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ON A WEAK TYPE ESTIMATE FOR SPARSE OPERATORS OF STRONG TYPE

G. A. KARAGULYAN, G. MNATSAKANYAN

Yerevan State University, Armenia[1]
E-mails: *g.karagulyan@ysu.am; mnatsakanyan_g@yahoo.com*

Abstract. We define sparse operators of strong type on abstract measure spaces with ball-bases. Weak and strong type inequalities for such operators are proved.

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1. Introduction

The sparse operators are very simple positive operators recently appeared in the study of weighted estimates of Calderón-Zygmund and other related operators. It was proved that some well-known operators (Calderón-Zygmund operators, martingale transforms, maximal function, Carleson operators, etc.) can be dominated by sparse operators, and this kind of dominations imply a series of deep results for the mentioned operators (see [1, 2, 4 – 7]). In particular, Lerner's [6] norm domination of the Calderón-Zygmund operators by sparse operators gave a simple alternative proof to the A_2-conjecture solved by Hytönen [3]. Lacey [5] established a pointwise sparse domination for the Calderón-Zygmund operators with an optimal condition (Dini condition) on the modulus of continuity, getting a logarithmic gain to the result previously proved by Conde-Alonso and Rey [1]. The paper [5] also proves a pointwise sparse domination for the martingale transforms, providing a short approach to the A_2-theorem proved by Treil-Thiele-Volberg [8]. For the Carleson operators norms sparse domination was proved by Di Plinio and Lerner [2], while the pointwise domination follows from a general result proved later in [4].

In this paper we consider sparse operators based on ball-bases in abstract measure spaces. The concept of ball-basis was introduced by the first author in [4]. Based on ball-basis the paper [4] defines a wide class of operators (including, in particular, the

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above mentioned operators) that can be pointwisely dominated by sparse operators. Some estimates of sparse operators in abstract spaces were obtained in [4]. In this paper we define a stronger version of sparse operators, and prove weak and strong type estimates for such operators.

We first recall the definition of the ball-basis from [4].

Definition 1.1. *Let $(X, \mathfrak{M}, \mu)$ be a measure space. A family of sets $\mathfrak{B} \subset \mathfrak{M}$ is said to be a ball-basis if it satisfies the following conditions.*

B1) $0 < \mu(B) < \infty$ *for any ball* $B \in \mathfrak{B}$.

B2) For any two points $x, y \in X$ *there exists a ball* $B \ni x, y$.

B3) If $E \in \mathfrak{M}$, *then for any* $\varepsilon > 0$ *there exists a finite or infinite sequence of balls* B_k, $k = 1, 2, \ldots$, *such that*

$$\mu\left(E \triangle \bigcup_k B_k\right) < \varepsilon.$$

B4) For any $B \in \mathfrak{B}$ *there is a ball* $B^* \in \mathfrak{B}$ *(called a* hull *of* B*) satisfying the conditions:*

$$\bigcup_{A \in \mathfrak{B}:\, \mu(A) \le 2\mu(B),\, A \cap B \neq \varnothing} A \subset B^*, \quad \mu(B^*) \le \mathcal{K}\mu(B),$$

where $\mathcal{K}$ *is a positive constant.*

A ball-basis $\mathfrak{B}$ is said to be doubling if there is a constant $\eta > 1$ such that for any $A \in \mathfrak{B}$, $A^* \neq X$, one can find a ball $B \in \mathfrak{B}$ to satisfy

$$A \subsetneq B, \quad \mu(B) \le \eta \cdot \mu(A). \tag{1.1}$$

In [4], it was shown that the condition (1.1) in the definition can equivalently be replaced by a stronger condition $\eta_1 \le \mu(B)/\mu(A) \le \eta_2$, where $\eta_2 > \eta_1 > 1$. It is well-known the non-standard features of non-doubling bases in many problems of analysis.

