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An exponential estimate for the cubic partial sums of multiple Fourier series

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Abstract. We prove an exponential integral estimate for the cubic partial sums of multiple Fourier series on sets of large measure. This estimate yields some new properties of Fourier series.

Keywords: multiple Fourier series, exponential integral estimates, cubic partial sums.

§ 1. Introduction

Put $\mathbb{T}=\mathbb{R}/2\pi$ and let $\mathbb{T}^d$ denote the d-dimensional torus. For every function $f\in L^1(\mathbb{T}^d)$ we consider the multiple Fourier series and its conjugate:

$$\sum_{\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d} a_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}, \tag{1}$$

$$\sum_{\mathbf{n}=(n_1,\dots,n_d)\in\mathbb{Z}^d} \biggl(\prod_{k=1}^{d}(-i\cdot\operatorname{sign} n_k)\biggr) a_{\mathbf{n}}e^{i\mathbf{n}\cdot\mathbf{x}}, \tag{2}$$

where

$$\mathbf{n}=(n_1,\dots,n_d),\qquad \mathbf{x}=(x_1,\dots,x_d),\qquad \mathbf{n}\cdot\mathbf{x}=n_1x_1+\dots+n_dx_d,$$

$$a_{\mathbf{n}}=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} f(\mathbf{x})e^{-i\mathbf{n}\cdot\mathbf{x}}\,d\mathbf{x}.$$

Denote the rectangular and cubic partial sums of the series (1) by

$$S_{\mathbf{n}}f(\mathbf{x})=\sum_{-n_i\leqslant k_i\leqslant n_i} a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}},\qquad \mathbf{n}\in\mathbb{Z}^d,$$

$$S_{n}f(\mathbf{x})=\sum_{-n\leqslant k_i\leqslant n} a_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}},\qquad n\in\mathbb{N},$$

and let $\widetilde{S}_{\mathbf{n}}$ and $\widetilde{S}_n$ be their conjugates.

We shall consider the Orlicz class of functions corresponding to the logarithmic function

$$\operatorname{Log}_k(u)=|u|\max\{0,\log^k|u|\},\qquad k=1,2,\dots. \tag{3}$$

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This is the Banach space of functions

$$\operatorname{Log}_k(L)(\mathbb{T}^d) = \left\{ f \in L^1(\mathbb{T}^d) \colon \int_{\mathbb{T}^d} \operatorname{Log}_k(f) < \infty \right\}$$

with the Luxemburg norm

$$\|f\|_{\operatorname{Log}_k(L)} = \inf\left\{ \lambda \colon \lambda > 0,\ \int_{\mathbb{T}^d} \operatorname{Log}_k\left(\frac{f}{\lambda}\right) \leqslant 1 \right\} < \infty.$$

It is well known that the rectangular partial sums of the d-dimensional Fourier series of any function $f \in \operatorname{Log}_{d-1}(L)(\mathbb{T}^d)$ converge in measure (see [1], [2]), that is,

$$\lim_{\min(\mathbf{n})\to\infty} \bigl|\{\mathbf{x} \in \mathbb{T}^d \colon |S_{\mathbf{n}} f(\mathbf{x}) - f(\mathbf{x})| > \varepsilon\}\bigr| = 0 \tag{4}$$

for every $\varepsilon > 0$, where

$$\min(\mathbf{n}) = \min_{1\leqslant i\leqslant d} n_i.$$

On the other hand, Konyagin [3] and Getsadze [4] established that $\operatorname{Log}_{d-1}(L)$ is the largest Orlicz space whose elements satisfy (4).

The following problem was considered in [5], [6]. Find an exact estimate for the growth of a function $\Phi \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to 0} \Phi(t) = 0$ such that for every function $f \in \operatorname{Log}_{d-1}(L)(\mathbb{T}^d)$ and every number $\varepsilon > 0$ one can find a set $E_{f,\varepsilon} \subset \mathbb{T}^d$, $|E_{f,\varepsilon}| > (2\pi)^d - \varepsilon$, satisfying the condition

$$\lim_{\min(\mathbf{n})\to\infty} \int_{E_{f,\varepsilon}} \Phi\bigl(|S_{\mathbf{n}} f(\mathbf{x}) - f(\mathbf{x})|\bigr)\, d\mathbf{x} = 0. \tag{5}$$

The expected sharp bound for the growth of such functions is

$$\limsup_{t\to\infty} \frac{\log \Phi(t)}{t^{1/d}} < \infty. \tag{6}$$

One can observe that (5) implies convergence in measure and, moreover, it gives a quantitative characterization of the convergence rate.

This problem was considered in [6] in the one-dimensional case. The following estimate for the conjugate function $\widetilde{f}$ was proved there:

$$\int_{\mathbb{T}} \exp\left(c_1 \frac{\widetilde{f}(x)}{Mf(x)} \right) dx < c_2, \tag{7}$$

where $Mf(x)$ is the Hardy–Littlewood maximal function. It then was used to derive the following exponential estimate for the one-dimensional partial sums of Fourier series, which in its turn yields (5) in the one-dimensional case.

Theorem A (see [6]). *For every $f \in L^1(\mathbb{T})$ we have*

$$\int_{\mathbb{T}} \exp\left(c_1 \frac{|S_n f(x)| + |\widetilde{S}_n f(x)|}{Mf(x)} \right) dx \leqslant c_2, \qquad n = 1, 2, \dots, \tag{8}$$

where c_1 and c_2 are absolute constants.

The sharpness of the exponent in (8) (and hence in (5)) was proved by Oskolkov [7].

