

Mathematics

GENERAL CLASSES OF TAYLOR–MACLAURIN TYPE FORMULAS
IN COMPLEX DOMAIN

B. A. SAHAKYAN*

Chair of Mathematical Analysis YSU, Armenia

In the present paper the results of [1] are generalized, more general systems of operators generated by Riemann–Liouville integral and derivative. General systems of functions generated by Mittag–Leffler type functions are introduced as well.

Keywords: Riemann–Liouville operators, Taylor–Maclaurin’s type formula.

§ 1. Preliminaries and Lemmas. Let $\alpha \in [0, 1)$, $1/\rho = 1 - \alpha$ ($\rho \geq 1$), $f(z)$ is a complex function satisfying $f(re^{i\varphi}) \in L(0, l(\varphi))$ for fixed $\varphi (-\pi < \varphi \leq \pi)$, $0 < r < +\infty$, where $(0, l(\varphi)) = \{z; \arg z = \varphi, 0 < |z| < l < +\infty\}$.

Then the function $D^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^z (z - \xi)^{\alpha-1} f(\xi) d\xi$, where the integration is taken along the line segment connecting points 0 and z , $\arg(z - \xi)^{\alpha-1} = (\alpha - 1)\arg z$, is called the Riemann–Liouville integral of order α of $f(z)$, and the function $D^{1/\rho} f(z) \equiv \frac{d}{dz} D^{-\alpha} f(z)$ is called the Riemann–Liouville derivative of order $1/\rho$ of $f(z)$.

The operators $D^{0/\rho} f(z) = f(z)$, $D^{1/\rho} f(z)$, $D^{n/\rho} f(z) = D^{1/\rho} D^{(n-1)/\rho} f(z)$, $n \geq 2$, are called the Riemann–Liouville operators of successive differentiation of order n/ρ , $n = 0, 1, \dots$, of function $f(z)$.

It is known, that if $\alpha \in (0, 1)$, then $D^\alpha f(z) \equiv \frac{d}{dz} D^{-(1-\alpha)} f(z)$.

If $f(x) \in L(0, l)$, then for any $\alpha \in (0, +\infty)$, a.e. $D^\alpha D^{-\alpha} f(x) = f(x)$. If $\alpha \in (0, 1]$, $D^\alpha f(x) \in L(0, l)$ and $[D^{-(1-\alpha)} f(x)]_{x=0} = 0$, then a.e. $D^{-\alpha} D^\alpha f(x) = f(x)$ (see [2], chap. IX, (1.11), (1.12')). These formulas are true also for complex variable z .

Let $\mu > 0$, $\alpha > 0$. Then the following formula is true:

$$\frac{1}{\Gamma(\alpha)} \int_0^z (z - \xi)^{\alpha-1} E_\rho(\lambda \xi^{1/\rho}; \mu) \xi^{\mu-1} d\xi = z^{\mu+\alpha-1} E_\rho(\lambda z^{1/\rho}; \mu + \alpha) \quad (1.1)$$

* E-mail: maneats@rambler.ru

(see [2], chap. III, (1.16)), where λ is an arbitrary parameter, the integration is taken along the line segment connecting points 0 and z , and

$E_\rho(z; \mu) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu + k/\rho)}$ ($\rho > 0$) is the ρ order entire function of Mittag-

Leffler type for any value of parameter μ . We put

$$e_{\rho, \mu}(z; \lambda) \equiv E_\rho(-\lambda z^{1/\rho}; \mu) z^{\mu-1}, \quad e_{\rho, 1/\rho}(z; \lambda) = E_\rho(-\lambda z^{1/\rho}; 1/\rho) z^{1/\rho-1} \equiv e_\rho(z; \lambda).$$

Lemma 1.1. Let $\rho \geq 1$, $1/\rho \leq \mu < 1 + 1/\rho$, λ is an arbitrary parameter. Then the following formula holds:

$$D^{(\mu-1/\rho)} \{e_{\rho, \mu}(z; \lambda)\} = e_\rho(z; \lambda). \quad (1.2)$$

