ISSN 1068-3623, Journal of Contemporary Mathematical Analysis, 2015, Vol. 50, No. 5, pp. 253–257. © Allerton Press, Inc., 2015.
Original Russian Text © A. G. Gasparyan, V. K. Ohanyan, 2015, published in Izvestiya NAN Armenii. Matematika, 2015, No. 5, pp. 17-23.

STOCHASTIC AND INTEGRAL GEOMETRY

Orientation-Dependent Chord Length Distribution as a Function of Maximal Chord

A. G. Gasparyan[1*] and V. K. Ohanyan[1]**

[1] *Yerevan State University, Armenia*

Received September 1, 2014

Abstract—In [3] and [4] have been proved that covariogram and orientation-dependent chord length distribution function of a triangle and an ellipse depends on maximal chord $t_{max}(u)$ in direction u. In this paper a necessary condition for orientation-dependent chord length distribution function as a function of maximal chord is obtained. A class of parallelograms is constructed for which this necessary condition is violated.

MSC2010 numbers: 60D05; 52A22; 53C65

DOI: 10.3103/S1068362315050064

Keywords: *Covariogram; maximal chord; orientation-dependent distribution; orientation-dependent chord length distribution.*

1. INTRODUCTION

Let[1] R^n $(n \geq 2)$ be the n-dimensional Euclidean space, $\mathbf{D} \subset R^n$ be a bounded convex body with inner points, and V_n be the n-dimensional Lebesgue measure in R^n. Denote by S^{n-1} the $(n-1)$-dimensional sphere of radius 1 centered at the origin in R^n. A random line which is parallel to $u \in S^{n-1}$ and intersects $\mathbf{D}$ has an intersection point (denoted by x) with $\Pi r_{u^\perp}\mathbf{D}$, where $\Pi r_{u^\perp}\mathbf{D}$ is the orthogonal projection of $\mathbf{D}$ onto the hyperplane $u^\perp$ ($u^\perp$ is the hyperplane with normal u and passing through the origin). We can identify the points of $\Pi r_{u^\perp}\mathbf{D}$ and the lines which intersect $\mathbf{D}$ and are parallel to u. Assuming that the intersection point x is uniformly distributed over the convex body $\Pi r_{u^\perp}\mathbf{D}$ we can define the following distribution function.

Definition 1.1. *The function*

$$F(u,t) = \frac{V_{n-1}\{x \in \Pi r_{u^\perp}\mathbf{D} : V_1(g(u,x) \cap \mathbf{D}) < t)\}}{b_{\mathbf{D}}(u)}$$

is called orientation-dependent chord length distribution function of $\mathbf{D}$ *in direction* u *at point* $t \in R^1$*, where* $g(u,x)$ *is the line which is parallel to* u *and intersects* $\Pi r_{u^\perp}\mathbf{D}$ *at point* x*, and* $b_{\mathbf{D}}(u) = V_{n-1}(\Pi r_{u^\perp}\mathbf{D})$.

Definition 1.2 (see [1]). *The function*

$$C(\mathbf{D}, h) = V_n(\mathbf{D} \cap (\mathbf{D} + h)), \quad h \in R^n, \tag{1.1}$$

is called the covariogram of $\mathbf{D}$. *Here* $\mathbf{D} + h = \{x + h : x \in \mathbf{D}\}$.

Notice that each vector $h \in R^n$ can be represented in the form $h = tu$, where u is the direction of h, and t is the length of h.

*E-mail: ara1987-87@mail.ru
**E-mail: victo@aua.am

[1] This research the first author was partially supported by the state committee of sciences of the Republic of Armenia, Grant number 14A-1a09

Lemma 1.1 (see [1]). *Let $u \in S^{n-1}$ and $t > 0$ be such that $\mathbf{D} \cap (\mathbf{D} + tu)$ contains inner points. Then $C(\mathbf{D}, u, t)$ is differentiable with respect to t and the following equality holds:*

$$-\frac{\partial C(\mathbf{D}, u, t)}{\partial t} = (1 - F(u,t)) \cdot b_{\mathbf{D}}(u). \tag{1.2}$$

At $t = 0$ the right-hand derivative of the left-hand side exists, and the equation (1.2) remains valid.