The relation (5) in the two-dimensional case with a function Φ satisfying (6) was established in [5]. The case $d \geqslant 3$ of this problem remains open, and so is the problem of the sharpness of (6) in the two-dimensional case.

Analogous estimates for the one-dimensional Walsh system and rearranged Haar systems were established in [8]. In [9], a similar problem was considered for general orthogonal L^2-series.

In this paper we consider a similar problem for cubic partial sums. Our main result is the following theorem.

Theorem 1. *For every $f \in \operatorname{Log}_{d-1}(L)(\mathbb{T}^d)$ there is a measurable function $F(\mathbf{x}) > 0$ on $\mathbb{T}^d$ such that*

$$|\{\mathbf{x} \in \mathbb{T}^d \colon F(\mathbf{x}) > \lambda\}| \lesssim \frac{\|f\|_{\operatorname{Log}_{d-1}(\mathbb{T}^d)}}{\lambda}, \tag{9}$$

$$\int_{\mathbb{T}^d} \exp\biggl(\frac{|S_n f(\mathbf{x})| + |\widetilde{S}_n f(\mathbf{x})|}{F(\mathbf{x})}\biggr) d\mathbf{x} \lesssim 1, \qquad n = 1, 2, \dots . \tag{10}$$

Here and in what follows, the relation $a \lesssim b$ stands for the inequality $a \leqslant c \cdot b$, where c is a constant depending only on the dimension d.

Corollary 1. *For every $f \in \operatorname{Log}_{d-1}(L)(\mathbb{T}^d)$ and every $\varepsilon > 0$ there is a set $E = E_{f,\varepsilon} \subset \mathbb{T}^d$ such that*

$$|E_{f,\varepsilon}| > (2\pi)^d - \varepsilon, \tag{11}$$

$$\int_{E_{f,\varepsilon}} \exp\biggl(\gamma\varepsilon \frac{|S_n f(\mathbf{x})| + |\widetilde{S}_n f(\mathbf{x})|}{\|f\|_{\operatorname{Log}_{d-1}(\mathbb{T}^d)}}\biggr) d\mathbf{x} \lesssim 1, \qquad n = 1, 2, \dots, \tag{12}$$

where $\gamma > 0$ is a constant depending only on d.

Corollary 2. *For every $f \in \operatorname{Log}_{d-1}(L)(\mathbb{T}^d)$ and every $\varepsilon > 0$ there is a set $E_{f,\varepsilon} \subset \mathbb{T}^d$ satisfying (11) and such that the relations*

$$\lim_{n\to\infty} \int_{E_{f,\varepsilon}} \bigl(\exp\bigl(A|S_n f(\mathbf{x}) - f(\mathbf{x})|\bigr) - 1\bigr)\, d\mathbf{x} = 0, \tag{13}$$

$$\lim_{n\to\infty} \int_{E_{f,\varepsilon}} \bigl(\exp\bigl(A|\widetilde{S}_n f(\mathbf{x}) - \widetilde{f}(\mathbf{x})|\bigr) - 1\bigr)\, d\mathbf{x} = 0 \tag{14}$$

hold for any $A > 0$, where $\widetilde{f}$ is the d-dimensional conjugate function of f.

Remark 1. The method used in our proof of Theorem 1 is also applicable to the mixed partial sums of multiple Fourier series defined by the formula

$$S^B_{\mathbf{n}} f(\mathbf{x}) = \sum_{-n_i \leqslant k_i \leqslant n_i} \biggl(\prod_{s\in B} (-i \cdot \operatorname{sign} n_s)\biggr) a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \mathbf{n} \in \mathbb{Z}^d,$$

where $B \subset \{1, 2, \dots, d\}$ (see [10], Ch. 8). Namely, given any $f \in \operatorname{Log}_{d-1}(L)(\mathbb{T}^d)$, one can find a function $F(\mathbf{x}) > 0$ satisfying (9) and

$$\int_{\mathbb{T}^d} \exp\biggl(\frac{|S^B_n f(\mathbf{x})|}{F(\mathbf{x})}\biggr) d\mathbf{x} \lesssim 1, \qquad n = 1, 2, \dots .$$

To avoid technical difficulties in the proofs, we consider only the typical cases when $B=\varnothing$ or $\{1,2,\dots,d\}$ (Theorem 1).

Remark 2. The counterexamples of Konyagin [3] and Getsadze [4] show that $\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)$ is the largest Orlicz class where such properties hold.

Remark 3. We prove Theorem 1 by reducing it to the one-dimensional case. This well-known approach was first used by Sjölin [11] to prove a multidimensional version of Carleson's theorem.

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§ 2. Notation and lemmas

By Theorem 9.5 in Ch. 2 of [12], the Luxemburg norm satisfies the relations

$$\|f\|_{\mathrm{Log}_k(L)} \leqslant 1 \quad \Longrightarrow \quad \int_{\mathbb{T}^d} \mathrm{Log}_k(f) \leqslant \|f\|_{\mathrm{Log}_k(L)}, \tag{15}$$

$$\|f\|_{\mathrm{Log}_k(L)} \geqslant 1 \quad \Longrightarrow \quad \int_{\mathbb{T}^d} \mathrm{Log}_k(f) \geqslant \|f\|_{\mathrm{Log}_k(L)}. \tag{16}$$

