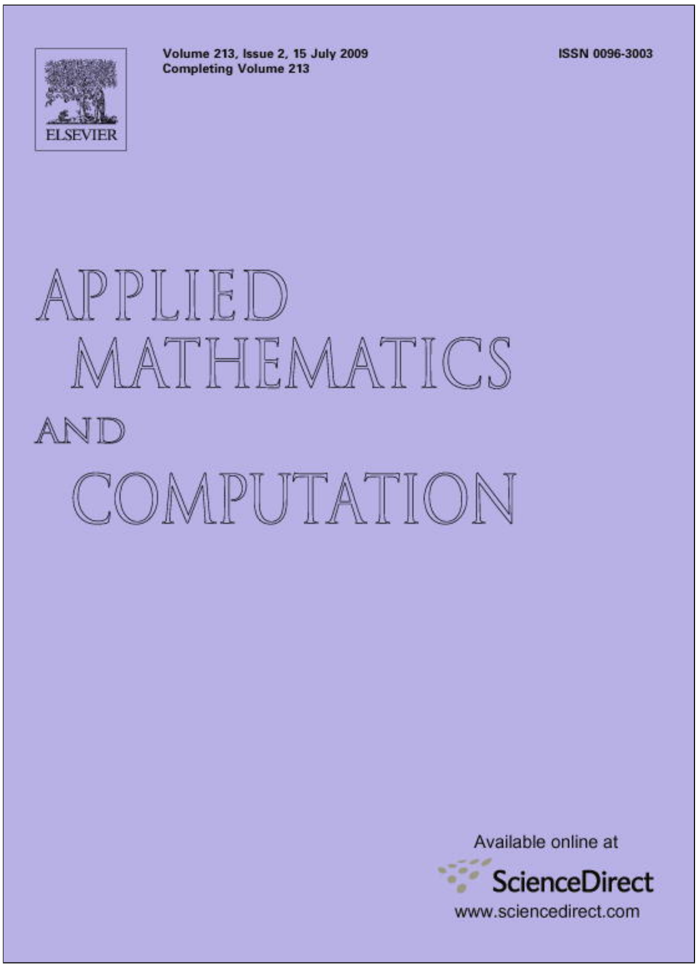
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The generalized Libera transform is bounded on the Besov mixed-norm, BMOA and VMOA spaces on the unit disc

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ABSTRACT

The main results of this note prove that the generalized Libera operator is bounded on the Besov mixed-norm space $B_\alpha^{p,q}(\mathbb{D})$ as well as on the spaces *BMOA* and *VMOA* on the unit disk. The compactness of the operator on $B_\alpha^{p,q}(\mathbb{D})$ is also studied.

1. Introduction and preliminaries

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}, \partial\mathbb{D}$ its boundary, $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$ and $dm(\cdot) = \frac{1}{\pi} r\,dr\,d\theta$ the normalized Lebesgue area measure on $\mathbb{D}$. For each complex γ with $\Re\gamma > -1$ and for each nonnegative integer k, let A_k^γ be defined as the kth coefficient in the expression

$$(1-x)^{-(\gamma+1)} = \sum_{k=0}^{\infty} A_k^\gamma x^k,$$

so that

$$A_k^\gamma = \frac{\Gamma(\gamma+k+1)}{\Gamma(\gamma+1)\Gamma(k+1)}.$$

Let $z_0 \in \mathbb{D}$ be fixed, then the following operator

$$\Lambda_{z_0}(f)(z) = \frac{1}{z-z_0}\int_{z_0}^{z} f(t)dt, \quad z \in \mathbb{D}, \tag{1}$$

where $f \in H(\mathbb{D})$, is one of the most natural averaging operators on $H(\mathbb{D})$, and for $z_0 = 0$ it is called the Libera transform [28]. Restricting the domain of the operator Λ_{z_0}, we can extend the definition of Λ_{z_0} to values of z_0 on $\partial\mathbb{D}$.

For some previous results in this area, see [2,10,30,33,41] and the references therein. The transform can be considered as a formal adjoint of Cesàro operator on $H^2(\mathbb{D})$, (see, for example [31]). Recent results on related integral-type operators can be found, for example, in [1,4,6–9,14–27,32,34–40,42,43,45], see also the references therein.