Proof. Since $0 \leq \mu - 1/\rho < 1$, then according to the definition of operator $D^{(\mu-1/\rho)}$, using formula (1.1) we have

$$\begin{aligned} D^{(\mu-1/\rho)} \{e_{\rho, \mu}(z; \lambda)\} &= \frac{d}{dz} D^{-(1+1/\rho-\mu)} \{e_{\rho, \mu}(z; \lambda)\} = \\ &= \frac{d}{dz} \left\{ \frac{1}{\Gamma(1+1/\rho-\mu)} \int_0^z (z-\xi)^{1/\rho-\mu} E_\rho(-\lambda \xi^{1/\rho}; \mu) \xi^{\mu-1} d\xi \right\} = \\ &= \frac{d}{dz} \{E_\rho(-\lambda z^{1/\rho}; 1+1/\rho) z^{1/\rho}\} = E_\rho(-\lambda z^{1/\rho}; 1/\rho) z^{1/\rho-1} = e_\rho(z; \lambda), \end{aligned}$$

$$z = re^{i\varphi}, \quad \xi = \tau e^{i\varphi}, \quad 0 < \tau < r < l < +\infty.$$

Lemma 1.2. Let $\rho \geq 1$, $1/\rho \leq \mu < 1 + 1/\rho$, λ is an arbitrary parameter. Then the following formula holds:

$$D^{-(\mu-1/\rho)} D^{(\mu-1/\rho)} \{e_{\rho, \mu}(z; \lambda)\} = e_{\rho, \mu}(z; \lambda). \quad (1.3)$$

Lemma 1.3. Let $\rho \geq 1$, $1/\rho \leq \mu < 1 + 1/\rho$, λ is an arbitrary parameter, $f(z) \in L(0; l(\varphi))$, $-\pi < \varphi \leq \pi$, is continuous on $(0; l(\varphi))$. Then the following formula holds:

$$D^{(\mu-1/\rho)} \left\{ \int_0^z e_{\rho, \mu}(z-\xi; \lambda) f(\xi) d\xi \right\} = \int_0^z e_{\rho, \mu}(z-\xi; \lambda) f(\xi) d\xi, \quad (1.4)$$

$$z = re^{i\varphi}, \quad \xi = \tau e^{i\varphi}, \quad 0 < \tau < r < l < +\infty.$$

Lemma 1.4. Let $\rho \geq 1$, $1/\rho \leq \mu < 1 + 1/\rho$, λ is an arbitrary parameter, $f(z) \in L(0; l(\varphi))$ is continuous on $(0; l(\varphi))$. Then the following formula holds:

$$D^{-(\mu-1/\rho)} \left\{ \int_0^z e_{\rho, \mu}(z-\xi; \lambda) f(\xi) d\xi \right\} = \int_0^z e_{\rho, \mu}(z-\xi; \lambda) f(\xi) d\xi, \quad (1.5)$$

$$z = re^{i\varphi}, \quad \xi = \tau e^{i\varphi}, \quad 0 < \tau < r < l < +\infty.$$

We note that these Lemmas for $z = x \in (0, +\infty)$ are proven in [4]. We remark also that these Lemmas are true for $\mu \geq 1 + 1/\rho$, but then $D^{(\mu-1/\rho)} f(z) =$

$$= \frac{d^P}{dz^P} \{D^{-(P+1/\rho-\mu)} f(z)\}, \quad \text{where } P \text{ is an integer satisfying } P-1 < \mu-1/\rho \leq P.$$

Lemma 1.5. Let $\alpha \in [0, 1)$, $1/\rho = 1 - \alpha$ ($\rho \geq 1$), $1/\rho \leq \mu < 1 + 1/\rho$, λ is an arbitrary parameter. Then in the class of functions satisfying $D^{(\mu-1/\rho)}y(re^{i\varphi}) \in L(0, l(\varphi))$, $D^{-\alpha}\{D^{(\mu-1/\rho)}y(re^{i\varphi})\} \in L(0, l(\varphi))$, $-\pi < \varphi \leq \pi$, the following of Cauchy type problem

$$\begin{aligned} (D^{1/\rho} + \lambda)D^{(\mu-1/\rho)}y(z) = 0, \quad D^{-\alpha}\{D^{(\mu-1/\rho)}y(z)\}\Big|_{z=0} = 0, \\ D^{-(1+1/\rho-\mu)}y(z)\Big|_{z=0} = 0 \end{aligned} \quad (1.6)$$

has a unique solution $y(z) \equiv 0$.

