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On Some Systems of Propositional Minimal Logic with Loop Detection

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Introduction. Backwards proof search and theorem proving with a standard cut-free system for the propositional fragment of minimal logic is inefficient because of three problems. Firstly, the proof search is not in general terminating caused by the possibility of looping. Secondly, it might generate proofs which are permutations of each other and represent the same natural deduction. Finally, during the proof some choice should be made to decide which rules to apply and where to use them.

The sequent system GM^- for minimal logic was introduced in [1]. GM^- is a permutation-free sequent system; it avoids the problems of permutations in the cut-free sequent system of Gentzen. This removes the second of the problems. But notice that permutations are avoided in GM^- by a focusing method – several choice points are removed. That is, GM^- partly addresses the third problem and hence is advantageous as a system for theorem proving. However, the naive implementation of GM^- will lead to the possibility of looping.

Some looping mechanisms have been considered earlier in [2-4].

In this paper following [2] the history mechanism is developed in two ways and applied to GM^-. Each of the constructed systems has advantages and disadvantages.

2. **Systems with history mechanism**. Further in the text we follow well known definitions of a formula, sequent, proof, context, stoup, equivalence of the systems as in [2, 5].

One way to prevent loops is to add a history to each sequent. The history is the set of all sequents that have occurred so far in a proof tree. After each backwards inference the new sequent (without its history) is checked to see whether it is a member of this set. If it is we have looping and we backtrack. If not the new history is the union of the new sequent (without its history) and the old history, and we try to prove the new sequent, and so on.

$$\frac{A,\Gamma \Rightarrow B;\ \varepsilon}{\Gamma \Rightarrow A \supset B;\ H}\ \left(\supset R_1\right),\ if\ A \notin \Gamma \qquad \frac{\Gamma \Rightarrow B;\ H}{\Gamma \Rightarrow A \supset B;\ H}\ \left(\supset R_2\right),\ if\ A \in \Gamma$$

$$\frac{A,\Gamma \Rightarrow \bot;\ \varepsilon}{\Gamma \Rightarrow \neg A;\ H}\ \left(\neg R_1\right),\ if\ A \notin \Gamma \qquad \frac{\Gamma \Rightarrow \bot;\ H}{\Gamma \Rightarrow \neg A;\ H}\ \left(\neg R_2\right),\ if\ A \in \Gamma$$

$$\frac{\Gamma \Rightarrow A;\ (C,H) \qquad \Gamma \xrightarrow{B} C;H}{\Gamma \xrightarrow{A \supset B} C;\ H}\ (\supset L),\ if\ C \notin H$$

$$\frac{\Gamma \Rightarrow A;\ (C,H)}{\Gamma \xrightarrow{\neg A} C;\ H}\ (\neg L),\ if\ C \notin H$$

$$\frac{\Gamma \xrightarrow{A} C;\ H}{\Gamma \xrightarrow{A \wedge B} C;\ H}\ \left(\wedge L_1\right) \qquad \frac{\Gamma \xrightarrow{B} C;\ H}{\Gamma \xrightarrow{A \wedge B} C;\ H}\ \left(\wedge L_2\right)$$

$$\frac{\Gamma \Rightarrow A;\ H \qquad \Gamma \Rightarrow B;\ H}{\Gamma \Rightarrow A \wedge B;\ H}\ (\wedge R)$$

$$\frac{A,\Gamma \Rightarrow C;\ \varepsilon \qquad B,\Gamma \Rightarrow C;\ \varepsilon}{\Gamma \xrightarrow{A \vee B} C;\ H}\ (\vee L),\ if\ A,B \notin \Gamma$$

$$\frac{\Gamma \Rightarrow A;\ H}{\Gamma \Rightarrow A \vee B;\ H}\ \left(\vee R_1\right) \qquad \frac{\Gamma \Rightarrow B;\ H}{\Gamma \Rightarrow A \vee B;\ H}\ \left(\vee R_2\right)$$

$$\frac{A,\Gamma \xrightarrow{A} B;\ H}{A,\Gamma \Rightarrow B;\ H}\ (C)^{*} \qquad \frac{\Gamma \Rightarrow A;\ (A,H)}{\Gamma \xrightarrow{\bot} A;\ H}\ (\bot) \qquad \frac{}{\Gamma \xrightarrow{A} A;\ H}\ (ax)$$

$*$ B *is either a propositional variable*, $\bot$ *or a disjunction*.