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Operator (1) can be generalized as follows. For $z_0 \in \overline{\mathbb{D}}$ fixed, $\gamma \in \mathbb{C}$ such that $\Re\gamma > -1$, and $f \in H(\mathbb{D})$ we define the linear operator $\Lambda_{z_0}^{\gamma}(f)$ by

$$\Lambda_{z_0}^{\gamma}(f)(z) = \sum_{m=0}^{\infty}\left(\sum_{k=m}^{\infty}\frac{A_{k-m}^{\gamma} z_0^{k-m}}{A_k^{\gamma+1}} a_k\right) z^m, \tag{2}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k, z \in \mathbb{D}$. Note that $\Lambda_{z_0}^{\gamma}$ is only formally defined and

$$\Lambda_{z_0}^{\gamma}(f)(z) = \frac{\gamma+1}{(z-z_0)^{\gamma+1}}\int_{z_0}^{z} f(\zeta)(z-\zeta)^{\gamma} d\zeta,$$

or, taking as a path the segment joining z_0 and z

$$\Lambda_{z_0}^{\gamma}(f)(z) = (\gamma+1)\int_0^1 f(\phi_t(z))(1-t)^{\gamma} dt, \tag{3}$$

where $\phi_t(z) = (1-t)z_0 + tz$. We call operator (3) *generalized Libera operator.*

Here, we investigate the generalized Libera operator on the Besov mixed-norm space (see, e.g. [44])

$$B_{\alpha}^{p,q}(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) \middle| \, \|f\|_{B_{\alpha}^{p,q}}^q = \int_0^1 M_p^q(f^{(k)}, r)(1-r)^{q(k-\alpha)-1} dr < \infty \right\},$$

where $p, q \in (0,\infty), k$ is an integer, $0 < \alpha < k$. The space $B_{\alpha}^{p,q}$ does not depend on k, and for different $k (k > \alpha)$ equivalent "norms" appear. It is easy to see that for $p = q > 1$ and $\alpha = 1/p$ this space is equivalent to the classical analytic Besov space $B^p(\mathbb{D})$.

As usual $M_p(f,r)$ denotes the pth integral mean of the function f, that is,

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}, \quad r \in [0,1).$$

The mixed-norm spaces $L_{\alpha}^{p,q}$ and $\mathscr{A}_{\alpha}^{p,q}, p, q \in (0,\infty), \alpha > -1$, are defined as follows:

$$L_{\alpha}^{p,q}(\mathbb{D}) = \left\{ f \text{ measurable on } \mathbb{D} \,\middle|\, \|f\|_{L_{\alpha}^{p,q}}^q := \int_0^1 M_p^q(f,r)(1-r)^{\alpha} dr < \infty \right\},$$

and $\mathscr{A}_{\alpha}^{p,q} = H(\mathbb{D}) \cap L_{\alpha}^{p,q}$. For $p = q$ the spaces $\mathscr{A}_{\alpha}^{p,p}$ coincide with the well-known weighted Bergman spaces, see [12] for the general theory of Bergman spaces.

This paper is organized as follows. In Section 2 we prove an auxiliary result which is used in the proof of the boundedness of operator (3) on the Besov mixed-norm space in Section 3. In Section 3 we also prove the boundedness of the operator on the BMOA as well as VMOA space. In Section 4 compactness of the operator (3) on the Besov mixed-norm space is proved under some conditions.

Throughout the paper, constants are denoted by C, they are positive and may differ from one occurence to the other. The notation $a \asymp b$ means that there is a positive constant C such that $\frac{1}{C}|a| \leqslant |b| \leqslant C|a|$.

2. Auxiliary results

The following lemma, regarding the boundedness of the composition operator on the mixed-norm space, was proved in [41]. We sketch its proof here for the completeness and for benefit of the reader.

Lemma 1. *Let $p, q \in (0,\infty), \alpha > -1, \varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant analytic function. Then the composition operator $C_{\varphi}(f) = f \circ \varphi$ on $\mathscr{A}_{\alpha}^{p,q}(\mathbb{D})$ satisfies the following inequality:*

$$\|C_{\varphi}(f)\|_{\mathscr{A}_{\alpha}^{p,q}}^q \leqslant 3^{\frac{q}{p}}\left(\frac{\|\varphi\|_{\infty} + |\varphi(0)|}{\|\varphi\|_{\infty} - |\varphi(0)|}\right)^{\frac{q}{p}+\alpha+1} \|f\|_{\mathscr{A}_{\alpha}^{p,q}}^q.$$

Proof. Let $a = |\varphi(0)|$ and $b = \|\varphi\|_{\infty} = \sup_{z\in\mathbb{D}}|\varphi(z)|$. By a well-known consequence of the Schwarz's Lemma (see, for example [11, p. 3]), we have

$$|\varphi(z)| \leqslant \frac{b(a+b|z|)}{b+a|z|} \quad \text{for } z \in \mathbb{D}. \tag{4}$$