In fact, these inequalities hold not only for logarithmic but also for general Luxemburg norms. Using (15) and (16), one can easily check that

$$\|f\|_{\mathrm{Log}_k(L)} \lesssim 1 + \int_{\mathbb{T}^d} \mathrm{Log}_k(f) \tag{17}$$

for every $f \in \mathrm{Log}_k(\mathbb{T}^d)$. Clearly, if $\|f\|_{\mathrm{Log}_k(L)} = 1$, then we have both upper and lower bounds

$$1 + \int_{\mathbb{T}^d} \mathrm{Log}_k(f) \lesssim \|f\|_{\mathrm{Log}_k(L)} = 1 \lesssim 1 + \int_{\mathbb{T}^d} \mathrm{Log}_k(f). \tag{18}$$

The one-dimensional conjugate function of $f \in L^1(\mathbb{T})$ is defined as

$$\widetilde{f}(x) = \text{p.v. } \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x+t)}{2\tan(t/2)}\, dt = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon<|t|<\pi} \frac{f(x+t)}{2\tan(t/2)}\, dt. \tag{19}$$

It is well known that $\widetilde{f}(x)$ is defined a.e. for every Lebesgue integrable function and satisfies the inequality

$$\int_{\mathbb{T}} \mathrm{Log}_{k-1}(\widetilde{f}) \lesssim 1 + \int_{\mathbb{T}} \mathrm{Log}_k(f), \qquad k = 1, 2, \dots \tag{20}$$

(see [13], Ch. 7). We shall need this inequality in the following form.

Lemma 1. *If* $f \in \mathrm{Log}_k(L)(\mathbb{T}^d)$, $k = 0, 1, \dots,$ *then the function*

$$g(x_1, x_2, \dots, x_d) = \text{p.v. } \int_{\mathbb{T}} \frac{f(x_1+t, x_2+t, x_3, \dots, x_d)}{\tan(t/2)}\, dt$$

is defined a.e. on $\mathbb{T}^d$ *and satisfies the bound*

$$\int_{\mathbb{T}^d} \mathrm{Log}_{k-1}(g) \lesssim 1 + \int_{\mathbb{T}^d} \mathrm{Log}_k(f).$$

We define the d-dimensional conjugate of a function $f \in \mathrm{Log}_{d-1}(\mathbb{T}^d)$ as an iterated integral:

$$\begin{aligned}\widetilde{f}(\mathbf{x}) &= \text{p.v. } \frac{1}{\pi^d}\int_{\mathbb{T}^d} f(\mathbf{x}+\mathbf{t})\prod_{k=1}^{d}\frac{1}{2\tan(t_k/2)}\,dt_1\ldots dt_d \\ &= \text{p.v. } \frac{1}{\pi}\int_{\mathbb{T}}\Big(\ldots\Big(\text{p.v. } \frac{1}{\pi}\int_{\mathbb{T}} f(\mathbf{x}+\mathbf{t})\prod_{k=1}^{d}\frac{1}{2\tan(t_k/2)}\,dt_d\Big)\ldots\Big)\,dt_1,\end{aligned}$$

where the variables of integration are taken in the reverse order $t_d, t_{d-1}, \ldots, t_1$. Note that the d-dimensional conjugate $\widetilde{f}$ is defined a.e. for $f \in \mathrm{Log}_{d-1}(\mathbb{T}^d)$. In what follows we understand all integrals in the sense of the principal value and omit the symbol p.v. before them. The two-dimensional case of the following lemma was proved in [14]. This lemma enables us to use the modified partial sums

$$\begin{aligned}S_n^* f(\mathbf{x}) &= \frac{1}{\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d}\frac{\sin nt_k}{2\tan(t_k/2)} f(\mathbf{x}+\mathbf{t})\,d\mathbf{t}, \\ \widetilde{S}_n^* f(\mathbf{x}) &= \frac{1}{\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d}\frac{\cos nt_k - 1}{2\tan(t_k/2)} f(\mathbf{x}+\mathbf{t})\,d\mathbf{t}\end{aligned}$$

in the proof of the theorem.

Lemma 2. *If* $f \in \mathrm{Log}_{d-1}(L)(\mathbb{T}^d)$, *then*

$$\int_{\mathbb{T}^d}\sup_n |S_n f(\mathbf{x}) - S_n^* f(\mathbf{x})|\,d\mathbf{x} \lesssim \|f\|_{\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)}, \tag{21}$$

$$\int_{\mathbb{T}^d}\sup_n |\widetilde{S}_n f(\mathbf{x}) - \widetilde{S}_n^* f(\mathbf{x})|\,d\mathbf{x} \lesssim \|f\|_{\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)}. \tag{22}$$

Proof. One can clearly assume that

$$\|f\|_{\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)} = 1. \tag{23}$$

We shall only prove (21). (22) can be proved similarly. We have

$$S_n f(\mathbf{x}) = \frac{1}{\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d} D_n(t_k) f(\mathbf{x}+\mathbf{t})\,d\mathbf{t}, \tag{24}$$

where

$$D_n(x) = \frac{\sin(n+1/2)x}{2\sin(x/2)} = \frac{\sin nx}{2\tan(x/2)} + \frac{1}{2}\cos nx \tag{25}$$

is the Dirichlet kernel. Substituting (25) into (24), we see that the difference $S_n f(\mathbf{x}) - S_n^* f(\mathbf{x})$ is the sum of several integrals of the form

$$\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d}\prod_{k\in A}\frac{\sin nt_k}{\tan(t_k/2)}\prod_{k\in A^c}\cos(nt_k)\cdot f(\mathbf{x}+\mathbf{t})\,d\mathbf{t}, \tag{26}$$

where $A \subsetneq \{1, 2, \dots, d\}$ is a subset of integers. Applying the product formulae for trigonometric functions, we split each integral (26) into a sum of integrals of the form

