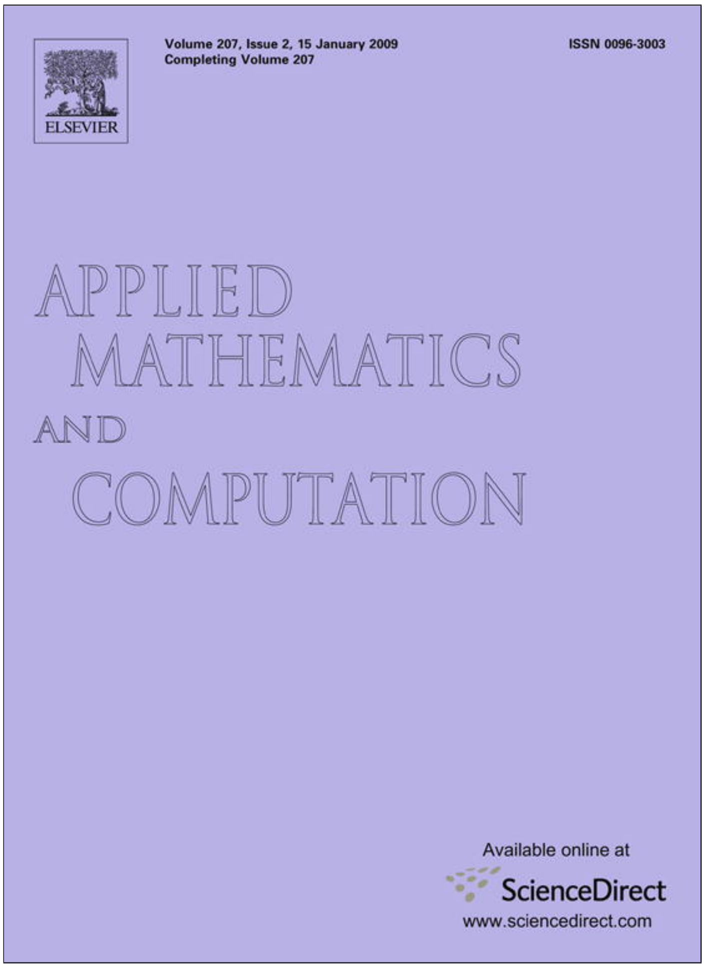
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Extended Cesàro operators between different Hardy spaces

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ABSTRACT

Let H^p denote the Hardy space of holomorphic functions on the unit ball $\mathbb{B}$. This note gives some sufficient and necessary conditions for the boundedness and compactness of the following extended Cesàro operators

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t} \quad \text{and} \quad L_g f(z) = \int_0^1 \Re f(tz) g(tz)\frac{dt}{t},$$

where $z \in \mathbb{B}$ and g is a fixed holomorphic map on $\mathbb{B}$, acting from the space H^p into the space H^q, for the case $p < q$. Our results extend and simplify some one-dimensional results.

1. Introduction and preliminaries

Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in the complex vector space $\mathbb{C}^n$, $S = \partial\mathbb{B} = \{z \in \mathbb{C}^n : |z| = 1\}$ its boundary, $d\sigma$ the normalized rotation invariant measure on S such that $\sigma(S) = 1$, dV is the normalized volume measure on $\mathbb{B}$ and $H(\mathbb{B})$ the class of all holomorphic functions on the unit ball. Let $z = (z_1, \dots, z_n)$ and $w = (w_1, \dots, w_n)$ be points in $\mathbb{C}^n$ and $\langle z, w\rangle = \sum_{k=1}^n z_k \bar{w}_k$. For $f \in H(\mathbb{B})$ with the Taylor expansion $f(z) = \sum_{|\beta|\geqslant 0} a_\beta z^\beta$, let

$$\Re f(z) = \sum_{|\beta|\geqslant 0} |\beta| a_\beta z^\beta$$

be the radial derivative of f, where $\beta = (\beta_1, \beta_2, \dots, \beta_n)$ is a multi-index and $z^\beta = z_1^{\beta_1} \cdots z_n^{\beta_n}$.