One can easily check that the family of Euclidean balls in $\mathbb{R}^n$ forms a ball-basis and it is doubling. An example of non-doubling ball-basis can serve us the martingale-basis defined as follows. Let $(X, \mathfrak{M}, \mu)$ be a measure space, and let $\{\mathfrak{B}_n : n \in \mathbb{Z}\}$ be a collection of measurable sets such that 1) each $\mathfrak{B}_n$ is a finite or countable partition of X, 2) for each n and $A \in \mathfrak{B}_n$ the set A is a union of sets $A' \in \mathfrak{B}_{n+1}$, 3) the collection $\mathfrak{B} = \cup_{n \in \mathbb{Z}} \mathfrak{B}_n$ generates the σ-algebra $\mathfrak{M}$, 4) for any points $x, y \in X$ there is a set $A \in \mathfrak{B}$ such that $x, y \in A$. One can easily check that $\mathfrak{B}$ satisfies all the ball-basis conditions B1)-B4). On the other hand, it is not always doubling. Obviously, it is doubling if and only if $\mu(\mathrm{pr}(B)) \le c\mu(B)$, $B \in \mathfrak{B}$, where $\mathrm{pr}(B)$ (parent of B) denotes the minimal ball satisfying $B \subsetneq \mathrm{pr}(B)$.

Let $\mathfrak{B}$ be a ball-basis in a measure space $(X, \mathfrak{M}, \mu)$. For $f \in L^r(X)$, $1 \le r < \infty$, and a ball $B \in \mathfrak{B}$ we set

$$\langle f\rangle_{B,r} = \left(\frac{1}{\mu(B)}\int_B |f|^r\right)^{1/r}, \quad \langle f\rangle^*_{B,r} = \sup_{A\in\mathfrak{B}: A\supset B} \langle f\rangle_{A,r}.$$

A collection of balls $\mathcal{S} \subset \mathfrak{B}$ is said to be sparse or γ-sparse if for any $B \in \mathcal{S}$ there is a set $E_B \subset B$ such that $\mu(E_B) \ge \gamma\mu(B)$ and the sets $\{E_B : B \in \mathcal{S}\}$ are pairwise disjoint, where $0 < \gamma < 1$ is a constant. We associate with $\mathcal{S}$ the operators:

$$\mathcal{A}_{\mathcal{S},r} f(x) = \sum_{A\in\mathcal{S}} \langle f\rangle_{A,r} \cdot \mathbb{I}_A(x), \quad \mathcal{A}^*_{\mathcal{S},r} f(x) = \sum_{A\in\mathcal{S}} \langle f\rangle^*_{A,r} \cdot \mathbb{I}_A(x),$$

called sparse and strong type sparse operators, respectively. The weak-L^1 estimate of $\mathcal{A}_{\mathcal{S},1}$ in $\mathbb{R}^n$ (case $r = 1$) as well as its boundedness on L^p ($1 < p < \infty$) were proved by Lerner [6]. The L^p-boundedness of $\mathcal{A}_{\mathcal{S},r}$ for general ball-bases was shown by the first author in [4].

We will say that a constant is admissible if it depends only on p and on the constants $\mathcal{K}$ and γ from the above definitions, and the notation $a \lesssim b$ will stand for the inequality $a \le c \cdot b$, where $c > 0$ is an admissible constant. The main result of this paper is the weak-L^r estimate of $\mathcal{A}^*_{\mathcal{S},r}$ generated by general ball-bases. More precisely, we have the following result.