Fix $r \in (0,1)$. For $R = R(r) = \frac{(b-a)r+2a}{a+b}$, from (4) it follows that $\varphi(\{|z| \leqslant r\}) \subset bR$. For $f \in \mathscr{A}_{\alpha}^{p,q}(\mathbb{D})$ let $h(z)$ be the harmonic extension of $|f(bRe^{i\theta})|^p$ on $|z| \leqslant bR$. Since $|f(z)|^p$ is subharmonic and $h(\varphi(z))$ is harmonic, we have

$$\int_0^{2\pi} |f(\varphi(re^{i\theta}))|^p d\theta \leqslant \int_0^{2\pi} h(\varphi(re^{i\theta})) d\theta = 2\pi h(\varphi(0)) \leqslant \frac{bR+a}{bR-a}\int_0^{2\pi} |f(bRe^{i\theta})|^p d\theta. \leqslant \frac{bR+a}{bR-a}\int_0^{2\pi} |f(Re^{i\theta})|^p d\theta. \tag{5}$$

Raising (5) to the q/pth power, multiplying obtained inequality by $(1-r)^\alpha\,dr$, integrating from 0 to 1, and then using the change $s = R(r)$ we obtain

$$\begin{aligned}\int_0^1 M_p^q(f\circ\varphi, r)(1-r)^\alpha\,dr &\leqslant \int_0^1 \left(\frac{bR+a}{bR-a}\right)^{q/p}\left(\int_0^{2\pi}|f(Re^{i\theta})|^p\,d\theta\right)^{q/p}(1-r)^\alpha\,dr\\ &= \int_{\frac{2a}{a+b}}^1 \left(\frac{bs+a}{bs-a}\right)^{q/p} M_p^q(f,s)\frac{(b+a)}{(b-a)}\left(\frac{(b+a)(1-s)}{b-a}\right)^\alpha ds\\ &\leqslant 3^{q/p}\left(\frac{b+a}{b-a}\right)^{q/p+\alpha+1}\int_{\frac{2a}{a+b}}^1 M_p^q(f,s)(1-s)^\alpha ds, \end{aligned} \tag{6}$$

from which the lemma follows. □

Remark 1. We would like to point out that it is not possible to take the change of variable

$$R(r) = \frac{a+br}{b+ar},$$

which seems more natural. Namely, for such a chosen R the first integral in (6), in this case, goes from a/b, but the integrand has the singularity at the point.

We also need the boundedness of the following well-known Bergman operator

$$(T_\beta f)(z) = (\beta+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{|1-\bar{w}z|^{\beta+2}}|f(w)|\,dm(w), \qquad z\in\mathbb{D}. \tag{7}$$

Lemma 2. *Let $\alpha > -1$ and $1 \leqslant p < \infty, 0 < q < 1$ or $0 < p < 1, 0 < q < \infty$. Then for any $\beta > -1 + \frac{\alpha+1}{q} + \max\left\{0, \frac{1}{p} - 1\right\}$, T_β is a bounded operator from $\mathscr{A}_\alpha^{p,q}$ to $L_\alpha^{p,q}$.*

For a proof, see [5, Lemma 4.1].

The following lemma can be found, for example, in [13, p. 128].

Lemma 3. *Let $p > 0$, f be a function holomorphic in the open disc $D(a,r)$ and continuous in $\overline{D(a,r)}$. Then for any circle Γ contained in $D(a,r)$*

$$\int_\Gamma |f(z)|^p|dz| \leqslant 2\int_{\partial D(a,r)} |f(z)|^p|dz|.$$

3. Boundedness of the generalized Libera transform on $B_\alpha^{p,q}$, *BMOA* and *VMOA*

In this section we prove the main results of this paper. Let

$$d\mu_\gamma(t) = (\gamma+1)(1-t)^\gamma dt \quad \text{and} \quad d\mu_{k,\alpha,q}(r) = (1-r)^{q(k-\alpha)-1}\,dr.$$

Theorem 1

(i) *For $z_0 \in \overline{\mathbb{D}}$ fixed, the generalized Libera transform (3) is bounded on the Besov mixed-norm space $B_\alpha^{p,q}$ if $p, q \in [1,\infty), \alpha > 0$.*
(ii) *For $z_0 \in \mathbb{D}$ fixed, the generalized Libera transform (3) is bounded on the Besov mixed-norm space $B_\alpha^{p,q}$ if $p, q \in (0,\infty), \alpha > 0$.*