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\phi(n(\pm t_1 \pm t_2 \pm \cdots \pm t_d))}{2^{d-1} \prod_{k \in A} \tan(t_k/2)} f(\mathbf{x} + \mathbf{t})\, d\mathbf{t}, \tag{27}$$

where the function ϕ is either the sine or cosine. This reduces the proof of the lemma to an estimation of the integrals (27). When $A = \varnothing$, the desired estimate is

$$\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} |f(\mathbf{x} + \mathbf{t})|\, d\mathbf{t} = \frac{\|f\|_{L^1}}{(2\pi)^d} \lesssim \|f\|_{L \log^{d-1} L}.$$

When $A \neq \varnothing$, the integrals (27) are estimated in a similar way. Therefore we estimate only the integral

$$I_n f(\mathbf{x}) = \int_{\mathbb{T}^d} \frac{\sin n(t_1 + t_2 + \cdots + t_d)}{\prod_{k=l+1}^d \tan(t_k/2)} f(\mathbf{x} + \mathbf{t})\, d\mathbf{t}, \tag{28}$$

which corresponds to $A = \{1, \dots, l\},\ l \geqslant 1$. After the change of variables

$$u_1 = t_1 + t_2 + \cdots + t_d, \quad u_2 = t_2, \quad \dots, \quad u_d = t_d \tag{29}$$

we obtain from (28) that

$$\begin{aligned} |I_n f(\mathbf{x})| &= \left| \int_{\mathbb{T}^d} \frac{\sin n u_1}{\prod_{k=l+1}^d \tan(u_k/2)} G(\mathbf{x}, \mathbf{u})\, d\mathbf{u} \right| \\ &= \left| \int_{\mathbb{T}^l} \sin n u_1 \left(\int_{\mathbb{T}^{d-l}} \frac{G(\mathbf{x}, \mathbf{u})}{\prod_{k=l+1}^d \tan(u_k/2)} du_{l+1} \dots du_d \right) du_1 \dots du_l \right| \\ &\leqslant \int_{\mathbb{T}^l} \left| \int_{\mathbb{T}^{d-l}} \frac{G(\mathbf{x}, \mathbf{u})}{\prod_{k=l+1}^d \tan(u_k/2)} du_{l+1} \dots du_d \right| du_1 \dots du_l, \end{aligned}$$

where

$$G(\mathbf{x}, \mathbf{u}) = f(x_1 + u_1 - u_2 - \cdots - u_d, x_2 + u_2, \dots, x_d + u_d). \tag{30}$$

The inner integral may be regarded as a function of the variables x_k, $k = 1, 2, \dots, d$, and u_j, $j = 1, 2, \dots, l$. Moreover, applying Lemma 1 $(d - l)$ times, we have

$$\begin{aligned} &\int_{\mathbb{T}^d} \sup_n |I_n(\mathbf{x})|\, d\mathbf{x} \\ &\qquad \leqslant \int_{\mathbb{T}^{d+l}} \left| \int_{\mathbb{T}^{d-l}} \frac{G(\mathbf{x}, \mathbf{u})}{\prod_{k=l+1}^d \tan(u_k/2)} du_{l+1} \dots du_d \right| du_1 \dots du_l\, dx_1 \dots dx_d \\ &\qquad \lesssim 1 + \int_{\mathbb{T}^{d+l}} \mathrm{Log}_{d-l}\left(|G(\mathbf{x}, u_1, \dots, u_l, 0, \dots, 0)|\right) du_1 \dots du_l\, dx_1 \dots dx_d \\ &\qquad = 1 + (2\pi)^l \int_{\mathbb{T}^d} \mathrm{Log}_{d-l}(f) \lesssim \|f\|_{\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)} = 1, \end{aligned}$$

which yields (21). Here we have used the inequality (18), which holds under the condition (23). □

§ 3. Proofs of the main results

Proof of Theorem 1. We first prove the estimate (10) for the operators

$$U_n f(\mathbf{x}) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{k=1}^{d} \phi_k(t_k) f(\mathbf{x}+\mathbf{t})\, d\mathbf{t}, \tag{31}$$

where each ϕ_k is one of the four functions

$$\frac{\sin nt}{2\tan(t/2)}, \qquad \frac{\cos nt}{2\tan(t/2)}, \tag{32}$$

$$\sin nt, \qquad \cos nt. \tag{33}$$

We call them operators of type U. When all the ϕ_k are of the form (33), the estimate (10) for U_n holds trivially. One can take $F(\mathbf{x}) \equiv c \cdot \|f\|_1$ with an appropriate absolute constant $c > 0$. It is also easy to prove (10) in the case when only one function of the form (32) occurs in (31). Indeed, we can assume without loss of generality that

$$U_n f(\mathbf{x}) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \frac{\sin nt_d}{2\tan(t_d/2)} \prod_{k=1}^{d-1} \sin nt_k \cdot f(\mathbf{x}+\mathbf{t})\, d\mathbf{t}. \tag{34}$$

Observe that

$$U_n f(\mathbf{x}) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{\sin n(t_d - x_d)}{2\tan((t_d - x_d)/2)} g(x_1, \dots, x_{d-1}, t_d)\, dt_d,$$

where

$$\begin{aligned} &g(x_1, \dots, x_{d-1}, t_d) \\ &\quad = \int_{\mathbb{T}^{d-1}} \prod_{k=1}^{d-1} \sin nt_k \cdot f(x_1 + t_1, \dots, x_{d-1} + t_{d-1}, t_d)\, dt_1 \dots dt_{d-1}. \end{aligned}$$