Theorem 1.1. *A sparse operator of strong type $\mathcal{A}^*_{\mathcal{S},r}$, $1 \le r < \infty$, corresponding to a general ball-basis, is a bounded operator on L^p for $r < p < \infty$, and satisfies the weak-L^r estimate, that is,*

$$\left\|\mathcal{A}^*_{\mathcal{S},r}(f)\right\|_p \lesssim \|f\|_p, \quad r < p < \infty, \tag{1.2}$$

$$\mu\left\{\mathcal{A}^*_{\mathcal{S},r}(f) > \lambda\right\} \lesssim \frac{\|f\|_r^r}{\lambda^r}, \quad \lambda > 0. \tag{1.3}$$

The proof of L^p-boundedness of $\mathcal{A}^*_{\mathcal{S},r}$ is simple and uses the duality argument as in [6]. Lerner's [6] proof of weak-L^1 estimate in $\mathbb{R}^n$ applies the standard Calderón-Zygmund decomposition argument. The Calderón-Zygmund decomposition may fail if the ball-basis is not doubling, so for the weak-L^r estimate in the case of general ball-basis we apply the function flattening technique displayed in Lemma 2.7. That is, we reconstruct the function $f \in L^r$ around the big values to get a λ-bounded function $g \in L^{2r}$, having ball averages of f dominated by those of g. As a result we will have $\|\mathcal{A}^*_{\mathcal{S},r} f\|_{r,\infty} \lesssim \|\mathcal{A}^*_{\mathcal{S},r} g\|_{2r,\infty}$, reducing the weak-$L^r$ estimate of $\mathcal{A}^*_{\mathcal{S},r}$ to weak-L^{2r}.

2. Auxiliary lemmas

Recall some definitions and propositions from [4]. We say that a set $E \subset X$ is bounded if $E \subset B$ for a ball $B \in \mathfrak{B}$.

Lemma 2.1 ([4]). *Let $(X, \mathfrak{M}, \mu)$ be a measure space with a ball-basis $\mathfrak{B}$. If $E \subset X$ is bounded and $\mathcal{G}$ is a family of balls with $E \subset \bigcup_{G \in \mathcal{G}} G$, then there exists a finite or infinite sequence of pairwise disjoint balls $G_k \in \mathcal{G}$ such that $E \subset \bigcup_k G_k^*$.*

Definition 2.1. *For a set $E \in \mathfrak{M}$ a point $x \in E$ is said to be a density point if for any $\varepsilon > 0$ there exists a ball $B \ni x$ such that $\mu(B \cap E) > (1 - \varepsilon)\mu(B)$. We say that a measure space $(X, \mathfrak{M}, \mu)$ satisfies the density property if almost all points of any measurable set are density points.*

Lemma 2.2 ([4]). *Any ball-basis satisfies the density property.*

The L^r maximal function associated to the ball-basis $\mathfrak{B}$ we denote by

$$M_r f(x) = \sup_{B \in \mathfrak{B}:\, x \in B} \langle f \rangle_{B,r}$$

Lemma 2.3 ([4]). *If $1 \leq r < p \leq \infty$, then the maximal function M_r satisfies the strong L^p and weak-L^r inequalities.*

Definition 2.2. *We say that $B \in \mathfrak{B}$ is a λ-ball for a function $f \in L^r(X)$ if*

$$\langle f \rangle_{B,r} > \lambda.$$

If, in addition, there is no λ-ball $A \supset B$ satisfying $\mu(A) \geq 2\mu(B)$, then B is said to be a maximal λ-ball for f.

Lemma 2.4. *Let the function $f \in L^r(X)$ have bounded support, and let $\lambda > 0$. There exist pairwise disjoint maximal λ-balls $\{B_k\}$ such that*

$$G_\lambda = \{x \in X : M_r f(x) > \lambda\} \subset \bigcup_k B_k^*. \tag{2.1}$$

Proof. Since f has bounded support, one can easily check that the set G_λ is also bounded. Besides, any λ-ball is in some maximal λ-ball. Thus we conclude that $G_\lambda = \bigcup_\alpha B_\alpha$, where each B_α is a maximal λ-ball. Applying the above covering lemma, we find a sequence of pairwise disjoint balls B_k such that

$$G_\lambda \subset \bigcup_k B_k^*$$

and so we have (2.1). $\square$

Let $B \subset (a,b)$ be a Lebesgue measurable set. For a given positive real $\kappa \le |B|$ denote

$$a(\kappa, B) = \inf\{a' : |(a,a') \cap B)| \ge \kappa\}, \quad L(\kappa, B) = (a, a(\kappa, G)) \cap B.$$

Observe that $L(\kappa, B)$ determines the "leftmost"set of measure κ in B and $a(\kappa, B)$ does not depend on the choice of a.