Proof. (i) We may assume that γ is a real number. Applying Minkowski's inequality twice, Lemma 1, with $\varphi = \phi_t$ and the fact that $\|\phi_t\|_\infty = (1-t)|z_0| + t$, we obtain

$$\begin{aligned}\|\Lambda_{z_0}^\gamma(f)\|_{B_\alpha^{p,q}} &= \left(\int_0^1 M_p^q\left(\int_0^1 (f\circ\phi_t)^{(k)}d\mu_\gamma(t), r\right)d\mu_{k,\alpha,q}(r)\right)^{1/q} \leqslant \left(\int_0^1\left(\int_0^1 M_p(f^{(k)}\circ\phi_t\cdot t^k, r)\,d\mu_\gamma(t)\right)^q d\mu_{k,\alpha,q}(r)\right)^{1/q}\\ &\leqslant \int_0^1\left(\int_0^1 M_p^q(f^{(k)}\circ\phi_t\cdot t^k, r)\,d\mu_{k,\alpha,q}(r)\right)^{1/q} d\mu_\gamma(t) = \int_0^1 \|f^{(k)}\circ\phi_t\|_{\mathscr{A}^{p,q}_{q(k-\alpha)-1}} t^k\,d\mu_\gamma(t)\\ &\leqslant 3^{1/p}\|f^{(k)}\|_{\mathscr{A}^{p,q}_{q(k-\alpha)-1}}\int_0^1\left(\frac{\|\phi_t\|_\infty + |\phi_t(0)|}{\|\phi_t\|_\infty - |\phi_t(0)|}\right)^{k-\alpha+1/p} t^k\,d\mu_\gamma(t) \leqslant 3^{1/p}2^{k-\alpha+1/p}\|f\|_{B_\alpha^{p,q}}\int_0^1 t^{\alpha-1/p}\,d\mu_\gamma(t)\\ &= (\gamma+1)3^{1/p}2^{k-\alpha+1/p}B(\alpha+1-1/p, \gamma+1)\|f\|_{B_\alpha^{p,q}}, \end{aligned}$$

where $B(\cdot,\cdot)$ is the Euler beta function, from which the result follows.

(ii) Let $z_0 \in \mathbb{D}, p, q \in (0, \infty), \alpha > 0$. In view of part (i) we may assume that $1 \leqslant p < \infty, 0 < q < 1$ or $0 < p < 1, 0 < q < \infty$. Let f be an arbitrary function of $B^{p,q}_\alpha$. This is equivalent to $f^{(k)} \in \mathscr{A}^{p,q}_{q(k-\alpha)-1}$ for some $k > \alpha$. By using the continuous inclusion

$$\mathscr{A}^{p,q}_\alpha \subset \mathscr{A}^{1,1}_\delta, \quad \delta > \frac{\alpha+1}{q} + \frac{1}{p} - 1, \tag{8}$$

(see [3, Theorem 1(v)]), we conclude that $f^{(k)}$ is in the Bergman space $\mathscr{A}^{1,1}_\beta$ for sufficiently large $\beta, \beta > k - \alpha + 1/p - 1$. Consequently, $f^{(k)}$ admits the integral representation (see, for example [12, p. 6])

$$f^{(k)}(z) = (\beta+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-\bar{w}z)^{\beta+2}} f^{(k)}(w)\,dm(w), \quad z \in \mathbb{D},$$

and so

$$\frac{d^k}{dz^k}(\Lambda^\gamma_{z_0} f)(z) = \int_0^1 f^{(k)}(\phi_t(z)) t^k\,d\mu_\gamma(t) = (\beta+1)\int_0^1 \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-\bar{w}\phi_t(z))^{\beta+2}} f^{(k)}(w)\,dm(w)\right) t^k\,d\mu_\gamma(t). \tag{9}$$

In order to estimate the integral (9) we need an estimate from below for the denominator in the integrand of (9), namely,

$$|1 - \bar{w}\phi_t(z)| \geqslant \frac{1-|z_0|}{2}|1 - \bar{w}z|. \tag{10}$$

Inequality (10) can be proved by repeated application of the triangle inequality:

$$|1 - \bar{w}\phi_t(z)| \geqslant 1 - |\phi_t(z)| \geqslant (1-t)(1-|z_0|) \geqslant \frac{1-|z_0|}{1+|z_0|}(1-t)|z-z_0|. \tag{11}$$

It follows from (11) that

$$\begin{aligned} |1 - \bar{w}\phi_t(z)| &= |1 - \bar{w}z + \bar{w}z - \bar{w}\phi_t(z)| \geqslant |1-\bar{w}z| - |w||z-\phi_t(z)| = |1-\bar{w}z| - |w|(1-t)|z-z_0| \\ &\geqslant |1-\bar{w}z| - \frac{1+|z_0|}{1-|z_0|}|1-\bar{w}\phi_t(z)|. \end{aligned}$$

