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Existence of Maximum Entropy Problem Solution in a General N-Dimensional Case

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Abstract

In the following paper, we will define conditions, which need to be satisfied in order for the maximum entropy problem applied in European call options to have a solution in a general n-dimensional case. We will also find a minimum right boundary for the price range in order to have at least one risk neutral measure satisfying the option pricing formula. The results significantly reduce the computational time of optimization algorithms used in maximum entropy problem.
Keywords: Entropy, Boundary, Distribution, Options.

1. Introduction

The maximum entropy methodology has recently started to become a quite popular tool with a huge potential of application in different fields [1, 2, 3, 4]. The core of the theory is based on Shannon's classical definition of information entropy [5], which is a crucial foundation in information theory. The maximum entropy approach has been broadly studied for its application in finance and financial extrapolation [6], and there have been significant contributions to its development since then, including the application of Legendre transforms [7], partially finite convex programming [8], the employment of risk neutral moments [9], as well as the application of the problem as a non-parametric approach in American options pricing [10]. By theory, in the discrete case, the price of a European call option should be equal to the mathematical expectation of future pay-offs' discounted value, thus lying in their convex hull. In reality, actual market prices may be biased from the theoretical ones [11] and lie out of the convex hull. We will concentrate on the derivation of conditions for the existence of solution which will not only reduce the computational time but will also result in an automated distribution recovery process [12, 13, 14] and will later allow us to develop algorithmic trading strategies that train on huge data sets.

2. Outline of the Problem

Consider having European call options for n different strike prices. Let us denote the vector of strike prices with K and the vector of future states with X. X and K needn't be of the same dimension. Maximum entropy methodology seeks a risk neutral probability measure p, such that

$$Ap = b, \tag{1}$$

$$\sum_{i=1}^{n} p_i = 1, \quad p_i \geq 0, \tag{2}$$

$$S(p) = \sum_{i=1}^{n} p_i ln(p_i) \quad \text{is maximal}, \tag{3}$$

where b is the vector of current option prices' future values for each strike and A is

$$\begin{bmatrix} (X_1 - K_1)^+ & (X_2 - K_1)^+ & \dots & (X_n - K_1)^+ \\ \vdots & \vdots & \ddots & \vdots \\ (X_1 - K_n)^+ & (X_2 - K_n)^+ & \dots & (X_n - K_n)^+ \end{bmatrix}, \tag{4}$$

where $(x)^+ = max(x, 0)$. The probability vector p and the vector of future states X have the same dimension, in fact p_i is the probability mass assigned to the future state X_i. The distribution of future states will change as we change the state vector X. The question that interests us is what kind of state vector should be considered in order for a probability measure satisfying (1), (2) to exist in the first place. It is obvious that the greater the number of A's linearly independent columns is, the bigger will their convex hull be, and so the more p vectors may exist satisfying (1), (2). So first of all we will consider the state vector $(K_1, \dots, K_n, K_n + t)$ for some arbitrary t. Matrix A will now have the form below.

$$\begin{bmatrix} 0 & K_2 - K_1 & \dots & K_n - K_1 & K_n - K_1 + t \\ 0 & 0 & \dots & K_n - K_2 & K_n - K_2 + t \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & K_n - K_{n-1} & K_n - K_{n-1} + t \\ 0 & 0 & \dots & 0 & t \end{bmatrix}. \tag{5}$$

We will denote A's columns by $a_0, a_1, \dots, a_n$. Let $\alpha(t)$ denote the angle between a_n and I, where I is the unit vector $(1, \dots, 1)$. It is easy to show that $\lim_{t\to\infty} \cos\alpha(t) = 1$, so in order to see if any t exists, s.t. (1), (2) are satisfied, we will consider I instead of a_n, assuming that the angle between b and I isn't 0 (this assumption holds throughout the text). Let's consider the following $n+1$ hyperplane - vector pairs (we denote hyperplanes by $hp(\cdot)$).

$$\begin{cases} hp(a_1, a_2, \dots, a_{n-1}, I), & a_0 \\ hp(a_0, a_2, \dots, a_{n-1}, I), & a_1 \\ \vdots & \\ hp(a_0, a_1, \dots, a_{n-2}, I), & a_{n-1} \\ hp(a_0, a_1, \dots, a_{n-2}, a_{n-1}), & I \end{cases}. \tag{6}$$

For each hyperplane above, we will denote by N_i its normal "pointing" in the direction of the associated vector a_i (note that $a_0 = (0, \dots, 0)$)

$$\begin{cases} \langle N_0 - a_1, a_0 - a_1 \rangle \geq 0 \\ \langle N_1, a_1 \rangle \geq 0 \\ \vdots \\ \langle N_n, a_n \rangle \geq 0 \end{cases}, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar (dot) product.

3. Existence of Solution

The following proposition is obvious.

Proposition 1. *There exists a finite t, s.t. (1), (2) are satisfied if and only if the following inequalities take place.*

$$\begin{cases} \langle N_0 - a_1, b - a_1\rangle \geq 0 \\ \langle N_1, b\rangle \geq 0 \\ \vdots \\ \langle N_n, b\rangle \geq 0 \end{cases} . \tag{8}$$

We now proceed to finding a minimal value for t, s.t. conditions (1) and (2) are satisfied. b represents the vector of prices and, thus its components are non-negative. Assume that (8) takes place. If the last component of b, b_n is 0, then the minimal value of t for which $b \in conv(a_0, a_1, \dots, a_{n-1}, a_n)$ is 0 ($conv(\cdot)$ denotes the convex hull). In case b_n is greater than 0, we will use the following lemmas (note that $a_n = a_{n-1} + tI$).

Lemma 1. *$\exists \mu > 0$, s.t. $\forall t$ for which $b \in conv(a_0, \dots, a_{n-1}, a_n)$, $t \geq \mu > 0$.*

Proof. Assume the opposite, then $\forall \epsilon > 0$ $\exists t_0 < \epsilon$, s.t. $\exists \gamma_0, \dots, \gamma_n$, $\gamma_i \geq 0$, $\sum_{i=0}^{n} \gamma_i = 1$, for which $\gamma_0 a_0 + \dots + \gamma_n a_n = b$. Let

$$r = \inf_{q \in conv(a_0, \dots, a_{n-1})} \rho(b, q),$$

$$\epsilon = \frac{r}{\rho(b, I)},$$

where ρ is the Euclidean distance. For the ϵ above there exists $0 < t_0 < \epsilon$, s.t. $\gamma_0 a_0 + \dots + \gamma_n a_n = b \Leftrightarrow \gamma_0 a_0 + \dots + (\gamma_{n-1} + \gamma_n) a_{n-1} + t_0 \gamma_n I = b \Rightarrow \rho(\gamma_0 a_0 + \dots + (\gamma_{n-1} + \gamma_n) a_{n-1}, b) = \rho(b - t_0 \gamma_n I, b) \leq t_0 \rho(I, b) < r$, resulting in a contradiction. ∎

Lemma 2. *If for some t_0 $b \in conv(a_0, \dots, a_n)$, then this also holds for any $t > t_0$.*

Proof. Let $t > t_0$, $\gamma'_{n-1} = \gamma_{n-1} + \gamma_n \frac{t - t_0}{t}$, $\gamma'_n = \gamma_n \frac{t_0}{t}$, then $\gamma'_{n-1} + \gamma'_n = \gamma_{n-1} + \gamma_n$ and $\gamma_0 a_0 + \dots + \gamma'_{n-1} a_{n-1} + \gamma'_n a_n = \gamma_0 a_0 + \dots + \gamma_n a_n = b$ ∎

We now know that the set T of all possible t's for which $b \in conv(a_0, \dots, a_n)$ is bounded from below by a positive number and unbounded from above. The next lemma proves that for $\underline{t} = \inf T$ b is again in the convex hull $conv(a_0, \dots, a_n)$.

Lemma 3. *Let T be the set of all t's, s.t. $b \in conv(a_0, \dots, a_n)$, then $\underline{t} = \inf T \in T$.*

Proof. Let's assume the opposite. As $\underline{t}$ is the infimum of T, then for $\forall \epsilon > 0$ $\exists t_0 \in T$, s.t. $0 < t_0 - \underline{t} < \epsilon$. Let

$$r = \inf_{q \in conv(a_0, \dots, a_{n-1}, a_{n-1} + \underline{t} I)} \rho(b, q),$$

$$\epsilon = \frac{r}{\rho(b, I)}.$$

By the assumption there exists a $t_0 < \underline{t} + \epsilon$, s.t. $b = \gamma_0 a_0 + \ldots + \gamma_n a_n$. Let $a_n^{'} = a_{n-1} + \underline{t}I$, then $\rho(\gamma_0 a_0 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n a_n^{'}, b) = \rho(b + (\underline{t} - t_0)I, b) \leq (t_0 - \underline{t})\rho(I, b) < r$, resulting in a contradiction. ∎

Based on the lemmas we may now formulate the main theorem of the article.

Theorem 1. *If condition (8) is satisfied, the angle between b and I isn't 0, then $b \in conv(a_0, \ldots, a_n)$, where $a_n = a_{n-1} + \underline{t}I$ and $\gamma_{n-1} = 0$ in the linear representation of b by vectors $a_0, \ldots a_n$. The minimal value of t, $\underline{t}$ is given by*

$$\underline{t} = \frac{b_n(K_n - K_{n-1})}{b_{n-1} - b_n}. \tag{9}$$

Proof. We only need to show that $\gamma_{n-1} = 0$. Assume it's not, then

$$b = \gamma_0 a_0 + \ldots + \gamma_n a_n = \gamma_0 a_0 + \ldots + \gamma_{n-1} a_{n-1} + \gamma_n (a_{n-1} + \underline{t}I) =$$

$$\gamma_0 a_0 + \ldots + (\gamma_n + \gamma_{n-1}) a_{n-1} + \frac{\underline{t}\gamma_n}{\gamma_n + \gamma_{n-1}} (\gamma_n + \gamma_{n-1}) I =$$

$$\gamma_0 a_0 + \ldots + (\gamma_n + \gamma_{n-1})(a_{n-1} + \frac{\underline{t}\gamma_n}{\gamma_n + \gamma_{n-1}} I).$$

As $\gamma_n > 0$, then

$$\frac{\underline{t}\gamma_n}{\gamma_n + \gamma_{n-1}} < \underline{t},$$

Which is a contradiction. Having known that a_{n-1} doesn't "participate" in the linear representation of b, we only need to find the value of $\underline{t}$, s.t. $b \in hp(a_0, \ldots, a_{n-2}, a_n)$. For that we will find the normal N of the hyperplane and solve $\langle N, b \rangle = 0$ for t. We find N by observing the determinant of the following matrix based on the vectors from the hyperplane.

$$\begin{bmatrix} K_2 - K_1 & 0 & \ldots & \ldots & 0 \\ K_3 - K_1 & K_3 - K_2 & 0 & \ldots & 0 \\ \vdots & & & & \\ K_{n-1} - K_1 & K_{n-1} - K_2 & \ldots & 0 & 0 \\ K_n - K_1 + t & K_n - K_2 + t & \ldots & K_n - K_{n-1} + t & t \\ e_1 & e_2 & \ldots & e_{n-1} & e_n \end{bmatrix}. \tag{10}$$

The determinant is

$$(-1)^{2n-1} t (K_2 - K_1) \ldots (K_{n-1} - K_{n-2}) e_{n-1} +$$

$$(-1)^{2n} (K_n - K_{n-1} + t)(K_2 - K_1) \ldots (K_{n-1} - K_{n-2}) e_n.$$

So $N = (0, \ldots, 0, -t, K_n - K_{n-1} + t)$, and therefore

$$\langle N, b \rangle = 0 \Leftrightarrow t = \frac{b_n(K_n - K_{n-1})}{b_{n-1} - b_n}.$$

∎

4. Conclusion

As a result we obtained a way of checking whether a solution to the maximum entropy problem applied in European call options exists, before starting the optimization. If (8) takes place, then in order for the solution to exist, the right bound of future states vector must be greater than or equal to the value of $\underline{t}$ described in the theorem above. Checking the existence of solution prevents the user from unknowingly proceeding to the stage of entropy maximization over an empty set of discrete probability distributions, which would yield unpredictable results.

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Ընդհանուր N-չափանի դեպքում առավելագույն էնտրոպիայի խնդրի լուծման գոյությունը

Ռ. Գևորգյան և Ն. Մարգարյան

Ամփոփում

Հետևյալ աշխատանքում կսահմանենք պայմաններ, որոնց բավարարվածությունն անհրաժեշտ է ընդհանուր n-չափանի դեպքում Եվրոպական օպցիոններում կիրառվող առավելագույն էնտրոպիայի խնդրի լուծման գոյության համար: Նաև կգտնենք գնային միջակայքի նվազագույն աջակողմյան սահմանը՝ օպցիոնների գնագոյացման բանաձևին բավարարող առնվազն մեկ ռիսկից չեզոք չափի գոյության համար: Ստացված արդյունքները զգալիորեն նվազեցնում են առավելագույն էնտրոպիայի խնդրի մեջ օգտագործվող օպտիմիզացիոն ալգորիթմների հաշվարկային ժամանակը:

Существование решения проблемы максимальной энтропии в общем N-мерном случае

Р. Геворгян и Н. Маргарян

Аннотация

В следующей статье мы определим условия, выполнение которых необходимо для существования решения проблемы максимальной энтропии, применяемой в Европейских опционах, в общем n-мерном случае. Мы также найдем минимальную правую границу для ценового диапазона, которая необходима для существования хотя бы одной риск-нейтральной меры удовлетворяющей формуле ценообразования опционов. Полученные результаты значительно уменьшают вычислительное время оптимизационных алгоритмов, используемых в задаче максимальной энтропии.