The α-Bloch space $\mathscr{B}^\alpha(\mathbb{B}) = \mathscr{B}^\alpha$, $\alpha > 0$, consists of all $f \in H(\mathbb{B})$ such that

$$\sup_{z\in\mathbb{B}} (1 - |z|^2)^\alpha |\Re f(z)| < \infty,$$

while the little α-Bloch space $\mathscr{B}_0^\alpha(\mathbb{B}) = \mathscr{B}_0^\alpha$, $\alpha > 0$, consists of all $f \in \mathscr{B}^\alpha$ such that

$$\lim_{|z|\to 1} (1 - |z|^2)^\alpha |\Re f(z)| = 0.$$

With the following norm:

$$\|f\|_{\mathscr{B}^\alpha} = |f(0)| + B_\alpha(f),$$

$\mathscr{B}^\alpha$ becomes a Banach space and $\mathscr{B}_0^\alpha$ is its closed subspace. For $\alpha = 1$, the spaces $\mathscr{B}^1$ and $\mathscr{B}_0^1$ become the Bloch and the little Bloch space (see, for example, [7,20,22,25,33,34] and the references therein).

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The Hardy space $H^p(\mathbb{B}) = H^p$, where $0 < p \leqslant \infty$, consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r) < \infty,$$

where

$$M_p(f,r) = \left(\int_S |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}$$

and

$$M_\infty(f,r) = \sup_{\zeta \in S} |f(r\zeta)|.$$

The weighted Bergman space $A^p_\alpha(\mathbb{B}) = A^p_\alpha$, $0 < p < \infty$, $\alpha \in (0,\infty)$, consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{A^p_\alpha} = \left(\int_{\mathbb{B}} (1-|z|^2)^{\alpha p - 1} |f(z)|^p dV(z)\right)^{1/p} < \infty.$$

Extended Cesàro operators with an analytic symbol g are defined as follows

$$T_g f(z) = \int_0^1 f(tz)\Re g(tz)\frac{dt}{t} \quad \text{and} \quad L_g f(z) = \int_0^1 \Re f(tz) g(tz)\frac{dt}{t}, \tag{1}$$

where $z \in \mathbb{B}$ and $f \in H(\mathbb{B})$.

The operator T_g was introduced by Hu in [9] and studied in [3,8–15,19,22,29,31], while L_g was introduced by Li and Stević in a private communication and studied in [3,11–15,19]. For closely related papers in the case of the unit polydisk see [3–6,23,24,30].

Here we study the boundedness and compactness of the operators T_g and L_g from H^p to H^q space. The case $p = q = 2$ was previously studied in [15]. We give some sufficient conditions for these operators to be bounded or compact, while in the case $p < q$, we show that these sufficient conditions are also necessary. Our main results partially extend the main results in paper [1], where the boundedness and compactness of the operator T_g between different Hardy spaces were investigated in the setting of the unit disk. We would like to point out that the results in [1] are based on some strongly one-dimensional results unlike the results in this paper. For some recent related results see also [16–18,28], as well as paper [32] concerning weighted composition operators between Hardy spaces on the unit ball $\mathbb{B}$.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $a \preceq b$ means that there is a positive constant C such that $a \leqslant Cb$. If both $a \preceq b$ and $b \preceq a$ hold, then one says that $a \asymp b$.

2. Auxiliary results

Several auxiliary results, which are used in the proofs of the main results, are quoted in this section.

Lemma 1. *For every $f, g \in H(\mathbb{B})$ it holds*

$$\Re[T_g(f)](z) = f(z)\Re g(z) \quad \textit{and} \quad \Re[L_g(f)](z) = \Re f(z) g(z).$$

A proof of the first identity can be found in [8]. The second identity is proved similarly and it was mentioned for the first time in [13].