From this, inequality (10) immediately follows. Therefore, an application of (10) leads (9) to the Bergman operator (7)

$$\begin{aligned} \left|\frac{d^k}{dz^k}(\Lambda^\gamma_{z_0} f)(z)\right| &\leqslant (\beta+1)\int_0^1 \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{|1-\bar{w}\phi_t(z)|^{\beta+2}}|f^{(k)}(w)|\,dm(w)\right) t^k\,d\mu_\gamma(t) \\ &\leqslant \frac{(\beta+1)(\gamma+1)2^{\beta+2}B(k+1,\gamma+1)}{(1-|z_0|)^{\beta+2}} \int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{|1-\bar{w}z|^{\beta+2}}|f^{(k)}(w)|\,dm(w) = C(\beta,\gamma,k,z_0)\,T_\beta(f^{(k)})(z). \end{aligned}$$

For β sufficiently large, Lemma 2 yields

$$\|\Lambda^\gamma_{z_0}(f)\|_{B^{p,q}_\alpha} = \left\|\frac{d^k}{dz^k}\Lambda^\gamma_{z_0}(f)\right\|_{\mathscr{A}^{p,q}_{q(k-\alpha)-1}} \leqslant C(\beta,\gamma,k,z_0)\|T_\beta(f^{(k)})\|_{L^{p,q}_{q(k-\alpha)-1}} \leqslant C\|f^{(k)}\|_{\mathscr{A}^{p,q}_{q(k-\alpha)-1}} = C\|f\|_{B^{p,q}_\alpha},$$

where the last constant C depends only on $p, q, \alpha, \beta, \gamma, k, z_0$. This completes the proof of Theorem 1. □

Remark 2. Theorem 1(i) fails in the cases

$$0 < p < \frac{1}{1+\alpha}, \quad 0 < q < \infty \quad \text{or} \quad p = \frac{1}{1+\alpha}, \quad 1 < q < \infty \ (0 < \alpha < 1). \tag{12}$$

This fact can be proved by the example

$$f_{z_0}(z) = (z_0 - z)^{-1}\left(\ln\frac{e}{z_0 - z}\right)^{-1}, \quad z \in \mathbb{D},$$

where $z_0 \in \partial\mathbb{D}$. Indeed, it is easily checked that

$$f'_{z_0}(z) \asymp (z_0 - z)^{-2}\left(\ln\frac{e}{z_0 - z}\right)^{-1}, \quad z \in \mathbb{D}.$$

Now we show that $f_{z_0}(z) \in B^{p,q}_\alpha$ if and only if (12) holds.

We may assume that $z_0 = 1$. For the expression $|1 - re^{i\theta}| = \sqrt{(1-r)^2 + 4r\sin^2\frac{\theta}{2}}$ we have the simple estimate

$$\frac{1}{\sqrt{2}}\left(1 - r + 2\sqrt{r}\frac{|\theta|}{\pi}\right) \leqslant |1 - re^{i\theta}| \leqslant 1 - r + |\theta|, \quad z = re^{i\theta} \in \mathbb{D},$$

in particular,

$$\frac{1}{\pi}(1 - r + |\theta|) \leqslant |1 - re^{i\theta}| \leqslant 1 - r + |\theta|, \quad \frac{1}{2} \leqslant r < 1. \tag{13}$$

Define the ring sector $E := \{z = re^{i\theta} \in \mathbb{D} : \frac{9}{10} < r < 1, |\theta| < \frac{1}{2}\}$, so that $|1 - z| < \frac{1}{2}$ $(z \in E)$, and the following inequalities are valid:

$$\left|\ln\frac{1}{1-z}\right| \leqslant \ln\frac{1}{|1-z|} + \frac{\pi}{2} \leqslant 5\ln\frac{1}{|1-z|}, \quad z \in E, \tag{14}$$

$$\left|\ln\frac{1}{1-z}\right| \geqslant \ln\frac{1}{|1-z|} \geqslant \ln\frac{1}{1-r+|\theta|} \geqslant \ln\frac{5}{3} > \frac{1}{2}, \quad z \in E. \tag{15}$$

Note that the integral $\|f_{z_0}\|_{B^{p,q}_\alpha} = \|f'_{z_0}\|_{\mathcal{A}^{p,q}_{q(1-\alpha)-1}}$ is equiconvergent to the integral

$$I := \int_{9/10}^{1}\left[\int_{-1/2}^{1/2}\frac{d\theta}{|1 - re^{i\theta}|^{2p}\left|\ln\frac{e}{1-re^{i\theta}}\right|^p}\right]^{q/p}(1-r)^{q(1-\alpha)-1}\,dr.$$

Now we estimate the integral

$$J_p(r) := \int_{-1/2}^{1/2}\frac{d\theta}{|1 - re^{i\theta}|^{2p}\left|\ln\frac{e}{1-re^{i\theta}}\right|^p}, \quad \frac{9}{10} < r < 1.$$

To this end, we distinguish three cases: $p > \frac{1}{2}, p < \frac{1}{2}, p = \frac{1}{2}$.

