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M a t h e m a t i c s

MÖBIUS-INVARIANT DIVISORS FOR THE SPACE A^p_α

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In the paper we introduce new Möbius-invariant and efficient divisors for A^p_α spaces. The method of construction of new divisors is shown.

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Introduction. Let B be a Banach space of analytic functions in the open unit disk $D=\{z\in\mathbb{C}:|z|<1\}$. For a sequence $\{a_k\}_1^\infty\subset D$ let

$$B_{\{a_k\}}=\{f\in B: f(a_k)=0,\ k=1,2,....\},$$

$\{a_k\}_1^\infty$ is called a B-zero set, if $B_{\{a_k\}}\neq\{0\}$.

An analytic function $g_a(z)$ in D is called an a-divisor for B, if $g_a(z)$ has a single simple zero at a, and the divisor operator $T_a: B_{\{a\}}\to B$ defined by

$$(T_a f)(z)=f(z)\big/g_a(z)$$

is bounded.

The Blaschke factor for a generated by a point $a\neq 0$ in D is defined as

$$b_a(z)=\frac{|a|}{a}\frac{a-z}{1-\bar{a}z},\quad z\in D,$$

and for $a=0$ we set $b_0(z)=z$.

A family of divisors $g_a(z)$, $a\in D$, is called Möbius invariant, if $g_a(z)=$ $=g_0(b_a(z))$.

A family of Möbius invariant divisors $g_a(z)$ $(a\in D)$ is called B-efficient, if it has the following properties:

a) for every B-zero set $\{a_k\}$ the product $g_{\{a_k\}}(z)=\prod\limits_k g_{a_k}(z)$ converges absolutely on D and uniformly on compact subsets of D,

b) the divisor operator $T_{\{a_k\}}$ $\left(T_{\{a_k\}}f\right)(z)=f(z)\big/g_{\{a_k\}}(z)$ maps $B_{\{a_k\}}$into B and $\left\|T_{\{a_k\}}\right\|\leq C$, where C is the same constant for all B-zero sets $\{a_k\}$.

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Let the normalized area measure in D be denoted by $d\sigma$:

$$d\sigma(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta, \ \ z = x+iy = re^{i\theta}.$$

For $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the Djrbashian space A^p_α [1,2] is the space of analytic functions in $L^p(D, d\sigma_\alpha)$, where $d\sigma_\alpha(z) = (\alpha+1)\left(1-|z|^2\right)^\alpha d\sigma(z)$.

If $f \in L^p(D, d\sigma_\alpha)$, we write $\|f\|_{p,\alpha} = \left(\int_D |f(z)|^p d\sigma(z)\right)^{\frac{1}{p}}$. For $1 \le p < +\infty$ the space $L^p(D, d\sigma_\alpha)$ is a Banach space with the above norm; and for $0 < p < 1$ the space is a metric space with the metric defined by

$$d(f,g) = \|f-g\|^p_{p,\alpha}.$$

We define [3,4] the class Φ of analytic functions φ in the unit disc D, satisfying $\varphi(0) = 1$ and such that the integrals converge $\int\limits_1^z \frac{\varphi(t)}{t}dt$ for every $0 < |z| < 1$, where the integral is taken along the contours in D with endpoints 1 and z that do not pass through zero.

For $\varphi \in \Phi$ we define

$$b_a^{(\varphi)}(z) = \exp\left\{\int\limits_1^{b_a(z)} \frac{\varphi(t)}{t}dt\right\} = b_a(z)\exp\left\{\int\limits_1^{b_a(z)} \frac{\varphi(t)-1}{t}dt\right\},$$

where $z, a \in D$ and integrals are taken along the contours in D with endpoints 1 and $b_a(z)$ that do not pass through zero for $z \ne a$.

In the case $\varphi(t) = (1-t)^\beta$ $(0 < \beta < +\infty)$, the function $b_a^{(\varphi)}(z) \equiv b_a^{(\beta)}(z)$ is the elementary factor of M.M. Djrbashian's infinite products [1,2].

Note that for $[m = 1,2,...]$ we have

$$b_a^{(m)}(z) = b_a(z)\exp\left\{1-b_a(z)+\frac{(1-b_a(z))^2}{2}+...+\frac{(1-b_a(z))^m}{m}\right\}.$$

In the case $\varphi(t) = \dfrac{2(1-t)}{2-t}$ the function $b_a^{(\varphi)}(z) \equiv h_a(z) = b_a(z)(2-b_a(z))$ is the elementary factor of Horowitz's infinite products [5].