Proof. We put $D^{(\mu-1/\rho)}y(z) \equiv \tilde{y}(z)$. Then the problem (1.6), according to Theorem 2.1 [5], has a unique solution $\tilde{y}(z) \equiv 0$. Now we note that

$$D^{(\mu-1/\rho)}y(z) = 0. \quad (1.7)$$

If we apply the operator $\tilde{D}^{(\mu-1/\rho)}$ to (1.7), then, using (1.6), we will have $y(z) \equiv 0$.

Lemma 1.6. Let $\alpha \in [0, 1)$, $1/\rho = 1 - \alpha$ ($\rho \geq 1$), $1/\rho \leq \mu < 1 + 1/\rho$, λ is an arbitrary parameter, the function $f(re^{i\varphi}) \in L(0, l(\varphi))$ is continuous on $(0, l(\varphi))$, $-\pi < \varphi \leq \pi$. Then in the class of functions satisfying $D^{(\mu-1/\rho)}y(re^{i\varphi}) \in L(0, l(\varphi))$, $D^{1/\rho}\{D^{(\mu-1/\rho)}y(re^{i\varphi})\} \in L(0, l(\varphi))$ the following Cauchy type problem

$$\begin{aligned} (D^{1/\rho} + \lambda)D^{(\mu-1/\rho)}y(z) = f(z), \quad D^{-\alpha}\{D^{(\mu-1/\rho)}y(z)\}\Big|_{z=0} = 0, \\ D^{-(1+1/\rho-\mu)}y(z)\Big|_{z=0} = 0, \end{aligned} \quad (1.8)$$

has a unique solution

$$y(z) = \int_0^{\bar{z}} e_{\rho, \mu}(z - \xi; \lambda) f(\xi) d\xi, \quad z = re^{i\varphi}, \quad \xi = \tau e^{i\varphi}, \quad 0 < \tau < r < l < +\infty. \quad (1.9)$$

Proof. We put $D^{(\mu-1/\rho)}y(z) = \tilde{y}(z)$. Then we get the following Cauchy type problem

$$(D^{1/\rho} + \lambda)\tilde{y}(z) = f(z), \quad D^{-\alpha}\tilde{y}(z)\Big|_{z=0} = 0. \quad (1.10)$$

According to Theorem 2.3 from [5], the problem (1.10) has a unique solution

$$\tilde{y}(z) = \int_0^{\bar{z}} e_{\rho}(z - \xi; \lambda) f(\xi) d\xi. \quad (1.11)$$

We apply operator $\tilde{D}^{(\mu-1/\rho)}$ to (1.11). Then, by the Lemmas 1.2–1.4, we get

$$D^{-(\mu-1/\rho)}\tilde{y}(z) = D^{-(\mu-1/\rho)}D^{(\mu-1/\rho)}y(z) = D^{-(\mu-1/\rho)}\left\{\int_0^{\bar{z}} e_{\rho}(z - \xi; \lambda) f(\xi) d\xi\right\}, \quad \text{i.e.}$$

$$y(z) = \int_0^{\bar{z}} e_{\rho, \mu}(z - \xi; \lambda) f(\xi) d\xi.$$

§ 2. Main Results. Let sequences $\{\lambda_j\}_0^\infty$, $\{\rho_j\}_0^\infty$, $\{\mu_j\}_0^\infty$ satisfy the conditions

$$\rho_j \geq 1, \quad 1/\rho_j \leq \mu_j < 1 + 1/\rho_j, \quad 0 < \lambda_j < \lambda_{j+1}, \quad j = 0, 1, \dots \quad (2.1)$$