Note that Lemma 1 is an analog of the following one-dimensional identities

$$\left(\int_0^z f(\zeta)g'(\zeta)d\zeta\right)' = f(z)g'(z), \quad \left(\int_0^z f'(\zeta)g(\zeta)d\zeta\right)' = f'(z)g(z).$$

By using the following inequality (see, e.g. [8])

$$(1-r)M_q(\Re f, r) \leqslant CM_q\left(f, \frac{1+r}{2}\right)$$

the next lemma easily follows:

Lemma 2. *There is a positive constant C independent of f such that*

$$|f(0)| + \sup_{0<r<1}(1-r)M_q(\Re f, r) \leqslant C \sup_{0<r<1} M_q(f,r). \tag{2}$$

The following result is proved in a standard way. See, for example, the proofs of the corresponding results in [13,23,24].

Lemma 3. *The operator $T_g(\text{or } L_g) : H^p \to H^q$ is compact if and only if $T_g(\text{or } L_g) : H^p \to H^q$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in H^p which converges to zero uniformly on compact subsets of $\mathbb{B}$ as $k \to \infty$, we have*

$$\|T_g f_k\|_{H^p} \to 0 \text{ as } k \to \infty (\text{or } \|L_g f_k\|_{H^q} \to 0 \text{ as } k \to \infty).$$

The following two inclusions go back to Hardy and Littlewood. Their proofs for the unit ball case can be found in [2, Theorems 3.7(ii) and 5.13].

Lemma 4. *Assume that $0 < p < q < \infty$ and $f \in H(\mathbb{B})$. Then*

(a)

$$H^p \subset A^q_{\frac{n}{p}-\frac{n}{q}},$$

moreover, there is a positive constant C such that for every $f \in H^p$,

$$\|f\|_{A^q_{n/p-n/q}} \leqslant C\|f\|_{H^p},$$

(b) *if further $f(0) = 0$, then*

$$\|f\|_{H^q} \leqslant C(p,q,n)\|\Re f\|_{A^p_\alpha}, \quad 0 < \alpha = 1 + \frac{n}{q} - \frac{n}{p}.$$

3. The boundedness and compactness of $T_g, L_g : H^p \to H^q$

In this section we consider the boundedness and compactness of the operators $T_g, L_g : H^p \to H^q$. The following results are main in this paper.

Theorem 1. *Assume that $0 < p < q < \infty$. Then $T_g : H^p \to H^q$ is bounded if and only if $g \in \mathscr{B}^{1+\frac{n}{q}-\frac{n}{p}}$. Moreover, if $T_g : H^p \to H^q$ is bounded then*

$$\|T_g\|_{H^p\to H^q} \asymp \sup_{z\in\mathbb{B}}(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(z)| =: M. \tag{3}$$

Proof. First assume that $T_g : H^p \to H^q$ is bounded and that $p, q \in (0,\infty)$ are arbitrary. Set

$$f_w(z) = \frac{(1-|w|^2)^a}{(1-\langle z,w\rangle)^{\frac{n}{p}+a}}, \quad w \in \mathbb{B}, \tag{4}$$

where $a > 0$. We have

$$f_w(w) = \frac{1}{(1-|w|^2)^{\frac{n}{p}}}, \quad \text{and} \quad |\Re f_w(w)| = \left(\frac{n}{p}+a\right)\frac{|w|^2}{(1-|w|^2)^{\frac{n}{p}+1}}. \tag{5}$$

By [21, Theorem 1.4.10], we know that

$$M_p(f_w, r) \leqslant C\frac{(1-|w|^2)^a}{(1-r|w|)^a} \leqslant C.$$

Therefore $f_w \in H^p$, and moreover $\sup_{w\in\mathbb{B}}\|f_w\|_{H^p} \leqslant C$.