Case $p > 1/2$. Using the inequalities (13)–(15), and also the inequalities $0 < \ln\frac{5}{3} < \ln\frac{1}{3/2-r} < \ln 2$ $\left(\frac{9}{10} < r < 1\right)$ we have

$$\begin{aligned} J_p(r) &\asymp \int_{-1/2}^{1/2}\frac{d\theta}{|1 - re^{i\theta}|^{2p}\left(\ln\frac{e}{|1-re^{i\theta}|}\right)^p} \asymp \int_0^{1/2}\frac{d\theta}{(1-r+\theta)^{2p}\left(\ln\frac{1}{1-r+\theta}\right)^p} = \int_{1-r}^{3/2-r}\frac{dx}{x^{2p}\left(\ln\frac{1}{x}\right)^p} \\ &= \int_{\ln\frac{1}{3/2-r}}^{\ln\frac{1}{1-r}}\frac{e^{(2p-1)t}}{t^p}\,dt \asymp \int_1^{\ln\frac{1}{1-r}}\frac{e^{(2p-1)t}}{t^p}\,dt, \end{aligned} \tag{16}$$

for all $r \in \left(\frac{9}{10}, 1\right)$. By the l'Hôpital rule it can be shown that

$$\int_1^x \frac{e^{(2p-1)t}}{t^p}\,dt \sim \frac{e^{(2p-1)x}}{(2p-1)x^p}, \quad \text{as } x \to +\infty.$$

Therefore, for all r sufficiently close to 1

$$J_p(r) \asymp C_p\frac{e^{(2p-1)\ln\frac{1}{1-r}}}{\left(\ln\frac{1}{1-r}\right)^p} = C_p\frac{1}{(1-r)^{2p-1}\left(\ln\frac{1}{1-r}\right)^p}.$$

Thus,

$$I \asymp \int_{9/10}^{1}\frac{dr}{(1-r)^{q(1+\alpha-1/p)+1}\left(\ln\frac{1}{1-r}\right)^q} = \int_0^{1/10}\frac{dx}{x^{q(1+\alpha-1/p)+1}\left(\ln\frac{1}{x}\right)^q}.$$

The last integral converges if and only if (12) holds.

Case $p < 1/2$ immediately follows from (16), $J_p(r) \asymp 1$, and the integral I converges.

Case $p = 1/2$. Using the inequalities (13)–(15), and also the inequalities

$$0 < \ln\frac{5}{3} < \ln\frac{1}{3/2-r} < \ln 2 < \frac{1}{2}\ln\frac{1}{1-r}, \quad \frac{9}{10} < r < 1,$$

we deduce that

$$J_{1/2}(r) \asymp \int_0^{1/2}\frac{d\theta}{(1-r+\theta)\left(\ln\frac{1}{1-r+\theta}\right)^{1/2}} = 2\left[\left(\ln\frac{1}{1-r}\right)^{1/2} - \left(\ln\frac{1}{3/2-r}\right)^{1/2}\right] \asymp \left(\ln\frac{1}{1-r}\right)^{1/2},$$

for all $r \in \left(\frac{9}{10}, 1\right)$. Thus, the integral I converges if and only if (12) holds.

On the other hand, $(\Lambda^\gamma_{z_0}f_{z_0})(z)$ makes no sense at any point $z \in \mathbb{D}$ because

$$(\Lambda^\gamma_{z_0}f_{z_0})(z) = \frac{\gamma+1}{z_0 - z}\int_0^1\frac{(1-t)^\gamma dt}{t\ln\frac{e}{t(z_0-z)}} = \infty.$$

Those cases when $z_0 \in \partial\mathbb{D}$ and $\frac{1}{1+\alpha} < p < 1$ remain open.