G. Matheron [1] conjectured that the covariogram of a convex body $\mathbf{D}$ determines $\mathbf{D}$ within the class of all convex bodies, up to translations and reflections in a point. G. Averkov and G. Bianchi [2] showed that every planar convex body is determined within all planar convex bodies by its covariogram and orientation-dependent chord length distribution function, up to translations and reflections.

Let $L(\omega)$ be a random segment of length $l > 0$, which is parallel to a fixed direction $u \in S^{n-1}$ and intersects the body $\mathbf{D}$. Consider the random variable $|L|(\omega) = V_1(L(\omega) \cap \mathbf{D})$ with $L(\omega) \in \Omega(u)$, where the set $\Omega(u)$ is defined as follows:

$$\Omega(u) = \{\text{segments with length } l\text{, which are parallel to } u \text{ and intersect } \mathbf{D}\}.$$

Let $g(u,x)$ denote the line with direction u and passing through the point $x \in \Pi r_{u^\perp}\mathbf{D}$. Observe that every random segment $L(\omega)$ lies on the line $g(u,x)$, and hence $L(\omega)$ can be specified by the coordinates $(g(u,x), y)$, where y is the one-dimensional coordinate of the center of $L(\omega)$ on the line $g(u,x)$. As the origin on the line $g(u,x)$ we take one of the intersection points of $g(u,x)$ and $\mathbf{D}$. Using the above notation, we can identify $\Omega(u)$ with the following set:

$$\Omega(u) = \left\{(x,y) : x \in \Pi r_{u^\perp}\mathbf{D}, \quad y \in \left[-\frac{l}{2}, \chi(u,x) + \frac{l}{2}\right]\right\},$$

where $\chi(u,x) = V_1(g(u,x) \cap \mathbf{D})$.

Definition 1.3. *Given a random segment L and a direction $u \in S^{n-1}$. The function*

$$F_{|L|}(u,t) = \frac{V_n(\{(x,y) \in \Omega(u) : \ |L|(x,y) < t\})}{V_n(\Omega)} \tag{1.3}$$

is called orientation-dependent distribution function of L in direction $u \in S^{n-1}$.

In [6] was obtained the following formula, which establishes a relationship between the distribution function of the random variable $|L|(\omega)$ and the orientation-dependent chord length distribution function in R^n:

$$F_{|L|}(u,t) = \begin{cases} 0 & \text{for} \quad t \le 0, \\ \dfrac{b_{\mathbf{D}}(u)\left[2t + F(u,t)(l-t) - \int_0^t F(u,z)dz\right]}{V_n(\mathbf{D}) + l b_{\mathbf{D}}(u)} & \text{for} \quad 0 \le t \le l, \\ 1 & \text{for} \quad t > l. \end{cases} \tag{1.4}$$

2. A NECESSARY CONDITION FOR $F(u,t)$

Given a body $\mathbf{D}$ and a direction $u \in S^{n-1}$, let $t_{max}(u)$ denote the maximal chord of $\mathbf{D}$ in direction $u \in S^{n-1}$. Consider a random segment $L(\omega)$ of length $t_{max}(u)$, which is parallel to the fixed direction $u \in S^{n-1}$ and intersects the body $\mathbf{D}$. Substituting in (1.4) $|L| = l = t_{max}(u)$, we obtain

$$F_{t_{max}(u)}(u,t) = \begin{cases} 0 & \text{for} \quad t \le 0, \\ C \cdot \left[2t + F(u,t)(t_{max}(u) - t) - \displaystyle\int_0^t F(u,z)dz\right] & \text{for} \quad 0 \le t \le t_{max}(u), \\ 1 & \text{for} \quad t > t_{max}(u), \end{cases} \tag{2.1}$$