Then we can write

$$\begin{aligned} U_n f(\mathbf{x}) = {} & \frac{\cos nx_d}{\pi} \int_{\mathbb{T}} \frac{\sin nt_d \cdot g(x_1, \dots, x_{d-1}, t_d)}{2\tan((t_d - x)/2)}\, dt_d \\ & - \frac{\sin nx_d}{\pi} \int_{\mathbb{T}} \frac{\cos nt_d \cdot g(x_1, \dots, x_{d-1}, t_d)}{2\tan((t_d - x)/2)}\, dt_d. \end{aligned}$$

Let $M_d g(\mathbf{x})$ be the maximal function of $g(\mathbf{x})$ with respect to the variable x_d. It follows easily from (7) that

$$\int_{\mathbb{T}^d} \exp\left(c_1 \frac{|U_n f(\mathbf{x})|}{M_d g(\mathbf{x})} \right) d\mathbf{x} < c_2.$$

Since the maximal functions satisfies the weak L^1 inequality, the operators (34) and the function $F(\mathbf{x}) = M_d g(\mathbf{x})$ satisfy (10) and (9), as required.

To prove this for the general operators (31), we use induction on the dimension d. According to the approach above, the required assertion holds when $d = 1$. To make the induction step, we assume that the exponential estimate holds for all operators (31) in dimension $d-1 \geqslant 1$. Take a function $f \in \mathrm{Log}_{d-1}(\mathbb{T}^d)$ such that

$$\|f\|_{\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)} = 1. \tag{35}$$

According to the approach above, we can assume that at least two functions ϕ_k of type (32) occur in (31). Hence there is no loss of generality in assuming that

$$U_n f(\mathbf{x}) = \frac{1}{\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(t_k)\frac{\sin(nt_{d-1})}{2\tan(t_{d-1}/2)}\,\frac{\sin(nt_d)}{2\tan(t_d/2)}\,f(\mathbf{x}+\mathbf{t})\,d\mathbf{t}.$$

Thus we obtain

$$\begin{aligned} U_n f(\mathbf{x}) &= \frac{1}{2\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(t_k)\frac{\cos n(t_{d-1}-t_d)}{4\tan(t_{d-1}/2)\tan(t_d/2)}\,f(\mathbf{x}+\mathbf{t})\,d\mathbf{t} \\ &\quad - \frac{1}{2\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(t_k)\frac{\cos n(t_{d-1}+t_d)}{4\tan(t_{d-1}/2)\tan(t_d/2)}\,f(\mathbf{x}+\mathbf{t})\,d\mathbf{t} \\ &= U_n^{(1)}f(\mathbf{x}) - U_n^{(2)}f(\mathbf{x}). \end{aligned}$$

We estimate only the first integral $U_n^{(1)}f(\mathbf{x})$. The second can be estimated in a similar way. By making the change of variables

$$u_1 = t_1, \quad u_2 = t_2, \quad \ldots, \quad u_{d-2} = t_{d-2}, \quad u_{d-1} = t_{d-1} - t_d, \quad u_d = t_d$$

in the expression for $U_n^{(1)}f(\mathbf{x})$, we obtain

$$U_n^{(1)}f(\mathbf{x}) = \frac{1}{2\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(u_k)\frac{\cos nu_{d-1}}{4\tan((u_{d-1}+u_d/2))\tan(u_d/2)}\,G(\mathbf{x},\mathbf{u})\,d\mathbf{u},$$

where

$$G(\mathbf{x},\mathbf{u}) = f(x_1+u_1,\ldots,x_{d-2}+u_{d-2},x_{d-1}+u_{d-1}+u_d,x_d+u_d). \tag{36}$$

Using the identity

$$\frac{1}{\tan(u+v)\tan v} = \frac{1}{\tan u\tan v} - \frac{1}{\tan u\tan(u+v)} - 1,$$

we obtain that

$$\begin{aligned} U_n^{(1)}f(\mathbf{x}) &= \frac{1}{2\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(u_k)\frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)}\,\frac{1}{2\tan(u_d/2)}\,G(\mathbf{x},\mathbf{u})\,d\mathbf{u} \\ &\quad - \frac{1}{2\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(u_k)\frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)}\,\frac{1}{2\tan((u_{d-1}+u_d)/2)}\,G(\mathbf{x},\mathbf{u})\,d\mathbf{u} \\ &\quad - \frac{1}{2\pi^d}\int_{\mathbb{T}^d}\prod_{k=1}^{d-2}\phi_k(u_k)\cos nu_{d-1}\,G(\mathbf{x},\mathbf{u})\,d\mathbf{u} \\ &= U_n^{(1,1)}f(\mathbf{x}) - U_n^{(1,2)}f(\mathbf{x}) - U_n^{(1,3)}f(\mathbf{x}). \end{aligned}$$

For each $i = 1, 2, 3$ we shall find a function $F^{(i)}(\mathbf{x}) \geqslant 0$ such that

$$|\{\mathbf{x} \in \mathbb{T}^d : F^{(i)}(\mathbf{x}) > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}{\lambda}, \tag{37}$$

$$\int_{\mathbb{T}^d} \exp\left(\frac{|U_n^{(1,i)} f(\mathbf{x})|}{F^{(i)}(\mathbf{x})}\right) d\mathbf{x} \lesssim 1. \tag{38}$$