Consider the sequence of operators on an admissible class of functions $f(z)$,

$\{\Delta^{(n)}f(z)\}_0^\infty$, $\{\tilde{\Delta}^{(n)}f(z)\}_0^\infty$ defined by

$$\Delta^{(0)} f(z) \equiv f(z), \quad \Delta^{(n)} f(z) = \left\{ \prod_{j=0}^{n-1} \tilde{D}_j \right\} f(z), \quad n \geq 1, \quad (2.2)$$

$$\tilde{\Delta}^{(n)} f(z) \equiv D^{(-\alpha_n)} D^{(\mu_n - 1/\rho_n)} \Delta^{(n)} f(z), \quad n \geq 0, \quad (2.3)$$

where ($j \geq 0$)

$$\begin{aligned} \tilde{D}_j f(z) &\equiv (D^{1/\rho_j} + \lambda_j) D^{(\mu_j - 1/\rho_j)} f(z), \quad D^{-\alpha_j} f(z) \equiv \frac{1}{\Gamma(\alpha_j)_0} \int_0^z (z - \xi)^{\alpha_j - 1} f(\xi) d\xi, \\ D^{1/\rho_j} f(z) &\equiv \frac{d}{dz} D^{-\alpha_j} f(z), \quad \alpha_j = 1 - (1/\rho_j), \quad D^{(\mu_j - 1/\rho_j)} f(z) \equiv \frac{d}{dz} D^{-(1+1/\rho_j - \mu_j)} f(z). \end{aligned}$$

Consider the sequence of functions $\{\Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n)\}_0^\infty$ given by

$$\begin{aligned} \Omega_0(z; \{\lambda_0, \rho_0, \mu_0\}) &\equiv e_{\rho_0, \mu_0}(z; \lambda_0), \\ \Omega_1(z; \{\lambda_j, \rho_j, \mu_j\}_0^1) &\equiv \int_0^z e_{\rho_0, \mu_0}(z - t_1; \lambda_0) e_{\rho_1, \mu_1}(t_1; \lambda_1) dt_1, \\ \Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n) &\equiv \int_0^z e_{\rho_0, \mu_0}(z - t_1; \lambda_0) dt_1 \int_0^{t_1} e_{\rho_1, \mu_1}(t_1 - t_2; \lambda_1) dt_2 \times \\ &\quad \times \int_0^{t_{n-1}} e_{\rho_{n-1}, \mu_{n-1}}(t_{n-1} - t_n; \lambda_{n-1}) e_{\rho_n, \mu_n}(t_n; \lambda_n) dt_n, \quad n \geq 2, \end{aligned} \quad (2.4)$$

where $z = re^{i\varphi}$, $t_k = \tau_k e^{i\varphi}$ ($k = 0, 1, \dots, n$), $0 < \tau_n < \tau_{n-1} < \dots < \tau_1 < r < l < +\infty$,

$$e_{\rho_j, \mu_j}(z; \lambda_j) \equiv E_{\rho_j}(-\lambda_j z^{1/\rho_j}; \mu_j) z^{\mu_j - 1} = \sum_{k=0}^{\infty} \frac{(-\lambda_j)^k z^{k/\rho_j}}{\Gamma(\mu_j + k/\rho_j)} z^{\mu_j - 1}, \quad j = 0, 1, \dots$$

Note that for $\rho_j = \rho \geq 1$, $\mu_j = 1/\rho$, $j = 0, 1, \dots$, operators (2.2), (2.3) and the system of functions (2.4) were introduced in [1], and for $z = x \in (0, +\infty)$ – in [3]. For $\rho_j \geq 1$, $\mu_j \geq 1/\rho_j$, $j = 0, 1, \dots$, $z = x \in (0, +\infty)$ these systems were introduced in [4].

Lemma 2.1.

1⁰. For any $n \geq 0$ the following relations hold:

$$\Delta^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} = \tilde{\Delta}^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} \equiv 0, \quad k \geq n + 1, \quad (2.5)$$

$$\tilde{\Delta}^{(n)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} = E_{\rho_n}(-\lambda_n z^{1/\rho_n}; 1). \quad (2.6)$$

$$2^0. \quad \tilde{\Delta}^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} \Big|_{z=0} = 0, \quad 0 \leq k \leq n - 1. \quad (2.7)$$

$$3^0. \quad D^{-(1+1/\rho_k - \mu_k)} \Delta^{(k)} \{\Omega_n(z; \{\lambda_j; \rho_j; \mu_j\}_0^n)\} \Big|_{z=0} = 0, \quad 0 \leq k \leq n. \quad (2.8)$$