A, B, C *are formulae*. Γ, H *are sets of formulae*.

B, Γ *is shorthand for* $\{B\} \cup \Gamma$.

Fig. 1. The propositional system $SwMin$.

$$\frac{A,\Gamma \Rightarrow B;\ \{B\}}{\Gamma \Rightarrow A \supset B;\ H}\ (\supset R_1),\ if\ A \notin \Gamma \qquad \frac{A,\Gamma \Rightarrow \bot;\ \{\bot\}}{\Gamma \Rightarrow \neg A;\ H}\ (\neg R_1),\ if\ A \notin \Gamma$$

$$\frac{\Gamma \Rightarrow B;\ (B,H)}{\Gamma \Rightarrow A \supset B;\ H}\ (\supset R_2),\ if\ A \in \Gamma,\ B \notin H$$

$$\frac{\Gamma \Rightarrow \bot;\ (\bot,H)}{\Gamma \Rightarrow \neg A;\ H}\ (\neg R_2),\ if\ A \in \Gamma,\ \bot \notin H$$

$$\frac{\Gamma \Rightarrow A;\ (A,H) \qquad \Gamma \xrightarrow{B} C; H}{\Gamma \xrightarrow{A \supset B} C;\ H}\ (\supset L),\ if\ A \notin H$$

$$\frac{\Gamma \Rightarrow A;\ (A,H)}{\Gamma \xrightarrow{\neg A} C;\ H}\ (\neg L),\ if\ A \notin H$$

$$\frac{\Gamma \xrightarrow{A} C;\ H}{\Gamma \xrightarrow{A \wedge B} C;\ H}\ (\wedge L_1) \qquad \frac{\Gamma \xrightarrow{B} C;\ H}{\Gamma \xrightarrow{A \wedge B} C;\ H}\ (\wedge L_2)$$

$$\frac{\Gamma \Rightarrow A;\ (A,H) \qquad \Gamma \Rightarrow B;\ (B,H)}{\Gamma \Rightarrow A \wedge B;\ H}\ (\wedge R),\ if\ A,B \notin H$$

$$\frac{A,\Gamma \Rightarrow C; \{C\} \quad B,\Gamma \Rightarrow C;\ \{C\}}{\Gamma \xrightarrow{A \vee B} C;\ H}\ (\vee L),\ if\ A,B \notin \Gamma$$

$$\frac{\Gamma \Rightarrow A;\ (A,H)}{\Gamma \Rightarrow A \vee B;\ H}\ (\vee R_1),\ if\ A \notin H \qquad \frac{\Gamma \Rightarrow B;\ (B,H)}{\Gamma \Rightarrow A \vee B;\ H}\ (\vee R_2),\ if\ B \notin H$$

$$\frac{A,\Gamma \xrightarrow{A} B;\ H}{A,\Gamma \Rightarrow B;\ H}\ (C)^{*} \qquad \frac{\Gamma \Rightarrow A;\ (A,H)}{\Gamma \xrightarrow{\bot} A;\ H}\ (\bot) \qquad \frac{}{\Gamma \xrightarrow{A} A;\ H}\ (ax)$$

* B is either a propositional variable, $\bot$ or a disjunction.

A, B, C are formulae. Γ, H are sets of formulae.

B, Γ is shorthand for $\{B\} \cup \Gamma$.

Fig. 2. The propositional system $ScMin$.

The approach requires lots of sequents to be stored and on every step the list should be used for specific checkings. All that is quite inefficient as the sequents being stored contain much more information than actually needed to proceed. To prevent looping we can keep few information and satisfy the requirements.

The main idea behind to reduce the history and check the loops is the fact that only goal formulae need to be stored. The rules of GM^{-} are such that the context cannot decrease; once a formula is in the context it will remain in the context of all sequents above it in the proof tree. For two sequents to be the same they obviously need to have the same context. We may empty the history every time the context is extended, since we will never get any of the sequents below the extended one again. Goal formulae are the only ones to be stored in the history. If we come across a goal already in the history we have the same goal and the same context as another sequent, that is, a loop.

There are two slightly different approaches to doing this. There is the straightforward extension and modification of the system which we shall call a $SwMin$, and there is an approach which involves storing more formulae in the history, but that detects loops more quickly. This we will call as $ScMin$, and the implementation is in some cases more efficient than the $SwMin$.