The space *BMOA* of functions $f \in H(\mathbb{D})$ can be defined by the seminorm ([11])

$$\|f\|_{BMOA} := \sup_{\zeta\in\mathbb{D}} \left(\int_{-\pi}^{\pi} |f(e^{i\theta}) - f(\zeta)|^2 P_\zeta(\theta) \frac{d\theta}{2\pi} \right)^{1/2},$$

where

$$P_\zeta(\theta) = \frac{1 - |\zeta|^2}{|1 - e^{-i\theta}\zeta|^2}$$

is the Poisson kernel. The space *VMOA* consists of the closure of polynomials in *BMOA*, or equivalently of those functions in *BMOA* for which

$$\int_{-\pi}^{\pi} |f(e^{i\theta}) - f(\zeta)|^2 P_\zeta(\theta) \frac{d\theta}{2\pi} = o(1) \quad \text{as } \zeta \to \partial\mathbb{D}.$$

Theorem 2. *For $z_0 \in \overline{\mathbb{D}}$, the generalized Libera transform preserves the spaces BMOA and VMOA.*

Proof. Assuming that $f \in BMOA$ and γ is a real number, we estimate

$$\begin{aligned}
\|\Lambda_{z_0}^{\gamma}(f)\|_{BMOA}^2 &= \sup_{\zeta\in\mathbb{D}} \int_{-\pi}^{\pi} |(\Lambda_{z_0}^{\gamma} f)(e^{i\theta}) - (\Lambda_{z_0}^{\gamma} f)(\zeta)|^2 P_\zeta(\theta) \frac{d\theta}{2\pi} = \sup_{\zeta\in\mathbb{D}} \int_{-\pi}^{\pi} \left| \int_0^1 \left((f\circ\phi_t)(e^{i\theta}) - (f\circ\phi_t)(\zeta) \right) d\mu_\gamma(t) \right|^2 P_\zeta(\theta) \frac{d\theta}{2\pi} \\
&\leqslant \sup_{\zeta\in\mathbb{D}} \int_{-\pi}^{\pi} \int_0^1 \left| (f\circ\phi_t)(e^{i\theta}) - (f\circ\phi_t)(\zeta) \right|^2 d\mu_\gamma(t) P_\zeta(\theta) \frac{d\theta}{2\pi} \\
&\leqslant \int_0^1 \left[\sup_{\zeta\in\mathbb{D}} \int_{-\pi}^{\pi} \left| (f\circ\phi_t)(e^{i\theta}) - (f\circ\phi_t)(\zeta) \right|^2 P_\zeta(\theta) \frac{d\theta}{2\pi} \right] d\mu_\gamma(t) = \int_0^1 \|f\circ\phi_t\|_{BMOA}^2 d\mu_\gamma(t).
\end{aligned}$$

On the other hand, for any $\phi = \phi_t$, the following inequality

$$\|f\circ\phi\|_{BMOA} \leqslant \|f\|_{BMOA}$$

was proved in [10]. Hence

$$\|\Lambda_{z_0}^{\gamma}(f)\|_{BMOA}^2 \leqslant \|f\|_{BMOA}^2 \int_0^1 d\mu_\gamma(t) = \|f\|_{BMOA}^2. \tag{17}$$

Assuming now $f \in VMOA$ and choosing a sequence of polynomials Q_n such that $\|f - Q_n\|_{BMOA} \to 0$ as $n \to \infty$. Using the estimate (17) we conclude that

$$\|\Lambda_{z_0}^{\gamma}(f - Q_n)\|_{BMOA} \leqslant \|f - Q_n\|_{BMOA} \to 0 \qquad \text{as } n \to \infty.$$

Since $\Lambda_{z_0}^{\gamma}(Q_n)$ are also polynomials we deduce that $\Lambda_{z_0}^{\gamma}(f) \in VMOA$. □

Remark 3. Note that from the proof of Theorem 2 it follows that

$$\|\Lambda_{z_0}^{\gamma}\|_{BMOA\to BMOA} \leqslant 1.$$

Remark 4. Theorem 2 for the operator (1) was proved in [10].

4. Compactness of the generalized Libera transform on $B_\alpha^{p,q}$

In this section we find some sufficient conditions for the generalized Libera transform (3) to be compact on the Besov mixed-norm space $B_\alpha^{p,q}$. Compactness of the operator (3) on $\mathscr{A}_\alpha^{p,q}$ is studied in [41].

Theorem 3. *For $z_0 \in \mathbb{D}$, the generalized Libera transform (3) is compact on the Besov mixed-norm space $B_\alpha^{p,q}$ if $1 \leqslant p < \infty, 0 < q < \infty, \alpha > 0$.*

Proof. Similarly to Lemmas 4 and 5 of [41] we can show that the operator $\Lambda_{z_0}^{\gamma} : B_\alpha^{p,q} \to B_\alpha^{p,q}$ is compact if and only if for every bounded sequence $(f_m)_{m\in\mathbb{N}}$ in $B_\alpha^{p,q}$ which converges to zero uniformly on compacts of $\mathbb{D}$ as $m \to \infty$, we have $\lim_{m\to\infty} \|\Lambda_{z_0}^{\gamma}(f_m)\|_{B_\alpha^{p,q}} = 0$.