The parts 1⁰, 2⁰ of Lemma 2.1 can be proved in the same way as the Lemma 3.1 for $z = x \in (0, +\infty)$ (see [4]). So, we need to prove only (2.8). For $0 < k < n$ denote

$$\begin{aligned} \Omega_{k,n}(z; \{\lambda_j, \rho_j, \mu_j\}_k^n) &= \int_0^z e_{\rho_k, \mu_k}(z - t_{k+1}; \lambda_k) dt_{k+1} \int_0^{t_{k+1}} e_{\rho_{k+1}, \mu_{k+1}}(t_{k+1} - t_{k+2}; \lambda_{k+1}) dt_{k+2} \dots \times \\ &\quad \times \dots \times \int_0^{t_{n-1}} e_{\rho_{n-1}, \mu_{n-1}}(t_{n-1} - t_n; \lambda_{n-1}) e_{\rho_n, \mu_n}(t_n; \lambda_n) dt_n, \quad z = re^{i\varphi}, \quad t_{k+1} = \tau_{k+1} e^{i\varphi}, \dots, \\ t_n &= \tau_n e^{i\varphi}, \quad \tau_n < \tau_{n-1} < \dots < t_{k+1} < r < +\infty. \end{aligned}$$

$$\Omega_{0,n}(z; \{\lambda_j, \rho_j, \mu_j\}_0^n) \equiv \Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n), \quad \Omega_{n,n}(z; \{\lambda_n, \rho_n, \mu_n\}) \equiv e_{\rho_n, \mu_n}(z; \lambda_n).$$

Let $0 \leq k \leq n-1$. Using the definition of operator $\Delta^{(k)}$ and Lemma 1.6, we easily obtain

$$\Delta^{(k)} \{\Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n)\} = \Omega_{k,n}(z; \{\lambda_j, \rho_j, \mu_j\}_k^n). \quad (2.9)$$

Now applying the operator $D^{-(1+1/\rho_k - \mu_k)}$ to (2.15), we get

$$\begin{aligned} D^{-(1+1/\rho_k - \mu_k)} \Delta^{(k)} \{\Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n)\} &= D^{-(1+1/\rho_k - \mu_k)} \{\Omega_{k,n}(z; \{\lambda_j, \rho_j, \mu_j\}_k^n)\} = \\ &= \int_0^z e_{\rho_k, 1+1/\rho_k} (z-t_{k+1}; \lambda_k) \Omega_{k+1,n}(t_{k+1}; \{\lambda_j, \rho_j, \mu_j\}_{k+1}^n) dt_{k+1}. \end{aligned} \quad (2.10)$$

For $k=n$ we will have

$$D^{-(1+1/\rho_n - \mu_n)} \Delta^{(n)} \{\Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n)\} = E_{\rho_n}(-\lambda_n z^{1/\rho_n}; 1+1/\rho_n) z^{1/\rho_n}. \quad (2.11)$$

From (2.10), (2.11) the formula (2.8) follows.

Lemma 2.2. Let the function $f(z) \in L(0, l(\varphi))$ be continuous on $(0, l(\varphi))$, $0 < l < +\infty$. Then for any $n \geq 1$ the following formula holds

$$\begin{aligned} \int_0^z e_{\rho_0, \mu_0} (z-t_1; \lambda_0) dt_1 \int_0^{t_1} e_{\rho_1, \mu_1} (t_1-t_2; \lambda_1) dt_2 \dots \int_0^{t_{n-1}} e_{\rho_{n-1}, \mu_{n-1}} (t_{n-1}-t_n; \lambda_{n-1}) dt_n \int_0^{t_n} e_{\rho_n, \mu_n} (t_n-t_{n+1}; \lambda_n) f(t_{n+1}) dt_{n+1} = \\ = \int_0^z \Omega_n(z-t_1; \{\lambda_j, \rho_j, \mu_j\}_0^n) f(t_1) dt_1, \quad z = re^{i\varphi}, \quad t_k = \tau_k e^{i\varphi}, \quad k=0, 1, \dots, n+1, \quad \tau_{n+1} < \tau_n < \dots < \tau_1 < r < +\infty. \end{aligned}$$

Lemma 2.2 can be proved in the same way as Lemma 4.1 from [4] for $z = x \in (0, +\infty)$.