Lemma 2.5. *Let $A \subset B \subset (a,b)$ be Lebesgue measurable sets on the real line, and let $0 < \kappa \le |A|$. Then we have*

$$|L(\kappa, B) \triangle L(\kappa, A)| \le 2|B \setminus A|.$$

Proof. Obviously, we have $a \le a(\kappa, B) \le a(\kappa, A) \le b$. Since $|L(\kappa, B)| = |L(\kappa, B)|$, the sets

$$L(\kappa, B) \setminus L(\kappa, A) = \big((a, a(\kappa, B)) \cap (B \setminus A)\big),$$
$$L(\kappa, A) \setminus L(\kappa, B) = \big((a(\kappa, B), a(\kappa, A)) \cap A\big).$$

have the same measure. So, we get

$$|L(\kappa, B) \triangle L(\kappa, A)| = 2\left|\big((a, a(\kappa, B)) \cap (B \setminus A)\big)\right| \le 2|B \setminus A|.$$

Lemma 2.6. *Let $(X, \mathfrak{M}, \mu)$ be a non-atomic measure space and G_k be a finite or infinite sequence of measurable sets in X. If a sequence of numbers $\xi_k \ge 0$ satisfies $\sum_k \xi_k < \infty$ and the condition*

$$\sum_{j:\, \mu(G_j) \le \mu(G_k),\, G_j \cap G_k \ne \varnothing} \xi_j \le \mu(G_k), \quad k = 1, 2, \ldots, \tag{2.2}$$

then there exist pairwise disjoint measurable sets $\tilde{G}_k \subset G_k$ such that

$$\mu(\tilde{G}_k) = \xi_k, k = 1, 2, \ldots. \tag{2.3}$$

Proof. Without loss of generality we can suppose that $\mu(G_k)$ is decreasing. Since the measure space is non-atomic, we can also suppose that G_k are Lebesgue measurable sets in $\mathbb{R}$. We first assume that the sequence G_k, $k = 1, 2, \ldots, n$, is finite. We apply backward induction. The existence of $\tilde{G}_n \subset G_n$ satisfying $\mu(\tilde{G}_n) = \xi_n$ follows from (2.2), since the latter implies $\xi_n \le \mu(G_n)$ and we have that the measure is non-atomic. We define $\tilde{G}_n$ to be the leftmost set in G_n, that is, $\tilde{G}_n = L(\xi_n, G_n)$. Suppose by induction we have defined pairwise disjoint sets $\tilde{G}_k \subset G_k$ satisfying (2.3) for $l \le k \le n$. From (2.2) it follows that

$$\mu\left(G_{l-1} \setminus \bigcup_{k=l}^{n} \tilde{G}_k\right) \ge \mu(G_{l-1}) - \sum_{l \le j \le n:\, G_j \cap G_{l-1} \ne \varnothing} \mu(\tilde{G}_j) \ge \xi_{l-1}.$$

Hence we can define $\tilde{G}_{l-1} = L(\xi_{l-1}, G_{l-1} \setminus \bigcup_{k=l}^{n} \tilde{G}_k)$. To proceed the general case we apply the finite case that we have proved. Then for each n we find a family of pairwise disjoint sets $G_k^{(n)}$, $k = 1, 2, \ldots, n$ such that $\mu(G_k^{(n)}) = \xi_k$ for $1 \le k \le n$. Applying Lemma 2.5 and analyzing once again the leftmost selection argument of the tilde sets, one can observe that

$$\mu(G_k^{(n+1)} \triangle G_k^{(n)}) \le \sum_{j=k}^{n} \mu(G_{n+1}^{(n+1)} \cap G_j^{(n)}) \le \xi_{n+1}.$$