For $\varphi(t) = \left(\dfrac{1-t}{1+t}\right)^2$ the function

$$b_a^{(\varphi)}(z) \equiv q_a(z) = b_a(z)\exp\left\{\frac{2(1-b_a(z))}{1+b_a(z)}\right\}$$

is the elementary factor of Korenblum's infinite products [6].

In the case $\varphi(t) = \dfrac{1-t}{1+t}$ we have $b_a^{(\varphi)}(z) \equiv p_a(z) = \dfrac{4b_a(z)}{(1+b_a(z))^2}$.

In general, one can write the functions $b_a^{(\varphi)}(z)$ in the form

$$b_a^{(\varphi)}(z) = \exp\left\{-(1-b_a(z))\int\limits_0^1 \frac{\varphi(1-x(1-b_a(z)))\,dx}{1-(1-b_a(z))x}\right\}. \tag{1}$$

Now for some $\beta > 0$ let

$$|\varphi(t)| = O\left(|1-t|^{\beta}\right), \quad t \to 1,\ |t| < 1. \tag{2}$$

It follows from (1) that

$$\left|\log b_a^{(\varphi)}(z)\right| \le O(1)\frac{|1-b_a(z)|^{1+\beta}}{1-|1-b_a(z)|}. \tag{3}$$

If $a \in D$, $|a| > \dfrac{1+r}{2}$, $|z| \le r < 1$, we get

$$|1-b_a(z)| = \frac{1-|a|^2}{|1-\overline{a}z|} \le \frac{2}{1-r}(1-|a|) < 1. \tag{4}$$

Now from (1)–(4) and taking into account the multiplicity of a_k, we get:

1) if $A = \{a_k\}_1^{\infty}$ is a sequence of points in D with $\sum\limits_{k=1}^{\infty}(1-|a_k|)^{\beta+1} < +\infty$, then the Djrbashian's product $B_{\{a_k\}}^{(\beta)}(z) = \prod\limits_{k=1}^{\infty} b_{a_k}^{(\beta)}(z)$ converges uniformly on every compact subset of D, and the zero set of $B_{\{a_k\}}^{(\beta)}(z)$ is exactly A;

2) if $A = \{a_k\}_1^{\infty}$ is a sequence of points in D with $\sum\limits_{k=1}^{\infty}(1-|a_k|)^2 < +\infty$, then the Horowitz's product $H_{\{a_k\}}(z) = \prod\limits_{k=1}^{\infty} h_{a_k}(z)$ converges uniformly on every compact subset of D, and the zero set of $H_{\{a_k\}}(z)$ is exactly A;

3) if $A = \{a_k\}_1^{\infty}$ is a sequence of points in D with $\sum\limits_{k=1}^{\infty}(1-|a_k|)^3 < +\infty$, then the Korenblum's product $Q_{\{a_k\}}(z) = \prod\limits_{k=1}^{\infty} q_{a_k}(z)$ converges uniformly on every compact subset of D, and the zero set of $Q_{\{a_k\}}(z)$ is exactly A;

4) if $A = \{a_k\}_1^{\infty}$ is a sequence of points in D with $\sum\limits_{k=1}^{\infty}(1-|a_k|)^2 < +\infty$, then the product $P_{\{a_k\}}(z) = \prod\limits_{k=1}^{\infty} p_{a_k}(z)$ converges uniformly on every compact subset of D, and the zero set of $P_{\{a_k\}}(z)$ is exactly A.

Lemma 1. Let $f \in A_{\alpha}^{p}$ satisfy $f(0) \ne 0$, and let $A = \{a_k\}_1^{\infty}$ be the sequence of its zeros, counted according to their multiplicity. If a function φ satisfies (2) and decreases in $(0;1)$, then there exists a positive constant $C = C(p,\alpha,\varphi)$ such that $\dfrac{|f(0)|}{B_{\{a_k\}}^{(\varphi)}(0)} \le C\|f\|_{p,\alpha}$, where $B_{\{a_k\}}^{(\varphi)}(z) = \prod\limits_{k=1}^{\infty} b_{a_k}^{(\varphi)}(z)$.