In scope of considered systems sequent $\Gamma \Rightarrow C; H$ has context Γ, goal C, history H and no stoup, and sequent $\Gamma \xrightarrow{A} C; H$ has context Γ, goal C, history H and stoup A. When the history has been extended we have parenthesised (C, H) for emphasis, while by ε we denote empty history. The $SwMin$ system is displayed in Figure 1, and the $ScMin$ system in Figure 2.

The proof of introduced systems equivalence is done in two stages.

Theorem 2.1. *A sequent S is provable in GM^- if and only if $S;\varepsilon$ is provable in SwMin/ScMin (without *).*

Proof. The ⇐direction is straightforward.

To prove the ⇒direction we take an GM^- proof tree and use it to build a $SwMin/ScMin$ proof tree.

We start at the root, $\Gamma \Rightarrow A$ in GM^- and we have root $\Gamma \Rightarrow A; \{A\}$ in $SwMin/ScMin$.

Given a fragment of GM^- proof tree with corresponding fragment of $SwMin/ScMin$ proof tree, we look at the next inference in the GM^- tree. We have a recipe which we can use to build a fragment of $SwMin/ScMin$ proof tree corresponding to a strictly larger fragment of the GM^- proof tree.

As proof trees are finite, this process must be terminating.

Theorem 2.2. *The system $SwMin/ScMin$ with condition $*$ placed on rule (C) is equivalent to $SwMin/ScMin$ without the extra condition.*

Proof. The ⇐direction is trivial.

To prove the ⇒ direction, we first prove that GM^- and GM^- with $(*)$ condition on the weakening rule are equivalent. This is done by a simple induction on the depth of the proof and on complexity of formulae.

For any $SwMin/ScMin$ (without $*$) proof that doesn't satisfy $*$, we can consider it as an GM^- proof. Then we can find an GM^- proof satisfying $*$. Using the procedure in the proof of theorem 2.1, we can build an $SwMin/ScMin$ (with $*$) proof tree.

Theorem 2.3. The system $SwMin/ScMin$ and GM^- are equivalent.

Proof. The proof immediately follows from theorem 2.1 and 2.2.

3. Conclusion. Two systems for propositional fragment of minimal logic ($SwMin$ and $ScMin$) which are slightly different are introduced. Both systems are based on the idea of adding context to the sequents. In one system, $SwMin$, the history is kept smaller, but $ScMin$ detects loops more quickly. The heart of the difference between the two systems is that in the $SwMin$ loop checking is done when a formula leaves the goal, whereas in the $ScMin$ it is done when it becomes the goal.

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On Some Systems of Propositional Minimal Logic with Loop Detection

There exists different systems of I. Johansson's minimal logic. Looping is the main issue in one of the Gentzen style system. One way to detect loops is adding history to each sequent though it is insufficient. We have illustrated the use of the two history mechanisms for minimal propositional logic. The two systems both have their good points. The *SwMin* system is efficient in terms of storage and checkings required by its history mechanism. The *ScMin* system is efficient in that it detects loops as they occur, avoiding unnecessary computations.

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Ցիկլերի հայտնաբերումով մինիմալ տրամաբանության ասույթային որոշ համակարգերի մասին

Յոհանսոնի մինիմալ տրամաբանության համար գոյություն ունեն տարբեր համակարգեր։ Հենցենյան տիպի որոշ համակարգերում հիմնական խոչընդոտ է ցիկլը։ Ցիկլերի հայտնաբերման տարբերակներից է «պատմության» ավելացումը յուրաքանչյուր սեկվենսին, որը խնդրի լիարժեք լուծում չի տալիս։ Մինիմալ տրամաբանության երկու ասույթային համակարգ «պատմության» մեխանիզմով դիտարկված են, որոնցից յուրաքանչյուրն ունի իր առավելությունները։ *SwMin*-ը արդյունավետ է հիշողության օգտագործման և կատարվող ստուգումների տեսանկյունից, իսկ *ScMin*-ը հայտնաբերում է ցիկլերը ավելի վաղ փուլում։

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О некоторых системах минимальной пропозициональной логики с выявлением циклов

Существуют различные системы минимальной логики Йоганссона. В некоторых генценовских системах циклы являются одной из основных проблем. Добавление «истории» – один из подходов обнаружения цикла. Рассматриваются две системы минимальной пропозициональной логики с добавлением «истории». Каждая из рассматриваемых систем имеет свои преимущества. *SwMin* эффективна

с точки зрения использования памяти и дополнительных проверок. *ScMin* выявляет циклы на более ранних стадиях без дополнительных вычислений.

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