So, we conclude that

$$\mu(G_k^{(m)} \triangle G_k^{(n)}) \le \sum_{k=n+1}^{m} \xi_k, \quad m > n \ge k.$$

The last inequality implies that for a fixed k the sequence $\mathbb{I}_{G_k^{(m)}}$ converges in L^1-norm as $m \to \infty$. Moreover, one can see that the limiting function is again an indicator function of a set $\tilde{G}_k$, and the sequence $\tilde{G}_k$ satisfies the conditions of the lemma. □

Lemma 2.7. *Let $(X, \mathfrak{M}, \mu)$ be a non-atomic measure space, and let $f \in L^r(X)$, $1 \le r < \infty$, be a boundedly supported positive function. Then for any $\lambda > 0$ there exists a measurable set $E_\lambda \subset X$ such that*

$$\mu(E_\lambda) \lesssim \|f\|_r^r / \lambda^r, \quad \{x \in X : M_r f(x) > \lambda\} \subset E_\lambda, \tag{2.4}$$

and the function

$$g(x) = f(x) \cdot \mathbb{I}_{X \setminus E_\lambda}(x) + \lambda \cdot \mathbb{I}_{E_\lambda}(x) \tag{2.5}$$

satisfies the conditions:

$$g(x) \le \lambda \text{ a.e. on } X, \quad \langle f \rangle_{B,r} \lesssim \langle g \rangle_{B^*,r} \text{ whenever } B \in \mathfrak{B},\ B \not\subset E_\lambda. \tag{2.6}$$

Proof. Applying Lemma 2.4 we find a sequence of pairwise disjoint maximal λ-balls B_k satisfying (2.1). Thus, applying the density property (Lemma 2.2), one can conclude that

$$f(x) \le \lambda \text{ for a.a. } x \in X \setminus \bigcup_k B_k^*. \tag{2.7}$$

Given B_k, we associate the family of balls

$$\mathfrak{B}_k = \{B \in \mathfrak{B} : B \cap B_k^* \ne \varnothing,\ \mu(B) > 2\mu(B_k^*)\}. \tag{2.8}$$

Observe that if one of these families, say $\mathfrak{B}_{k_0}$, is empty, then in view of conditions B2) and B4), one can easily check that $X \subset B_{k_0}^{**}$. Then defining $E_\lambda = X$, the claim of the lemma will be satisfied. Hence we can assume that each $\mathfrak{B}_k$ is nonempty, and so, there is a ball $G_k \in \mathfrak{B}_k$ such that

$$\mu(G_k) \le 2 \inf_{B\in\mathfrak{B}_k} \mu(B). \tag{2.9}$$

From λ-maximality of B_k and the inequality $\mu(G_k) > 2\mu(B_k^*)$, we get $B_k^* \subset G_k^*$, $\langle f\rangle_{G_k^*,r} \le \lambda$. This implies

$$\frac{1}{\lambda^r}\int_{G_k^*} f^r \le \mu(G_k^*) \le c\cdot \mu(G_k), \tag{2.10}$$

where $c > 0$ is an admissible constant. Denote

$$D_1 = B_1^*, \quad D_k = B_k^* \setminus \cup_{1\le j\le k-1} B_j^*,\ k \ge 2,$$

and consider the numerical sequence $\xi_k = \frac{\delta}{\lambda^r}\int_{D_k} f^r$, $k = 1, 2, \ldots$, for some constant $\delta > 0$. Taking into account (2.10), for a small admissible constant $\delta > 0$ we obtain

$$\begin{aligned}\bigcup_{j:\,\mu(G_j)\le\mu(G_k),\, G_j\cap G_k\ne\varnothing} \xi_j &= \frac{\delta}{\lambda^r}\bigcup_{j:\,\mu(G_j)\le\mu(G_k),\, G_j\cap G_k\ne\varnothing}\int_{D_j} f^r\\ &\le \frac{\delta}{\lambda^r}\int_{G_k^*} f^r \le c\delta\mu(G_k) \le \mu(G_k),\end{aligned}$$

which gives condition (2.2). Since our measure space in non-atomic, applying Lemma 2.6, we find pairwise disjoint subsets $\tilde G_k \subset G_k$ such that

$$\mu(\tilde G_k) = \frac{\delta}{\lambda^r}\int_{D_k} f^r, \quad k = 1, 2, \ldots. \tag{2.11}$$