Proof. We can assume $f(0) = 1$. Let $n = n_f$ be the usual zero counting functions and $N = N(r) = \int\limits_0^r \dfrac{n(t)}{t}dt$.

If $f \in A_{\alpha}^{p}$, then there exists a positive constant C such that for all $r \in (0,1)$

$$n(r) \le \frac{C}{1-r}\log\frac{1}{1-r}, \tag{5}$$

$$N(r) \le C + \frac{\alpha+1}{p} \log \frac{1}{1-r}. \tag{6}$$

If $\{a_k\}_1^\infty$ is a zero set for some $f \in A^p_\alpha$, then for every $\beta > 0$

$$\sum_{k=1}^{\infty} (1-|a_k|)^{\beta+1} < +\infty. \tag{7}$$

For the proof of (5)–(7) see [7, 8].

We consider the expression

$$S = \sum_{k=1}^{\infty} \log \frac{1}{b_{a_k}^{(\varphi)}(0)} = -\sum_{k=1}^{\infty} \int_1^{|z_k|} \varphi(t) \frac{dt}{t} = -\int_0^1 \int_1^r \varphi(t) \frac{dt}{t} dn(r).$$

Using (5), (6) and twice integrating by parts, we will get

$$S = \int_0^1 \varphi(t) dN(r) = -\int_0^1 \varphi'(t) N(r)\, dr.$$

Since $f(0) = 1$, Jensen's formula gives $N(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \left| f\left(re^{i\theta}\right)\right| d\theta$. It follows that

$$Sp = -\int_D \log |f(z)|^p \varphi'(|z|)\, d\sigma(z) = -\int_0^1 \varphi'(r) \log \frac{1}{(1-r^2)^\alpha} dr +$$
$$+ \int_D \log\left(|f(z)|^p \left(1-|z|^2\right)^\alpha \right) d\mu(z),$$

where

$$d\mu(z) = -\frac{\varphi'(|z|)}{|z|} d\sigma(z). \tag{8}$$

(8) is a probability measure on D, because $-\int_D \frac{\varphi'(|z|)}{|z|} d\sigma(z) = -\int_0^1 \varphi'(r)\, dr = 1$.

Then we apply the arithmetic-geometric mean inequality

$$\int_D \log\left(|f(z)|^p \left(1-|z|^2\right)\right) d\mu(z) \le \log\left(\int_D |f(z)|^p \left(1-|z|^2\right) d\mu(z) \right),$$

note that $0 < -\int_0^1 \varphi'(r) \log \frac{1}{(1-r^2)^\alpha} dr = 2\alpha \int_0^1 \frac{\varphi(r)}{1-r^2} r dr < +\infty$ and using (6) we obtain the desired result.

Lemma 2. Let $|z| < 1$. Then

1) the inequality

$$\left| \exp\left(1 - z + \frac{(1-z)^2}{2} + \ldots + \frac{(1-z)^k}{k}\right)\right| \ge$$
$$\ge \exp\left(1 - |z| + \frac{(1-|z|)^2}{2} + \ldots + \frac{(1-|z|)^k}{k}\right), \tag{9}$$

takes hold for $k=1,2$ and is not true for $k=3,4$;

2) $\operatorname{Re}\frac{1-z}{1+z}\geq\frac{1-|z|}{1+|z|}$.

Proof. 1) For $|z|<1$, we have

$$|\exp(1-z)|=\exp(1-\operatorname{Re}z)\geq\exp(1-|z|), \tag{10}$$

$$\left|\exp\left(1-z+\frac{(1-z)^2}{2}\right)\right|=\exp\frac{1}{2}\left(\operatorname{Re}\left(z^2-4z+3\right)\right)\geq$$

$$\geq\exp\frac{1}{2}\left(|z|^2-4|z|+3\right)=\exp\left(1-|z|+\frac{(1-|z|)^2}{2}\right). \tag{11}$$

In the case $k=3$ we have

$$1-z+\frac{(1-z)^2}{2}+\frac{(1-z)^3}{3}=\frac{1}{6}\left(-2z^3+9z^2-18z+11\right),$$

and the inequality (9) is equivalent to

$$2r^2(1-\cos 3\theta)-9r(1-\cos 2\theta)+18(1-\cos\theta)\geq 0, \tag{12}$$

where $z=re^{i\theta}$, $0<r<1$.

The inequality (12) is not true, for example, when $\theta=\frac{\pi}{3}$ and $r=0.95$.

In the case $k=4$ easy computation shows that

$$1-z+\frac{(1-z)^2}{2}+\frac{(1-z)^3}{3}+\frac{(1-z)^4}{4}=\frac{3z^4-16z^3+36z^2-48z+25}{12}$$

Let $z=ir$, then $\operatorname{Re}\left(3z^4-16z^3+36z^2-48z+25\right)=3r^4-36r^2+25$ and the inequality $3r^4-36r^2\geq 3r^4-16r^3+36r^2-48r$ doesn't holds when $r=0.9$.

2) If $|z|<1$, then we have

$$\operatorname{Re}\frac{1-z}{1+z}=\frac{1-|z|^2}{1+2\operatorname{Re}z+|z|^2}\geq\frac{1-|z|^2}{1+2|z|+|z|^2}=\frac{1-|z|}{1+|z|}.$$

Theorem . Let $f\in A_\alpha^p$ has a zero set $A=\{a_k\}_1^\infty$. If a decreasing in $(0;1)$ function φ satisfies (2), and

$$\left|b_a^{(\varphi)}(z)\right|\geq\exp\left\{\int_1^{|b_a(z)|}\frac{\varphi(t)}{t}dt\right\},\quad |z|<1, \tag{13}$$

then there exists a positive constant $C=C(p,\alpha,\varphi)$ such that

$$\left\|f\big/B_{\{a_k\}}^{(\varphi)}\right\|\leq C\|f\|_{p,\alpha},\ \text{ where }\ B_{\{a_k\}}^{(\varphi)}(z)=\prod_{k=1}^{\infty}b_{a_k}^{(\varphi)}(z). \tag{14}$$

Proof. For every $w\in D\backslash Z$ let $f_w=f\circ\varsigma_w$, where $\varsigma_w(z)=\frac{w-z}{1-\overline{w}z}$, $z\in D$.

Then $f\in A_\alpha^p$ and its zero set is $\{\varphi_w(a_k)\}_1^\infty$, which does not contain 0. Fix any $\gamma>\alpha$, and apply Lemma 1 to the function f_w. Then there exists a positive constant $C = C(p,\gamma,\varphi)$ such that $\dfrac{|f(w)|}{\prod\limits_{k=1}^{\infty} b^{(\varphi)}_{\varsigma_w(z_k)}(0)} \le C\|f_w\|_{p,\gamma}$.

Since by (13) $\prod\limits_{k=1}^{\infty} b^{(\varphi)}_{\varsigma_w(z_k)}(0) = \prod\limits_{k=1}^{\infty} \exp\left\{\int\limits_1^{|\varsigma_w(z_k)|} \frac{\varphi(t)}{t}dt\right\} \le \left|B^{(\varphi)}_{\{a_k\}}(w)\right|$ for all $w \in D\backslash Z$,

we obtain $\left|\dfrac{f(w)}{B^{(\varphi)}_{\{a_k\}}(w)}\right|^p \le C^p(\gamma+1)\int\limits_D |f_w(z)|^p\left(1-|z|^2\right)^\gamma d\sigma(z) =$

$$= C^p(\gamma+1)\int\limits_D |f_w(z)|^p \frac{\left(1-|z|^2\right)^\gamma\left(1-|w|^2\right)^{\gamma+2}}{|1-\overline{w}z|^{2\gamma+4}} d\sigma(z).$$

By continuity, the above inequality also holds for other w in D. Now we obtain the norm estimate (14) arguing as in [7] (Theorem 1.7) or in [8] (Theorem 3.6).

It follows from Lemma 2 that functions $b^{(\beta)}_a$ $(\beta = 1,2)$, h, q and p satisfy the condition (13).

Corollary. The families $\left\{b^{(\beta)}_a\right\}$ $(\beta = 1,2)$, $\{h_a(z)\}, \{q_a(z)\}$, $\{p_a(z)\}$ are Möbius invariant and A^p_α-efficient families of divisors.

Remark. In [7] it is proved that the system $\{h_a(z)\}$ is a Möbius invariant and A^p_α-efficient family of divisors, and in [5] it is proved that $\{q_a(z)\}$ is a Möbius invariant and A^2_0-efficient family of divisors (with $C = 1$).

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