Using the boundedness of $T_g : H^p \to H^q$, Lemmas 2, 1 and Theorem 7.2.5 in [21], we have

$$\begin{aligned}\infty > C\|T_g\|_{H^p\to H^q} &\geqslant \|f_w\|_{H^p}\|T_g\|_{H^p\to H^q} \geqslant \|T_g(f_w)\|_{H^q} \geqslant C\sup_{0<r<1}(1-r)M_q(\Re(T_g f_w), r) = C\sup_{0<r<1}(1-r)M_q(f_w\Re g, r)\\ &\geqslant C\left[(1-|w|^2)^{\frac{n}{q}}|\Re g(w)||f_w(w)|\right](1-|w|^2) = C(1-|w|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(w)|.\end{aligned} \tag{6}$$

From (6) it follows $g \in \mathscr{B}^{1+\frac{n}{q}-\frac{n}{p}}$, moreover

$$M \leqslant C\|T_g\|_{H^p\to H^q} \tag{7}$$

for some positive C.

Now assume that $g \in \mathscr{B}^{1+\frac{n}{q}-\frac{n}{p}}$ and $1 + \frac{n}{q} - \frac{n}{p} \geqslant 0$. Choosing $s(p < s < q)$, using the fact $T_g f(0) = 0$, Lemma 4 (b), Lemma 1, and finally the continuous inclusion $H^p \subset A^s_{n/p-n/s}$ from Lemma 4 (a), we get

$$\begin{aligned}\|T_g f\|_{H^q} &\leqslant C\|\Re(T_g f)\|_{A^s_{1+n/q-n/s}} = C\left(\int_{\mathbb{B}}(1-|z|^2)^{\left(1+\frac{n}{q}-\frac{n}{s}\right)s-1}|f(z)\Re g(z)|^s dV(z)\right)^{1/s} \leqslant C\|f\|_{A^s_{n/p-n/s}}\sup_{z\in\mathbb{B}}(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(z)|\\ &= CM\|f\|_{A^s_{n/p-n/s}} \leqslant CM\|f\|_{H^p}\end{aligned} \tag{8}$$

from which it follows that $\|T_g\|_{H^p\to H^q} \leqslant CM$. This along with (7) gives the asymptotic relation (3). □

Remark 1. Note that if $1+\frac{n}{q}-\frac{n}{p}<0$, then by the maximum modulus theorem it follows that $g\equiv const$.

The following open problem is challenging.

Open problem. Assume that $0<p<q<\infty$ and $T_g : H^p \to H^q$ is bounded. Find the exact value of the norm $\|T_g\|_{H^p\to H^q}$.

For some recent results in the topic see [26,27].

Theorem 2. *Assume that $0<p<q\leqslant\infty$. Then $L_g : H^p\to H^q$ is bounded, if and only if $g(z)\equiv 0$.*

Proof. First assume that $L_g : H^p\to H^q$ is bounded. Let $f_w, w\in\mathbb{B}$, be defined by (4). We know that $\sup_{w\in\mathbb{B}}\|f_w\|_{H^p}\leqslant C$.

By Lemmas 2,1 and Theorem 7.2.5 in [21], we have

$$\infty > \|L_g(f_w)\|_{H^q} \geqslant C\sup_{0<r<1}(1-r)M_q(\Re(L_g f_w),r) \geqslant C\Big[(1-|w|^2)^{\frac{n}{q}}|g(w)||\Re f_w(w)|\Big](1-|w|^2)$$
$$\geqslant C|w|^2(1-|w|^2)^{\frac{n}{q}-\frac{n}{p}}|g(w)|. \tag{9}$$

From (9) it follows that

$$C|w|^2|g(w)| \leqslant (1-|w|^2)^{\frac{n}{p}-\frac{n}{q}}\|L_g(f_w)\|_{H^q}. \tag{10}$$

By letting $|w|\to 1$ in (10), noticing that $\frac{n}{p}-\frac{n}{q}>0$ and employing the maximum modulus theorem we obtain that $g(z)=0$, for each $z\in\mathbb{B}$, as claimed.

