)(\tilde{F}(y),\nabla\varrho_\delta(y))dy.$

$$|\tilde{I}_{12}^{(m)}(\delta)|\le\int_{Q_m}|u(x)|\,|F(x)|\,dx\le\int_{Q_m}\frac{u^2(x)dx}{r(x)\left(1+|\ln r(x)|\right)^{\frac{3}{2}}}+\|F\|^2\le$$

$$\le\frac{1}{2\ell_0}\int_Q u^2(x)dx+\frac{\sqrt{5}}{2}M\int_0^h\frac{dy_n}{y_n\left(1+|\ln y_n|\right)^{3/2}}+\|F\|^2=I_{12}^{(m)}.$$

Substituting the above obtained estimates in the equality (17), we get

$$\gamma_1\int_{Q_m'} r_\delta(x)\,|\nabla u(x)|^2\,dx\le\tilde{I}_1^{(m)}(\delta)\le\sum_{k=2}^{12}|\tilde{I}_k^{(m)}(\delta)|\le5\varepsilon\int_Q r_\delta(x)\,|\nabla u(x)|^2\,dx+I^{(m)}(\varepsilon),$$

where $I^{(m)}(\varepsilon)=\sum_{k=2}^{12} I_k^{(m)}$. Summing over all m with $1\le m\le p$, we get

$$\gamma_1\int_Q r_\delta(x)\,|\nabla u(x)|^2\,dx\le\gamma_1\sum_{m=1}^{p}\int_{Q'_m} r_\delta(x)\,|\nabla u(x)|^2\,dx\le 5\varepsilon p\int_Q r_\delta(x)\,|\nabla u(x)|^2\,dx+\sum_{m=1}^{p} I^{(m)}(\varepsilon).$$

Choosing $\varepsilon<\dfrac{\gamma_1}{10p}$, we get $\int_Q r_\delta(x)\,|\nabla u(x)|^2\,dx\le\dfrac{2}{\gamma_1}\sum_{m=1}^{p} I^{(m)}(\varepsilon)$.

Since the right-hand side of the last inequality does not depend on δ, $0<\delta<\delta_0$, then it is obviously that the function $r(x)\,|\nabla u(x)|^2$ is integrable over Q.

The Theorem is proved.

Received 27.02.2009

REFERENCES

1. **Gushchin A.K.** Mat. Sb., 1988, v. 137, № 179, p. 19–64; English transl. in Math. USSR Sb., 1990, v. 65.
2. **Gushchin A.K. and Mikhailov V.P.** Mat. Sb., 1991, v. 182, № 6, p. 787–810; English transl. in Math. USSR Sb., 1992, v. 73, № 1.
3. **Dumanyan V.Zh.** DAN RAN, 2002, v. 386, № 6, p. 735–737 (in Russian).
4. **Dumanyan V.Zh.** Note di Matematica, 2002/2003, v.21, № 2, p. 99–118.
5. **Dumanyan V.Zh.** DNAN Armenia, 2008, v. 108, № 1, p. 45–49 (in Russian).
6. **Dumanyan V.Zh.** DNAN Armenia, 2008, v. 108, № 2, p. 110–116 (in Russian).
7. **Mikhailov V.P.** Mat. Sb., 1976, v. 101, № 143, p. 163–188; English transl. in Math. USSR Sb., 1976, v. 30.
8. **Mikhailov V.P.** Differentsial'nye Uravneniya, 1976, v. 12, p. 1877–1891; English transl. in Differential Equations, 1976, v. 12.
9. **Mikhailov V.P.** Partial Differential Equations, 2nd ed. M.: Nauka, 1983; English transl. of 1st ed. M.: Mir, 1978.
10. **Petrushko I.M.** Mat. Sb., 1982, v. 119, № 161, p. 48–77; English transl. in Math. USSR Sb., 1984, v. 47.
11. **Petrushko I.M.** Mat. Sb., 1983, v. 120, № 162, p. 569–588; English transl. in Math. USSR Sb., 1984, v. 48.
12. **Mikhailov V.P.** Mat. Zametki, 1980, v. 27, p. 137–145; English transl. in Math. Notes, 1980, v. 27.
13. **Gushchin A.K. and Mikhailov V.P.** On Boundary Values of Solutions of Elliptic Equations, Generalized Functions and Their Applications in Mathematical Physics (Proc. Internat. Conf., Moscow, 1980; editor V.S. Vladimirov). Vychisl. Tsentr Akad. Nauk SSSR, M., 1981, p. 189–205 (in Russian).
14. **Bogoyavlenskiy O.I., Vladimirov V.S., Volovich I.V., Grushchin A.K., Drozhzhinov Yu.N., Zharinov V.V., Mikhailov V.P.** Boundary Value Problems of Math. Physics, Trudy Mat. Inst. Steklov, 1986, v. 175, p. 63–102; English transl. in Proc. Steklov Inst. Math., 1988, № 2 (175).
15. **Dumanyan V.Zh.** Uch. Zapiski EGU, 2008, № 2, p. 145–147 (in Russian).

Proc. of the Yerevan State Univ. Phys. and Mathem. Sci., 2009, № 3, p. 10–21.

Դիրիխլեի կշռային ինտեգրալի գնահատականը երկրորդ կարգի ընդհանուր էլիպսային հավասարման Դիրիխլեի խնդրի համար

Դիտարկվում է Դիրիխլեի խնդիրը երկրորդ կարգի գծային էլիպսային հավասարման համար $Q \subset R_n$, $\partial Q \in C^1$, սահմանափակ տիրույթում.

$$-\sum_{i,j=1}^{n}\left(a_{ij}(x)u_{x_i}\right)_{x_j}+\sum_{i=1}^{n}b_i(x)u_{x_i}-\sum_{i=1}^{n}\left(c_i(x)u\right)_{x_i}+d(x)u=f(x)-divF(x),\ x\in Q,$$

$$u|_{\partial Q}=u_0:$$

Լուծման համար ցույց է տրված $r(x)$ կշռով Դիրիխլեի ինտեգրալի սահմանափակությունը, այսինքն՝ Q տիրույթով $r(x)\left|\nabla u(x)\right|^2$ ֆունկցիայի ինտեգրելիությունը, որտեղ $r(x)$-ը $x \in Q$ կետի հեռավորությունն է ∂Q եզրից:

Оценка весового интеграла Дирихле для решения задачи Дирихле для общего эллиптического уравнения второго порядка

В ограниченной области $Q \subset R_n, \partial Q \in C^1$, рассматривается задача Дирихле для линейного эллиптического уравнения второго порядка

$$-\sum_{i,j=1}^{n}(a_{ij}(x)u_{x_i})_{x_j}+\sum_{i=1}^{n}b_i(x)u_{x_i}-\sum_{i=1}^{n}(c_i(x)u)_{x_i}+d(x)u=f(x)-divF(x),\ x\in Q,$$

$$u|_{\partial Q}=u_0.$$

Для решения установлена ограниченность интеграла Дирихле с весом $r(x)$, т.е. интегрируемость по Q функции $r(x)\left|\nabla u(x)\right|^2$, где $r(x)$ – расстояние точки $x \in Q$ до границы ∂Q.