Case $i = 1$. Consider the operator

$$U_n' g(x_1, \ldots, x_d) = \frac{1}{2\pi^{d-1}} \int_{\mathbb{T}^{d-1}} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos n u_{d-1}}{2\tan(u_{d-1}/2)} \times g(x_1 + u_1, \ldots, x_{d-1} + u_{d-1}, x_d)\, du_1 \ldots du_{d-1}$$

acting on the function

$$g(x_1, \ldots, x_d) = \frac{1}{\pi} \int_{\mathbb{T}} \frac{f(x_1, \ldots, x_{d-2}, x_{d-1} + t, x_d + t)}{2\tan(t/2)}\, dt. \tag{39}$$

In view of (36), we get

$$\begin{aligned} U_n^{(1,1)} f(\mathbf{x}) &= \frac{1}{2\pi^{d-1}} \int_{\mathbb{T}^{d-1}} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos n u_{d-1}}{2\tan(u_{d-1}/2)} \\ &\quad \times \left(\frac{1}{\pi} \int_{\mathbb{T}} \frac{1}{2\tan(u_d/2)} G(\mathbf{x}, \mathbf{u})\, du_d\right) du_1 \ldots du_{d-1} \\ &= \frac{1}{2\pi^{d-1}} \int_{\mathbb{T}^{d-1}} \prod_{k=1}^{d-2} \phi_k(u_k) \frac{\cos n u_{d-1}}{2\tan(u_{d-1}/2)} \\ &\quad \times g(x_1 + u_1, \ldots, x_{d-1} + u_{d-1}, x_d)\, du_1 \ldots du_{d-1} \\ &= U_n' g(x_1, \ldots, x_{d-1}, x_d). \end{aligned} \tag{40}$$

For every fixed x_d we may regard U_n' as a $(d-1)$-dimensional operator (31) of type U. Thus, by the induction hypothesis, for every $x_d \in \mathbb{T}$ there is a function $F_{x_d}(x_1, \ldots, x_{d-1}) = F^{(1)}(x_1, \ldots, x_d)$ such that

$$|\{(x_1, \ldots, x_{d-1}) \in \mathbb{T}^{d-1} \colon F_{x_d}(x_1, \ldots, x_{d-1}) > \lambda\}| \lesssim \frac{\|g_{x_d}\|_{\mathrm{Log}_{d-2}(\mathbb{T}^{d-1})}}{\lambda}, \tag{41}$$

$$\int_{\mathbb{T}^{d-1}} \exp\left(\frac{c|U_n' g_{x_d}(x_1, \ldots, x_{d-1})|}{F_{x_d}(x_1, \ldots, x_{d-1})}\right) dx_1 \ldots dx_{d-1} \lesssim 1, \qquad n = 1, 2, \ldots. \tag{42}$$

Here g_{x_d} is the function $g(x_1, \ldots, x_d)$ regarded as a function of the variables $x_1, \ldots, x_{d-1}$. On the other hand, it follows from Lemma 1 that

$$\int_{\mathbb{T}^d} \mathrm{Log}_{d-2}(g) \lesssim 1 + \int_{\mathbb{T}^d} \mathrm{Log}_{d-1}(f) \lesssim \|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)} = 1. \tag{43}$$

Applying (17), (18), (43) and (41), we obtain

$$\begin{aligned}
|\{\mathbf{x}\in\mathbb{T}^d : F^{(1)}(\mathbf{x})>\lambda\}| &\lesssim \frac{1}{\lambda}\int_{\mathbb{T}} \|g_{x_d}\|_{\mathrm{Log}_{d-2}(\mathbb{T}^{d-1})}\,dx_d\\
&\lesssim \frac{1}{\lambda}\int_{\mathbb{T}}\left(1+\int_{\mathbb{T}^{d-1}}\mathrm{Log}_{d-2}(g)\,dx_1\dots dx_{d-1}\right)dx_d\\
&\lesssim \frac{1}{\lambda}\left(1+\int_{\mathbb{T}^d}\mathrm{Log}_{d-2}(g)\right)\\
&\lesssim \frac{1}{\lambda}\left(1+\int_{\mathbb{T}^d}\mathrm{Log}_{d-1}(f)\right)\\
&\lesssim \frac{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}{\lambda}.
\end{aligned}$$

Using (40) and integrating the inequality (42) with respect to x_d, we get

$$\int_{\mathbb{T}^d}\exp\left(\frac{c|U_n^{(1,1)}f(\mathbf{x})|}{F^{(1)}(\mathbf{x})}\right)d\mathbf{x}\lesssim 1.$$

This yields (37) and (38) when $i=1$.

Case $i=2$. The estimate for $U_n^{(1,2)}f(\mathbf{x})$ can be proved in a similar way. We have

$$\begin{aligned}
U_n^{(1,2)}f(\mathbf{x}) = \frac{1}{2\pi^{d-1}}&\int_{\mathbb{T}^{d-1}}\prod_{k=1}^{d-2}\phi_k(u_k)\frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)}\\
&\times\left(\frac{1}{\pi}\int_{\mathbb{T}}\frac{G(\mathbf{x},\mathbf{u})}{2\tan((u_{d-1}+u_d)/2)}\,du_d\right)du_1\dots du_{d-1}.
\end{aligned}$$

The change of the variable $t=u_d+u_{d-1}$ in the inner integral yields that

$$\begin{aligned}
\frac{1}{\pi}&\int_{\mathbb{T}}\frac{G(\mathbf{x},\mathbf{u})}{2\tan((u_{d-1}+u_d)/2)}\,du_d\\
&=\frac{1}{\pi}\int_{\mathbb{T}}\frac{f(x_1+u_1,\dots,x_{d-2}+u_{d-2},x_{d-1}+t,x_d-u_{d-1}+t)}{2\tan(t/2)}\,dt\\
&=g(x_1+u_1,\dots,x_{d-2}+u_{d-2},x_{d-1},x_d-u_{d-1}),
\end{aligned}$$

where g is again the function (39). Thus we obtain

$$\begin{aligned}
U_n^{(1,2)}f(\mathbf{x})=U_n''g(x_1,\dots,x_{d-1},x_d)=\frac{1}{2\pi^{d-1}}&\int_{\mathbb{T}^{d-1}}\prod_{k=1}^{d-2}\phi_k(u_k)\frac{\cos nu_{d-1}}{2\tan(u_{d-1}/2)}\\
&\times g(x_1+u_1,\dots,x_{d-2}+u_{d-2},x_{d-1},x_d-u_{d-1})\,du_1\dots du_{d-1}.
\end{aligned}$$

For every fixed x_{d-1}, we may regard this as a $(d-1)$-dimensional operator of type U acting on the function g of the remaining variables $x_1,\dots,x_{d-2},x_d$. By the induction hypothesis, as in the case when $i = 1$, we obtain a function $F^{(2)}(\mathbf{x})$ satisfying (37) and (38) when $i = 2$.

Case $i = 3$. Observe that $U_n^{(1,3)}$ is also a $(d-1)$-dimensional operator of type U acting on the function (36). As in the previous cases, we can then easily obtain (37) and (38) when $i = 3$.

Thus we have established the desired estimate for U_n.

Since S_n^* is an operator of type U, we can find a function $F_1(\mathbf{x})$ such that

$$|\{\mathbf{x} \in \mathbb{T}^d \colon F_1(\mathbf{x}) > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}{\lambda} \tag{44}$$

$$\int_{\mathbb{T}^d} \exp\biggl(\frac{|S_n^* f(\mathbf{x})|}{F_1(\mathbf{x})}\biggr)\, d\mathbf{x} \lesssim 1, \qquad n = 1, 2, \dots . \tag{45}$$

As to $\widetilde{S}_n^*$, we have the bound

$$|\widetilde{S}_n^* f(\mathbf{x})| \leqslant |U_n f(\mathbf{x})| + G(\mathbf{x}),$$

where

$$U_n f(\mathbf{x}) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{k=1}^{d} \frac{\cos n t_k}{2\tan(t_k/2)} f(\mathbf{x} + \mathbf{t})\, d\mathbf{t},$$

$$G(\mathbf{x}) = \frac{1}{\pi^d} \biggl| \mathrm{p.v.} \int_{\mathbb{T}^d} \frac{f(\mathbf{x} + \mathbf{t})}{\prod_{k=1}^{d} 2\tan(t_k/2)}\, d\mathbf{t} \biggr|.$$

It is well known that $G(\mathbf{x})$ satisfies

$$|\{G(\mathbf{x}) > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}{\lambda}. \tag{46}$$

Since U_n is an operator of type U, there is a function $F_2(\mathbf{x})$ such that

$$|\{\mathbf{x} \in \mathbb{T}^d \colon F_2(\mathbf{x}) > \lambda\}| \lesssim \frac{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}{\lambda}, \tag{47}$$

$$\int_{\mathbb{T}^d} \exp\biggl(\frac{|U_n f(\mathbf{x})|}{F_2(\mathbf{x})}\biggr)\, d\mathbf{x} \lesssim 1, \qquad n = 1, 2, \dots . \tag{48}$$

Finally, by Lemma 2 we have

$$\begin{aligned} |S_n f(\mathbf{x})| + |\widetilde{S}_n f(\mathbf{x})| &\leqslant |S_n^* f(\mathbf{x})| + |\widetilde{S}_n^* f(\mathbf{x})| + F_3(\mathbf{x}) \\ &\leqslant |S_n^* f(\mathbf{x})| + |U_n f(\mathbf{x})| + G(\mathbf{x}) + F_3(\mathbf{x}), \end{aligned}$$

where the function $F_3(\mathbf{x}) \geqslant 0$ satisfies

$$\|F_3\|_{L^1(\mathbb{T}^d)} \lesssim \|f\|_{\mathrm{Log}_{d-1}(L)(\mathbb{T}^d)}. \tag{49}$$

We claim that all the conclusions of Theorem 1 hold for $F = 4(F_1 + F_2 + F_3 + G)$. Indeed, (9) follows immediately from (44), (46), (47) and (49) (using Chebyshev's inequality for F_3). To prove (10), observe that

$$\exp\left(\frac{|S_n f(\mathbf{x})| + |\widetilde{S}_n f(\mathbf{x})|}{F(\mathbf{x})}\right) \leqslant \exp\left(\frac{|S_n^* f(\mathbf{x})| + |U_n f(\mathbf{x})| + G(\mathbf{x}) + F_3(\mathbf{x})}{F(\mathbf{x})}\right)$$
$$\leqslant \exp\left(\frac{4|S_n^* f(\mathbf{x})|}{F(\mathbf{x})}\right) + \exp\left(\frac{4|U_n f(\mathbf{x})|}{F(\mathbf{x})}\right) + \exp\left(4\frac{G(\mathbf{x})}{F(\mathbf{x})}\right) + \exp\left(4\frac{F_3(\mathbf{x})}{F(\mathbf{x})}\right)$$
$$\leqslant \exp\left(\frac{|S_n^* f(\mathbf{x})|}{F_1(\mathbf{x})}\right) + \exp\left(\frac{|U_n f(\mathbf{x})|}{F_2(\mathbf{x})}\right) + 2e.$$

Combining this with (45) and (48), we complete the proof of the theorem. □

Proof of Corollary 1. Suppose that $f \in \mathrm{Log}_{d-1}(L)(\mathbb{T}^d)$ and let $F(x)$ be the function provided by Theorem 1. We define

$$E_{f,\varepsilon} = \left\{\mathbf{x} \in \mathbb{T}^d \colon F(\mathbf{x}) \leqslant \frac{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}{\gamma\varepsilon}\right\},$$

where γ is a constant. By (9), there is a constant γ (depending only on d) such that $|(E_{f,\varepsilon})^c| < \varepsilon$. This yields (11). Moreover, it follows from (10) that

$$\int_{E_{f,\varepsilon}} \exp\left(\gamma\varepsilon\frac{|S_n f(\mathbf{x})| + |\widetilde{S}_n f(\mathbf{x})|}{\|f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}\right) d\mathbf{x} \leqslant \int_{\mathbb{T}^d} \exp\left(\frac{|S_n f(\mathbf{x})| + |\widetilde{S}_n f(\mathbf{x})|}{F(\mathbf{x})}\right) d\mathbf{x} \lesssim 1,$$

so that (12) holds. □

Proof of Corollary 2. Suppose that $f \in \mathrm{Log}_{d-1}(L)(\mathbb{T}^d)$. It is well known that the $(C,1)$-means $\sigma_{\mathbf{n}} f$ of the Fourier series (1) of f and its conjugate (2) converge almost everywhere to f and $\widetilde{f}$ respectively. There is also the convergence in norm

$$\lim_{\min(\mathbf{n})\to\infty} \|\sigma_{\mathbf{n}} f - f\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)} = 0.$$

Using this, one can find a set $G \subset \mathbb{T}^d$ and a d-dimensional trigonometric polynomial P_k such that

$$|G| > (2\pi)^d - \frac{\varepsilon}{2}, \tag{50}$$

$$\|f - P_k\|_{L^\infty(G)} < \frac{1}{2k}, \tag{51}$$

$$\|\widetilde{f} - \widetilde{P}_k\|_{L^\infty(G)} < \frac{1}{2k}, \tag{52}$$

$$\|f - P_k\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)} < \frac{\gamma\varepsilon_k}{2k}. \tag{53}$$

Applying Corollary 1 with $\varepsilon_k = \varepsilon/2^{k+1}$, we find sets $E_k \subset \mathbb{T}^d$ with

$$|E_k| > (2\pi)^d - \varepsilon_k, \qquad k = 1, 2, \ldots, \tag{54}$$

$$\int_{E_k} \exp\left(\gamma\varepsilon_k\frac{|S_n(f - P_k)| + |\widetilde{S}_n(f - P_k)|}{\|f - P_k\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}\right) \leqslant c, \qquad n = 1, 2, \ldots. \tag{55}$$

Define

$$E_{f,\varepsilon} = G \cap \biggl(\bigcap_k E_k\biggr).$$

Then (11) follows from (50) and (54). Put $\phi(t) = \exp t - 1$. We easily see that $\phi(ab) \leqslant a\phi(b)$ for $0 < a < 1$ and $b > 0$. Thus, using (51), (53) and (55), we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_{E_{f,\varepsilon}} \bigl(\exp(A|S_n f - f|) - 1\bigr) \\
&\qquad = \lim_{n\to\infty} \int_{E_{f,\varepsilon}} \bigl(\exp\bigl(A|S_n(f-P_k) - (f-P_k)|\bigr) - 1\bigr) \\
&\qquad \leqslant \frac{A}{k} \sup_n \int_{E_{f,\varepsilon}} \bigl(\exp\bigl(k(|S_n(f-P_k)| + |f-P_k|)\bigr)\bigr) \\
&\qquad \leqslant \frac{A}{k}\biggl(\sup_n \int_{E_{f,\varepsilon}} \exp\bigl(2k|S_n(f-P_k)|\bigr) + \int_{E_{f,\varepsilon}} \exp\bigl(2k|f-P_k|\bigr)\biggr) \\
&\qquad \leqslant \frac{A}{k}\biggl(\sup_n \int_{E_{f,\varepsilon}} \exp\biggl(\frac{\gamma\varepsilon_k|S_n(f-P_k)|}{\|f-P_k\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}\biggr) + \int_{E_{f,\varepsilon}} \exp(2k|f-P_k|)\biggr) \lesssim \frac{A}{k}.
\end{aligned}$$

Since the last quantity can be arbitrarily small, we obtain (13). In a similar way, we arrive at the inequalities

$$\begin{aligned}
&\lim_{n\to\infty} \int_{E_{f,\varepsilon}} \bigl(\exp(A|\widetilde{S}_n f - \widetilde{f}|) - 1\bigr) \\
&\qquad \leqslant \frac{A}{k}\biggl(\sup_n \int_{E_{f,\varepsilon}} \exp\biggl(\frac{\gamma\varepsilon_k|\widetilde{S}_n(f-P_k)|}{\|f-P_k\|_{\mathrm{Log}_{d-1}(\mathbb{T}^d)}}\biggr) + \int_{E_{f,\varepsilon}} \exp(2k|\widetilde{f}-\widetilde{P}_k|)\biggr) \lesssim \frac{A}{k}
\end{aligned}$$

and, therefore, at (14). □

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