The reverse statement is trivial. □

Theorem 3. *Assume that $0<p<q<\infty$. Then $T_g : H^p\to H^q$ is compact if and only if $g\in\mathscr{B}_0^{1+\frac{n}{q}-\frac{n}{p}}$.*

Proof. Assume that $T_g : H^p\to H^q$ is compact. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $|z_k|\to 1$ as $k\to\infty$, and $h_k(z)=f_{z_k}(z)$, $k\in\mathbb{N}$ (where f_w is defined in (4)). Then by the proof of Theorem 1 we know that $\sup_{k\in\mathbb{N}}\|h_k\|_{H^p}\leqslant C$ and h_k converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k\to\infty$. Since T_g is compact, by using Lemma 3 it follows that $\lim_{k\to\infty}\|T_g h_k\|_{H^q}=0$. From this and since in view of (6) we have that

$$\|T_g h_k\|_{H^q} \geqslant C(1-|z_k|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(z_k)|,$$

we obtain $g\in\mathscr{B}_0^{1+\frac{n}{q}-\frac{n}{p}}$.

Now assume $g\in\mathscr{B}_0^{1+\frac{n}{q}-\frac{n}{p}}$ and $1+\frac{n}{q}-\frac{n}{p}>0$. Then for every $\varepsilon>0$ there is an $\delta\in(0,1)$ such that

$$(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(z)|<\varepsilon, \tag{11}$$

when $\delta\leqslant|z|<1$.

Assume that a sequence $(f_k)_{k\in\mathbb{N}}$ in H^p is such that $\sup_{k\in\mathbb{N}}\|f_k\|_{H^p}\leqslant L$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{B}$ as $k\to\infty$.

By using Lemmas 4, 1 and (11), and for any fixed $s\in(p,q)$, we obtain

$$\|T_g f_k\|_{H^q} \leqslant C\|\Re(T_g f_k)\|_{A^s_{1+n/q-n/s}} = C\left[\left(\int_{|z|<\delta}+\int_{\delta<|z|<1}\right)(1-|z|^2)^{\left(1+\frac{n}{q}-\frac{n}{s}\right)s-1}|f_k(z)\Re g(z)|^s dV(z)\right]^{1/s}$$
$$\leqslant C\sup_{|z|<\delta}|f_k(z)|\sup_{|z|<\delta}(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(z)| + +C\|f_k\|_{A^s_{n/p-n/s}}\sup_{\delta<|z|<1}(1-|z|^2)^{1+\frac{n}{q}-\frac{n}{p}}|\Re g(z)|$$
$$\leqslant C\sup_{|z|<\delta}|f_k(z)|+C\varepsilon\|f_k\|_{H^p}, \leqslant C\sup_{|z|<\delta}|f_k(z)|+CL\varepsilon. \tag{12}$$

Letting $k\to\infty$ in (12), using the fact that ε is an arbitrary positive number and by employing Lemma 3 it follows that $T_g : H^p\to H^q$ is compact. □

Remark 2. Note that if $1+\frac{n}{q}-\frac{n}{p}\leqslant 0$, then by the maximum modulus theorem it follows that $g\equiv const$.

The following theorem is a direct consequence of Theorem 2.

Theorem 4. *Assume that $0<p<q\leqslant\infty$. Then $L_g : H^p\to H^q$ is compact, if and only if $g(z)\equiv 0$.*

For the readers interested in this research area we leave the next research project.

Research project. Let $p,q>0$. Find a necessary and sufficient condition for the operator $T_g : H^p\to H^q$, (corresp. $L_g : H^p\to H^q$), $p\geqslant q$, to be bounded (or compact). Recall that the case $p=q=2$ was solved in [15].

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