Lemma 2.3. For any $n \geq 0$ the coefficients $\{a_k\}_0^n$ of the sum

$$P_n(z) = \sum_{k=0}^n a_k \Omega_k(z; \{\lambda_j, \rho_j, \mu_j\}_0^k) \quad (2.12)$$

can be recovered by the formula $a_k = \tilde{\Delta}^{(k)} P_n(0)$, $k=0, 1, \dots, n$.

Proof. Let $0 \leq j \leq n-1$. Applying the operator $\tilde{\Delta}^{(j)}$ to function $P_n(z)$ and using (2.5)–(2.8), we obtain

$$\tilde{\Delta}^{(j)} P_n(z) = a_j E_{\rho_j}(-\lambda_j z^{1/\rho_j}; 1) + \sum_{k=j+1}^n a_k \Omega_k(z; \{\lambda_j, \rho_j, \mu_j\}_0^k). \quad (2.13)$$

But since $E_{\rho_j}(0; 1) = 1$, then from (2.13) and (2.7) we obtain $a_j = \tilde{\Delta}^{(j)} P_n(0)$,

$j=0, 1, \dots, n-1$. Now we apply the operator $\tilde{\Delta}^{(n)}$ to (2.12):

$$\tilde{\Delta}^{(n)} P_n(z) = a_n \tilde{\Delta}^{(n)} \{\Omega_n(z; \{\lambda_j, \rho_j, \mu_j\}_0^n)\} = a_n E_{\rho_n}(-\lambda_n z^{1/\rho_n}; 1), \text{ hence } a_n = \tilde{\Delta}^{(n)} P_n(0).$$

Lemma 2.3 is proved.

Lemma 2.4. Let $P_n(z) = \sum_{m=0}^n a_m \Omega_m(z; \{\lambda_j, \rho_j, \mu_j\}_0^m)$. Then for any $n \geq 0$

$$D^{-(1+1/\rho_k - \mu_k)} \Delta^{(k)} \{P_n(z)\} \Big|_{z=0} = 0, \quad k=0, 1, \dots, n. \quad (2.14)$$

We note that from (2.5)–(2.8) it is easy to obtain (2.14). Now denote by $C_{n+1} \{[0, l(\varphi)], \langle \lambda_j \rangle, \langle \rho_j \rangle, \langle \mu_j \rangle\}$ the set of functions $f(z)$ satisfying the following conditions:

- 1) the functions $\tilde{\Delta}^{(k)} f(z)$, $k = 0, 1, \dots, n$ are continuous on $[0, l(\varphi))$, $-\pi < \varphi \leq \pi$;
- 2) the functions $\Delta^{(k)} f(z)$ ($k = 0, 1, \dots, n+1$) are continuous on $[0, l(\varphi))$ and $\Delta^{(k)} f(z) \in L(0, l(\varphi))$;
- 3) $[D^{-(1+1/\rho_k - \mu_k)} \Delta^{(k)} f(z)]|_{z=0} = 0$, $k = 0, 1, \dots, n$.

(2.15)

Note that any function $P_n(z) = \sum_{k=0}^n a_k \Omega_k(z; \{\lambda_j, \rho_j, \mu_j\}_0^k)$ belongs to $C_{n+1} \{[0, l(\varphi)), < \lambda_j >, < \rho_j >, < \mu_j >\}$.

Theorem. If $f(z) \in C_{n+1} \{[0, l(\varphi)), < \lambda_j >, < \rho_j >, < \mu_j >\}$, then

$$f(z) = \sum_{k=0}^n \tilde{\Delta}^k f(0) \Omega_k(z; \{\lambda_j, \rho_j, \mu_j\}_0^k) + R_n(z), \quad (2.16)$$

where

$$R_n(z) = \int_0^z \Omega_n(z - \xi; \{\lambda_j, \rho_j, \mu_j\}_0^n) \Delta^{(n+1)} f(\xi) d\xi, \quad (2.17)$$

$$z \in (0, l(\varphi)), \quad -\pi < \varphi \leq \pi, \quad z = re^{i\varphi}, \quad \xi = \tau e^{i\varphi}, \quad 0 < \tau < r < +\infty.$$

Proof. Let $P_n(z) = \sum_{k=0}^n \tilde{\Delta}^{(k)} f(0) \Omega_k(z; \{\lambda_j, \rho_j, \mu_j\}_0^k)$, $R_n(z) = f(z) - P_n(z)$.