For any $\varepsilon > 0$ choose $\delta \in (0,1)$ close to 1 such that $\int_\delta^1 t^k\, d\mu_\gamma(t) < \varepsilon$ and $|z_0| \leqslant \delta$. Assuming that $\sup_{m\in\mathbb{N}} \|f_m\|_{B^{p,q}_\alpha} \leqslant K$ and $f_m \to 0$ uniformly on compacts of $\mathbb{D}$ as $m \to \infty$, by Weierstrass theorem on uniform convergence [29, Theorem 10.27], we conclude that the same is true for the derivatives of f_m, that is, $f_m^{(k)} \to 0$ uniformly on compacts of $\mathbb{D}$ as $m \to \infty$.

For $t \in [0,\delta]$, we have

$$|\phi_t(z)| \leqslant (1-t)|z_0| + t = |z_0| + t(1-|z_0|) \leqslant |z_0| + \delta(1-|z_0|) =: r_0 < 1.$$

Consequently, there exists a positive integer m_0 such that for all $m > m_0$

$$\sup_{z\in\mathbb{D},\, t\in[0,\delta]} \left|(f_m^{(k)} \circ \phi_t)(z)\right| \leqslant \sup_{|z|\leqslant r_0} \left|f_m^{(k)}(z)\right| < \varepsilon. \tag{18}$$

Furthermore, for $|z_0| \leqslant \delta < r < 1$ and $\delta < t < 1$, the disc centered at $(1-t)z_0$ and of radius rt is contained in $\{z : |z| < r\}$. Hence, by Lemma 3,

$$rtM_p^p(f_m^{(k)} \circ \phi_t, r) = rt\int_{-\pi}^{\pi} \left|f_m^{(k)}\left((1-t)z_0 + tre^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} \leqslant 2r\int_{-\pi}^{\pi} \left|f_m^{(k)}(re^{i\theta})\right|^p \frac{d\theta}{2\pi} = 2rM_p^p(f_m^{(k)}, r). \tag{19}$$

By Minkowski's inequality, and inequalities (18) and (19), we have that

$$\begin{aligned}
\|\Lambda^\gamma_{z_0}(f_m)\|^q_{B^{p,q}_\alpha} &= \int_0^1 M_p^q\left(\int_0^1 (f_m\circ\phi_t)^{(k)}\, d\mu_\gamma(t), r\right) d\mu_{k,\alpha,q}(r) \leqslant \int_0^1 \left(\int_0^1 M_p(f_m^{(k)}\circ\phi_t \cdot t^k, r)\, d\mu_\gamma(t)\right)^q d\mu_{k,\alpha,q}(r) \\
&= \int_0^1 \left(\int_0^1 M_p(f_m^{(k)}\circ\phi_t, r) t^k\, d\mu_\gamma(t)\right)^q d\mu_{k,\alpha,q}(r) \\
&= \int_\delta^1 \left(\int_\delta^1 M_p(f_m^{(k)}\circ\phi_t, r) t^k\, d\mu_\gamma(t)\right)^q d\mu_{k,\alpha,q}(r) + \int_0^1 \left(\int_0^1 \chi_{[0,1)^2\setminus[\delta,1)^2}(t,r) M_p(f_m^{(k)}\circ\phi_t, r) t^k\, d\mu_\gamma(t)\right)^q d\mu_{k,\alpha,q}(r) \\
&\leqslant \left(\frac{2}{\delta}\right)^q \int_\delta^1 M_p^q(f_m^{(k)}, r)\left(\int_\delta^1 t^k\, d\mu_\gamma(t)\right)^q d\mu_{k,\alpha,q}(r) + C(k,\alpha,\gamma,q) \sup_{z\in\mathbb{D},\, t\in[0,\delta]} \left|(f_m^{(k)}\circ\phi_t)(z)\right|^q \\
&\leqslant \varepsilon^q C\int_\delta^1 M_p^q(f_m^{(k)}, r)\, d\mu_{k,\alpha,q}(r) + C\varepsilon^q \leqslant \varepsilon^q C\|f_m\|^q_{B^{p,q}_\alpha} + C\varepsilon^q \leqslant \varepsilon^q C\left(K^q + 1\right).
\end{aligned}$$

Thus, $\|\Lambda^\gamma_{z_0}(f_m)\|_{B^{p,q}_\alpha} \to 0$ as $m \to \infty$, as desired. □

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