The disjointness of the sets D_k implies

$$\sum_k \mu(\tilde G_k) = \frac{\delta}{\lambda^r}\sum_k \int_{D_k} f^r \lesssim \frac{\|f\|_r^r}{\lambda^r}. \tag{2.12}$$

From the λ-maximality and disjointness property of B_k, we get

$$\mu\left(\bigcup_k B_k^{**}\right) \lesssim \sum_k \mu\left(B_k\right) \le \frac{1}{\lambda^r}\sum_k\int_{B_k} f^r \le \frac{\|f\|_r^r}{\lambda^r}. \tag{2.13}$$

Denote $E_\lambda = \left(\bigcup_k \tilde G_k\right) \bigcup \left(\bigcup_k B_k^{**}\right)$. From (2.12) and (2.13) we get $\mu(E_\lambda) \lesssim \|f\|_r^r/\lambda^r$, and (2.7) implies (2.6). Hence it remains to prove that the function g satisfies (2.6). Take a ball $B \in \mathfrak{B}$ with $B \not\subset E_\lambda$. First of all observe that for each B_k satisfying $B \cap B_k^* \ne \varnothing$ we have $\mu(B) > 2\mu(B_k^*)$, since otherwise we would have $B \subset B_k^{**} \subset E_\lambda$, which is not true. Thus, whenever $B \cap B_k^* \ne \varnothing$ we have $B \in \mathfrak{B}_k$, then we get $\mu(G_k) \le 2\mu(B)$, and so $\tilde G_k \subset G_k \subset B^*$. Besides, from (2.7) and the definition of g it follows that $f(x) \le g(x)$ a.e. on $X \setminus \cup_k B_k^*$. Hence, using (2.11) and the disjointness of $\tilde{G}_k$, we can write

$$\langle f\rangle_{B,r}^r = \frac{1}{\mu(B)}\left(\int_{B\cap(\cup_k B_k^*)} f^r + \int_{B\setminus\cup_k B_k^*} f^r\right) \le \frac{1}{\mu(B)}\left(\sum_{k\colon B_k^*\cap B\neq\varnothing}\int_{B\cap D_k} f^r + \int_{B\setminus\cup_k B_k^*} g^r\right)$$
$$\le \frac{1}{\mu(B)}\left(\sum_{k\colon B_k^*\cap B\neq\varnothing}\int_{D_k} f^r + \int_B g^r\right) = \frac{1}{\mu(B)}\left(\sum_{k\colon B_k^*\cap B\neq\varnothing}\frac{\lambda^r\mu(\tilde{G}_k)}{\delta} + \int_B g^r\right)$$
$$= \frac{1}{\delta\mu(B^*)}\left(\sum_{k\colon B_k^*\cap B\neq\varnothing}\int_{\tilde{G}_k} g^r + \int_{B^*} g^r\right) \lesssim \langle g\rangle_{B^*,r}^r.$$

This implies (2.6). □

3. Proof of Theorem 1.1

Proof of L^p-boundedness. For any $B\in\mathcal{S}$ we have $\langle f\rangle_{B,r}^* \le M_r f(x)$ for all $x\in B$, and therefore $\langle f\rangle_{B,r}^* \le \langle M_r f\rangle_{B,r}$, $B\in\mathfrak{B}$. Let E_B be the disjoint portions of the sparse collection of balls satisfying $\mu(E_B)\ge\gamma\cdot\mu(B)$. Also, suppose that $r<p<\infty$ and $q=p/(p-1)$. Thus, for positive functions $f\in L^p$ and $g\in L^q(X)$, we can write

$$\int_X \mathcal{A}_{\mathcal{S},r}^* f\cdot g d\mu \le \sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}\int_B g d\mu = \sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}\cdot\langle g\rangle_{B,1}\cdot\mu(B)$$
$$\le \gamma^{-1}\sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}\cdot(\mu(E_B))^{1/p}\cdot\langle g\rangle_{B,1}\cdot(\mu(E_B))^{1/q}$$
$$\le \gamma^{-1}\left(\sum_{B\in\mathcal{S}}\langle M_r f\rangle_{B,r}^p\cdot\mu(E_B)\right)^{1/p}\cdot\left(\sum_{B\in\mathcal{S}}\langle g\rangle_{B,1}^q\cdot\mu(E_B)\right)^{1/q}$$
$$\le \gamma^{-1}\|M_r(M_r f)\|_p\|M_1(g)\|_q \lesssim \|M_r f\|_p\cdot\|g\|_q \lesssim \|f\|_p\cdot\|g\|_q,$$

which completes the proof of L^p-boundedness. □

Proof of weak-L^r estimate. Without loss of generality, we can assume that our measure space $(X,\mathfrak{M},\mu)$ is non-atomic, since any measure space can be extended to a non-atomic measure space by splitting the atoms as follows. Suppose $A\subset\mathfrak{M}$ is the family of atomic elements of the measure space $(X,\mathfrak{M},\mu)$, that is, for any $a\in A$ we have $\mu(a)>0$ and there is no proper $\mathfrak{M}$-measurable set in a. We can suppose that each atom is continuum and let $(a,\mathfrak{M}_a,\mu_a)$ be a a non-atomic measure space on $a\in A$ such that $\mu_a(a)=\mu(a)$. Denote by $\mathfrak{M}'$ the σ-algebra on X generated by $\mathfrak{M}$ and by all $\mathfrak{M}_a$, $a\in A$. Let μ' be an extension of μ such that $\mu'(E)=\mu_a(E)$ for any $\mathfrak{M}_a$-measurable set $E \subset a$. Hence, $(X, \mathfrak{M}', \mu')$ provides a non-atomic extension of the measure space $(X, \mathfrak{M}, \mu)$.

Now let f be a $\mathfrak{M}$-measurable function. The balls are $\mathfrak{M}$-measurable, and so they can not contain an atom a partially. Thus, the left and right sides of inequality (1.3) are not changed if we consider $(X, \mathfrak{M}', \mu')$ instead of the initial measure space. Hence, we can suppose that $(X, \mathfrak{M}, \mu)$ is itself non-atomic. Applying Lemma 2.7, we find a function g satisfying the conditions of the lemma. From (2.6) we get $\langle f\rangle^*_{B,r} \le \langle g\rangle^*_{B,r}$ for any $B \in \mathcal{S}$ with $B \not\subset E_\lambda$ and hence, $\mathcal{A}^*_{\mathcal{S},r} f(x) \le \mathcal{A}^*_{\mathcal{S},r} g(x)$, $x \in X \setminus E_\lambda$. Therefore, using the L^{2r} bound of $\mathcal{A}^*_{\mathcal{S},r}$, we obtain

$$\mu\{x \in X : \mathcal{A}^*_{\mathcal{S},r} f(x) > \lambda\} \le \mu(E_\lambda) + \mu\{x \in X \setminus E_\lambda : \mathcal{A}^*_{\mathcal{S},r} g(x) > \lambda\}$$
$$\lesssim \frac{\|f\|_r^r}{\lambda^r} + \frac{1}{\lambda^{2r}} \int_{X\setminus E_\lambda} |g|^{2r} \le \frac{\|f\|_r^r}{\lambda^r} + \frac{\lambda^r}{\lambda^{2r}} \int_{X\setminus E_\lambda} f^r \le \frac{2\|f\|_r^r}{\lambda^r}.$$

This completes the proof of theorem 1.1. □

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