We note that according to Lemmas 2.1, 2.3 and 2.4, from (2.15) we deduce that function $R_n(z)$ satisfies the following conditions:

$$\Delta^{(n+1)} R_n(z) = \Delta^{(n+1)} f(z) \equiv \psi(z), \quad z \in (0, l(\varphi)), \quad (2.18)$$

$$[\tilde{\Delta}^{(k)} R_n(z)]|_{z=0} = 0, \quad k = 0, 1, \dots, n, \quad (2.19)$$

$$[D^{-(1+1/\rho_k - \mu_k)} R_n(z)]|_{z=0} = 0, \quad k = 0, 1, \dots, n. \quad (2.20)$$

Using the definition of operator $\Delta^{(n+1)}$ we can write (2.18) in the form

$$(D^{1/\rho_n} + \lambda_n) D^{(\mu_n - 1/\rho_n)} \Delta^{(n)} R_n(z) = \psi(z). \quad (2.18')$$

Note that the function $\Delta^{(n)} R_n(z)$ satisfies the conditions of Lemma 1.6, so, the function $\Delta^{(n)} R_n(z)$ is uniquely determined from (2.18') by means of the integral formula

$$\Delta^{(n)} R_n(z) = \int_0^z e_{\rho_n, \mu_n}(z - t_{n+1}; \lambda_n) \psi(t_{n+1}) dt_{n+1}, \quad z = re^{i\varphi}, \quad t_{n+1} = \tau_{n+1} e^{i\varphi}, \quad 0 < \tau_{n+1} < r < l < +\infty.$$

Further we consider the equation

$$(D^{1/\rho_{n-1}} + \lambda_{n-1}) D^{(\mu_{n-1} - 1/\rho_{n-1})} \Delta^{(n-1)} R_n(z) = \int_0^z e_{\rho_n, \mu_n}(z - t_{n+1}; \lambda_n) \psi(t_{n+1}) dt_{n+1}.$$

Using Lemma 1.6 and (2.19), (2.20), we get

$$\Delta^{(n-1)} R_n(z) = \int_0^z e_{\rho_{n-1}, \mu_{n-1}}(z - t_n; \lambda_{n-1}) dt_n \int_0^{t_n} e_{\rho_n, \mu_n}(t_n - t_{n+1}; \lambda_n) \psi(t_{n+1}) dt_{n+1}.$$

Repeating our argument successively exhausting all the initial conditions (2.19), (2.20), we arrive at the identity

$$\begin{aligned}
R_n(z) &= \int_0^z e_{\rho_0, \mu_0}(z-t_1; \lambda_0) dt_1 \int_0^{t_1} e_{\rho_1, \mu_1}(t_1-t_2; \lambda_1) dt_2 \times \\
&\times \int_0^{t_n} e_{\rho_n, \mu_n}(t_n-t_{n+1}; \lambda_n) \psi(t_{n+1}) dt_{n+1} = \int_0^z \Omega_n(z-t_1; \{\lambda_j, \rho_j, \mu_j\}_0^n) \Delta^{(n+1)} f(t_1) dt_1, \\
z &= re^{i\varphi}, \quad t_j = \tau_j e^{i\varphi}, \quad \tau_{n+1} < \tau_n < \tau_1 < r < l < +\infty.
\end{aligned}$$

Theorem is proved.

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Բ. Հ. Մահակյան. Թեյլորի–Մակլորենի տիպի բանաձևերի ընդհանուր դասեր կոմպլեքս տիրույթում Էջ. 20–26

Ներկա աշխատանքում ընդհանրացվել են [1]-ում բերված արդյունք-ները, ներմուծվել են օպերատորների ավելի ընդհանուր համակարգեր, որոնք ծնվում են Ռիմանի–Լիուվիլի ինտեգրալով և ածանցյալով: Ներմուծվել են նաև ֆունկցիաների ընդհանուր համակարգեր, որոնք ծնվում են Միթագ–Լեֆլերի տիպի ֆունկցիաներով: