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A set-theoretical representation for weakly idempotent lattices and interlaced weakly idempotent bilattices

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Abstract In this paper we give set-theoretical characterizations both for weakly idempotent lattices and interlaced weakly idempotent bilattices. In particular, we obtain a set-theoretical representation for interlaced bilattices and distributive bilattices (without bounds).

Keywords Weakly idempotent semilattice · Weakly idempotent lattice · Weakly idempotent bilattice · Filter and ideal of the weakly idempotent lattice · Filter and ideal of the weakly idempotent bilattice

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1 Introduction

In [9], arbitrary lattices were characterized by sets, which generalize the well-known classical Birkhoff–Stone set-theoretical representation theorem for distributive lattices and Boolean algebras [8], [75] (for another generalization see [10], [12], [13], [20], [51]). A set-theoretical representation for bounded distributive bilattices is well known and were given in [26]. However, the same problem is still opened for interlaced bilattices, which have applications in multi-valued logic, artificial intelligence, logic programming and other directions of applied

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There exist various extensions of the concept of a lattice. For example, in [24], [25] the weakly associative lattices were introduced. In [39] an algebra with a system of identities was introduced, which we call a weakly idempotent lattice (see also [31], [62], [63]). The variety of weakly idempotent lattices is a nilpotent shift of the variety of lattices [39]. In paper [18], we studied weakly idempotent lattices with an additional interlaced operation. In this paper we give set-theoretical representations both for arbitrary weakly idempotent lattices and for interlaced weakly idempotent bilattices. In particular, we obtain a set-theoretical representation for interlaced bilattices, too. This problem was left open in the paper [26].
In the next section, we give the definitions of a weakly idempotent semilattice, a weakly idempotent lattice, a quasiordered set, an inf-quasiordered set, a sup-quasiordered set and an $infsup$-quasiordered set. We establish a connection among these weakly idempotent structures and the corresponding quasiorders (Lemmas 1–4, Corollaries 1, 2). In the third section we prove inequalities valid for weakly idempotent lattices. Further, we give definitions of a filter and ideal of a weakly idempotent lattice, then we characterize the filter and the ideal of a weakly idempotent lattice that are generated by a non-empty subset of the weakly idempotent lattice. On a subset of the set of all filters (ideals) of a weakly idempotent lattice, we define binary operations $\cap^*$ and $\cup^*$. In section four we prove a set-theoretical representation theorem for arbitrary weakly idempotent lattices (Theorem 1). Then we define the prime filter and the prime ideal of a weakly idempotent lattice. We redefine the binary operations $\cap^*$ and $\cup^*$ on a subset of the set of all filters (ideals) of a weakly idempotent lattice by the new operations $\cap^{**}$ and $\cup^{**}$. We get as a corollary from Theorem 1 set-theoretical representations for distributive weakly idempotent lattices and for arbitrary lattices (Corollaries 6, 7). In the fifth section we define a weakly idempotent bilattice, an interlaced weakly idempotent bilattice, a distributive weakly idempotent bilattice, a hyperidentity and superproduct, a filter and an ideal of the weakly idempotent bilattice. Then we prove certain inequalities valid on a weakly idempotent bilattice. Furthermore, we characterize the filter (ideal) of a weakly idempotent bilattice that is generated by a non-empty subset of the weakly idempotent bilattice (Lemmas 14, 15). Also we define the binary operations $\cap^*$ and $\cup^*$ on a subset of the set of all filters (ideals) of a weakly idempotent bilattice. In Theorem 2 we give a set-theoretical representation for interlaced weakly idempotent bilattices. Then we define a prime filter and prime ideal of weakly idempotent bilattices, also we redefine the binary operations $\cap^*$ and $\cup^*$ on a subset of the set of all filters (ideals) of a weakly idempotent bilattice by the operations $\cap^{**}$ and $\cup^{**}$. Finally, we get as corollaries set-theoretical representations for distributive weakly idempotent bilattices, interlaced bilattices and distributive bilattices without bounds (Corollaries 10, 11, 12).

2 Preliminary results

Definition 1 An algebra $(L; \wedge)$ with one binary operation is called a weakly idempotent semilattice, if it satisfies the following identities:

$$a \wedge b = b \wedge a; \text{ (commutativity)} \tag{1}$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c); \text{ (associativity)} \tag{2}$$

$$a \wedge (b \wedge b) = a \wedge b. \text{ (weak idempotency)} \tag{3}$$

The operation $\wedge$ is called a product. Adding the idempotence identity $a \wedge a = a$ we obtain a semilattice. The element $a \in L$ is called an idempotent of the weakly idempotent semilattice $(L; \wedge)$ if $a \wedge a = a$. The set of the idempotent elements of each weakly idempotent semilattice form a semilattice, i.e. the product of any two idempotent elements in the weakly idempotent semilattice is an idempotent element. Namely:
$(x \wedge y) \wedge (x \wedge y) \overset{(2)}{=} x \wedge (y \wedge (x \wedge y)) \overset{(1)}{=} x \wedge (y \wedge (y \wedge x)) \overset{(2)}{=} x \wedge ((y \wedge y) \wedge x) \overset{(1),(3)}{=} x \wedge (x \wedge y) \overset{(2)}{=} (x \wedge x) \wedge y \overset{(1),(3)}{=} y \wedge x = x \wedge y$.

Definition 2 The relation $\theta \subseteq L \times L$ is called a quasiorder if it is reflexive and transitive.

Example 1 The cover of a set L is a family $P = \{X_i\}_{i \in I}$ of subsets of L such that $\cup_{i \in I} X_i = L$. A relation Q, defined on the set of all covers of the set L:

$$P_1 Q P_2 \iff \forall X \in P_1 \exists Y \in P_2 (X \subseteq Y)$$

is a quasiorder. It is not an order, because there exist such different covers P_1 and P_2 that $P_1 Q P_2$ and $P_2 Q P_1$.

Example 2 The classical relation of divisibility on $\mathbf{Z}$ is a quasiorder.

So, the results of this paper are related to the number theory too.

Every quasiorder generates an order as follows.
Let θ be a quasiorder on the set $L \neq \emptyset$; then $E_\theta = \theta \cap \theta^{-1} \subseteq L \times L$ is an equivalence. For any element $x \in L$ let us denote by $[x]$ the class of the relation E_θ which contains the element x. Let $\leq_\theta$ be a relation induced on the set L/E_θ from θ in the following manner: for $[a], [b] \in L/E_\theta$

$$[a] \leq_\theta [b] \iff a\theta b.$$

A straightforward argument show that this definition is correct and that it is an order.

The function $K : L/E_\theta \mapsto L$ is called a choice function, if $K([a]) \in [a]$ for each $[a] \in L/E_\theta$.

Definition 3 The pair $(L; \theta)$ is called an inf-quasiordered (a sup-quasiordered) set, if for any two classes of equivalences $[a], [b] \in L/E_\theta$ there exists $inf([a], [b]) = [a] \wedge [b]$ (dually $sup([a], [b]) = [a] \vee [b]$), i.e. if $(L/E_\theta; \leq_\theta)$ is a lower (an upper) semilattice.

Lemma 1 *Let* $(L;\theta)$ *be an* inf*-quasiordered set and let* $K : L/E_\theta \to L$ *be an arbitrary choice function. If for* $x, y \in L$, $x \wedge y = K(inf([x],[y])) = K([x] \wedge [y])$, *then the algebra* $(L;\wedge)$ *is a weakly idempotent semilattice, which we call a lower weakly idempotent semilattice.*

Proof It is straightforward to check that $(L;\wedge)$ satisfies the identities (1)–(3).

Lemma 2 *Let* $(L;\wedge)$ *be a weakly idempotent semilattice. Then the relation* $a\theta b \leftrightarrow a \wedge b = a \wedge a$ *is a quasiorder on the set* L*, the mapping* $K : L/E_\theta \mapsto L$, $[a] \mapsto a \wedge a$ *is a choice function, and the pair* $(L;\theta)$ *is an* inf*-quasiordered set with* $inf([a],[b]) = [a \wedge b]$*, and* $x \wedge y = K(inf([x],[y]))$.

Proof The first part is obvious. Let us show that $inf([a],[b]) = [a \wedge b]$. First note that $[a \wedge b] \leq_\theta [a]$ and $[a \wedge b] \leq_\theta [b]$. Indeed $(a \wedge b) \wedge a = a \wedge b = (a \wedge b) \wedge (a \wedge b)$, then $[a \wedge b] \leq_\theta [a]$. Similarly for $[a \wedge b] \leq_\theta [b]$. Consider $[c] \in L/E_\theta$ a lower bound for $[a],[b] \in L/E_\theta$, then $c\theta a$ and $c\theta b$, thus $c \wedge (a \wedge b) = (c \wedge a) \wedge b = (c \wedge c) \wedge b = c \wedge (c \wedge c) = c \wedge c$, yields $[c] \leq_\theta [a \wedge b]$ and $inf([a],[b]) = [a \wedge b]$.

The following two lemmas are dual to Lemma 1 and Lemma 2, respectively.

Lemma 3 *Let* $(L;\theta)$ *be a sup-quasiordered set and let* $K : L/E_\theta \mapsto L$ *be an arbitrary choice function. If for each two elements,* $x, y \in L$*:*

$$x \vee y = K(sup([x],[y])) = K([x] \vee [y]),$$

then the algebra $(L;\vee)$ *is a weakly idempotent semilattice, which we call an upper weakly idempotent semilattice.*

Lemma 4 *Let* $(L;\vee)$ *be a weakly idempotent semilattice. Then the relation* $a\theta b \leftrightarrow a \vee b = b \vee b$ *is a quasiorder on the set* L*, the mapping* $K : L/E_\theta \mapsto L$, $[a] \mapsto a \vee a$ *is a choice function, and the pair* $(L;\theta)$ *is a sup-quasiordered set with* $sup([a],[b]) = [a \vee b]$*, and* $x \vee y = K(sup([x],[y]))$.

Definition 4 An algebra $(L;\wedge,\vee)$ with two binary operations is called a weakly idempotent lattice, if the reducts $(L;\wedge)$ and $(L;\vee)$ are weakly idempotent semilattices and the following identities are valid:

$$a \wedge (b \vee a) = a \wedge a, a \vee (b \wedge a) = a \vee a, \text{ (weak absorption)} \qquad (4)$$

$$a \wedge a = a \vee a. \text{ (equalization)} \qquad (5)$$

The operation $\vee$ is called a sum.

Example 3 Let us consider the relation of divisibility on the set $\mathbf{Z}\backslash\{0\}$, which is a quasiorder on $\mathbf{Z}\backslash\{0\}$. The corresponding equivalence classes are the following sets $\{x, -x\}$. Define a choice function as $K([x]) = |x|$. Then we have: $x \wedge_1 y = \gcd(|x|,|y|)$ and $x \vee_1 y = \text{lcm}(|x|,|y|)$. Thus, the algebra $(Z \setminus \{0\};\wedge_1,\vee_1)$ is a weakly idempotent lattice, which is not a lattice, since $x \wedge_1 x \neq x$ for negative x.

Example 4 If we define the choice function on $\mathbf{Z} \setminus \{0\}$ as follows $K([x]) = -|x|$, then we have: $x \wedge_2 y = -\gcd(|x|, |y|)$ and $x \vee_2 y = -\mathrm{lcm}[|x|, |y|]$. Hence, the algebra $(Z \setminus \{0\}; \wedge_2, \vee_2)$ also is a weakly idempotent lattice.

The element $a \in L$ is called an idempotent of a weakly idempotent lattice $(L; \wedge, \vee)$ if $a \wedge a = a$ and $a \vee a = a$.

Remark 1 The product (sum) of any two elements of a weakly idempotent lattice $(L; \wedge, \vee)$ is an idempotent element:
$(x \wedge y) \vee (x \wedge y) \overset{(5)}{=} (x \wedge y) \wedge (x \wedge y) \overset{(2)}{=} ((x \wedge y) \wedge x) \wedge y \overset{(1)}{=} ((y \wedge x) \wedge x) \wedge y \overset{(2)}{=} (y \wedge (x \wedge x)) \wedge y \overset{(3)}{=} (y \wedge x) \wedge y \overset{(1)}{=} (x \wedge y) \wedge y \overset{(2)}{=} x \wedge (y \wedge y) \overset{(3)}{=} x \wedge y.$
The other condition is proved similarly. So, the set of all idempotent elements of a weakly idempotent lattice is a lattice.

Definition 5 A pair $(L; \theta)$ is called an $infsup$-quasiordered set, if for any two classes of equivalences $[a], [b] \in L/E_\theta$ there exist both $inf([a], [b]) = [a] \wedge [b]$ and $sup([a], [b]) = [a] \vee [b]$, i.e. if $(L/E_\theta; \leq_\theta)$ is a lattice.

The following two corollaries immediately follow from Lemmas 1 − 4.

Corollary 1 *Let $(L; \theta)$ be an $infsup$-quasiordered set and let $K : L/E_\theta \mapsto L$ be an arbitrary choice function. If for any two elements $x, y \in L$:*

$$x \wedge y = K(inf([x], [y])) = K([x] \wedge [y]),$$

$$x \vee y = K(sup([x], [y])) = K([x] \vee [y]),$$

then the algebra $(L; \wedge, \vee)$ is a weakly idempotent lattice.

Corollary 2 *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice. Then the relation $a\theta b \leftrightarrow a \wedge b = a \wedge a \leftrightarrow a \vee b = b \vee b$ is a quasiorder on the set L, the mapping $K : L/E_\theta \mapsto L$; $[a] \mapsto a \vee a$ is a choice function, and the pair $(L; \theta)$ is an $infsup$-quasiordered set with $inf([a], [b]) = [a \wedge b], sup([a], [b]) = [a \vee b]$ and*

$$x \wedge y = K(inf([x], [y])), \quad x \vee y = K(sup([x], [y])). \tag{6}$$

Definition 6 A weakly idempotent lattice $(L; \wedge, \vee)$ is called bounded, if the corresponding lattice $(L/E_\theta; \leq_\theta)$ is bounded.

3 Weakly idempotent lattices

Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice. By Corollary 2, each weakly idempotent lattice defines the quasiorder θ (which is denoted by $\leq$ in this section) and it is defined by the following rule:

$$x \leq y \leftrightarrow x \wedge y = x \wedge x. \tag{7}$$

Remark 2 Note, that the operations of the weakly idempotent lattice preserve the corresponding quasiorder. Indeed, let $a \leq b$ and $c \leq d$, i.e. $a \vee b = b \vee b$ and $c \vee d = d \vee d$, then $(a \vee c) \vee (b \vee d) = (a \vee b) \vee (c \vee d) = (b \vee b) \vee (d \vee d) = b \vee d$, thus $a \vee c \leq b \vee d$. In the same way we show that $a \wedge c \leq b \wedge d$.

Lemma 5 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice. Then:*

$$a \leq b \leq a \leftrightarrow a \wedge a = b \wedge b.$$

Proof Let $a \leq b \leq a$, then by (7) $a \wedge b = a \wedge a$ and $a \wedge b = b \wedge b$, hence, $a \wedge a = b \wedge b$. Conversely, let $a \wedge a = b \wedge b$, then $a \wedge b \overset{(3)}{=} a \wedge (b \wedge b) = a \wedge (a \wedge a) \overset{(3)}{=} a \wedge a$ and $a \vee b \overset{(3)}{=} a \vee (b \vee b) = a \vee (a \vee a) \overset{(3)}{=} a \vee a$, thus, by (7) $a \leq b$ and $b \leq a$.

Lemma 6 *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice. Then the following inequality is valid on L:*

$$x \wedge y \leq x \leq x \vee y.$$

Proof From the identities (1)–(3) and Remark 1 we have: $(x \wedge y) \wedge x \overset{(1)}{=} (y \wedge x) \wedge x \overset{(2)}{=} y \wedge (x \wedge x) \overset{(3)}{=} y \wedge x \overset{(1)}{=} x \wedge y = (x \wedge y) \wedge (x \wedge y)$. Hence, by (7) we obtain: $x \wedge y \leq x$. The second side of the inequality proved in a similar way.

We say that a nonempty subset A of a weakly idempotent lattice $(L; \wedge, \vee)$ is a weakly idempotent sublattice if it is closed under the weakly idempotent lattice operations, i.e. if the following condition is valid:

$$x, y \in A \rightarrow x \wedge y, x \vee y \in A. \tag{8}$$

Definition 7 The weakly idempotent sublattice F of a weakly idempotent lattice $L = (L; \wedge, \vee)$ is called a filter of L if the following condition is satisfied: $x \in F, y \in L \rightarrow x \vee y \in F$.

A filter is called proper if it is different from L. The set of all the filters of the weakly idempotent lattice L is denoted by $F(L)$.

Definition 8 A weakly idempotent sublattice I of a weakly idempotent lattice $L = (L; \wedge, \vee)$ is called an ideal of L if the following condition is satisfied: $x \in I, y \in L \rightarrow x \wedge y \in I$.

An ideal is called proper if it is different from L. The set of all the ideals of the weakly idempotent lattice L is denoted by $I(L)$.

Lemma 7 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice. The following statements are equivalent for L:*
1. F is a filter of L;
2. If $x, y \in F$, then $x \wedge y \in F$ and if $x \in F, y \in L, x \leq y$, then $y \vee y \in F$.

Proof (1→2) Let F be a filter on L; then, for any $x, y \in F$, $x \wedge y \in F$ (by (8)) and for any $x \in F$, $y \in L$ $x \vee y \in F$. Assume that $x \leq y$, (i.e. $x \vee y = y \vee y$), then $y \vee y \in F$.
(2→1) For $x \in F$, $y \in L$, using Lemma 6, we get: $(x \vee y) \vee (x \vee y) \in F$, since $x \leq x \vee y$. From Remark 1, we obtain that $(x \vee y) \vee (x \vee y) = x \vee y$. Thus, $x \vee y \in F$ for each $x \in F, y \in L$.

Lemma 8 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice. The following statements are equivalent for L:*
1. I is an ideal of L;
2. If $x, y \in I$, then $x \vee y \in F$ and if $x \in F, y \in L, x \leq y$, then $x \wedge x \in I$.

Lemma 9 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice and $X \subseteq L$. Then*

$$[X) = X \cup \{x \in L | \exists x_1, \ldots, x_n \in X, x_1 \wedge \ldots \wedge x_n \leq x \;\&\; x = K([x])\}$$

is the smallest filter that contains $X \subseteq L$.

Proof Let us show that $[X)$ is a filter. Take $x, y \in [X)$, then consider the following cases:
• $x, y \in [X) \setminus X$, then there exist $x_1, \ldots, x_n, y_1, \ldots, y_m \in X$ such that $x_1 \wedge \ldots \wedge x_n \leq x = K([x])$ and $y_1 \wedge \ldots \wedge y_m \leq y = K([y])$. By Remark 2 and the equality (6) we get: $x_1 \wedge \ldots \wedge x_n \wedge y_1 \wedge \ldots \wedge y_m \leq x \wedge y = K(inf([x], [y]))$; hence, $x \wedge y \in [X)$.
The following cases are proved similarly:
• $x \in [X) \setminus X, y \in X$;
• $y \in [X) \setminus X, x \in X$;
• $x, y \in X$.

Thus, for each $x, y \in [X)$ we obtain that $x \wedge y \in [X)$. Similarly, we prove the second condition of Lemma 7; hence, $[X)$ is a filter.

Suppose $F \neq [X)$ is a filter of the weakly idempotent lattice L, which contains X. Let $x \in [X) \setminus X$, then there exist $x_1, \ldots, x_n \in X$ such that $x_1 \wedge \ldots \wedge x_n \leq x = K([x])$. Since F is a filter which contains X, we obtain that $x_1 \wedge \ldots \wedge x_n \in F$ and $x \wedge x \in F$. By equality (6), we have: $x \wedge x = K(inf([x], [x])) = K([x]) = x$, and $x \in F$. Hence, $[X) \subseteq F$ and $[X)$ is the smallest filter that contains X.

Lemma 10 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice and $X \subseteq L$. Then*

$$(X] = X \cup \{x \in L | \exists x_1, \ldots, x_n \in X, x \leq x_1 \wedge \ldots \wedge x_n \;\&\; x = K([x])\}$$

is the smallest ideal that contains $X \subseteq L$.

Corollary 3 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice and $a \in L$. Then*

$$[a) = \{a\} \cup \{x \in L | a \leq x \;\&\; x = K([x])\}$$

is the smallest filter that contains $a \in L$.

Proof Take $X = \{a\}$ in Lemma 9.

Corollary 4 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice and $a \in L$. Then*

$$(a] = \{a\} \cup \{x \in L | x \leq a \ \&\ x = K([x])\}$$

is the smallest ideal that contains $a \in L$.

Proof Take $X = \{a\}$ in Lemma 10.

Lemma 11 *Let $L = (L; \wedge, \vee)$ be a weakly idempotent lattice. Then the following assertions are valid on L:*

$$[a \wedge a) \cup [b \wedge b) \subseteq [a \wedge b), \quad [a \vee b) = [a \vee a) \cap [b \vee b),$$

$$(a \wedge b] = (a \wedge a] \cap (b \wedge b], \quad (a \vee a] \cup (b \vee b] \subseteq (a \vee b].$$

Proof Let us show that $[a \wedge a) \cup [b \wedge b) \subseteq [a \wedge b)$. Suppose $y \in [a \wedge a) \cup [b \wedge b)$, then let us consider the following possible cases:
1) In the first case: $y = a \wedge a$; from Lemma 6 and the equality (6) we obtain: $a \wedge b \leq a \leq a \wedge a = K(inf([a], [a])) = K([a])$. Hence, $y = a \wedge a \in (a \wedge b]$.
2) In the second case: $a \leq y = K([y])$; by Lemma 6 we obtain: $a \wedge b \leq a \leq y$, i.e. $y \in [a \wedge b)$.
The following two cases are proved similarly:
3) $y = b \wedge b$,
4) $b \leq y = K([y])$.
Thus, $[a \wedge a) \cup [b \wedge b) \subseteq [a \wedge b)$.

For each $a \in L$, let $f(a) = \{F \in F(L) | a \in F\}$ and $i(a) = \{I \in I(L) | a \in I\}$. Define on the sets $f(L) = \{f(a) | a \in L\}$ and $i(L) = \{i(a) | a \in L\}$ the operations $\cap^*$ and $\cup^*$, by the following rules:

$$f(a) \cap^* f(b) = \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cup Y \subseteq F\};$$

$$f(a) \cup^* f(b) = \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cap Y \subseteq F\};$$

$$i(a) \cap^* i(b) = \{I \in I(L) | \exists X \in i(a \wedge a), \exists Y \in i(b \wedge b), X \cup Y \subseteq I\};$$

$$i(a) \cup^* i(b) = \{I \in I(L) | \exists X \in i(a \wedge a), \exists Y \in i(b \wedge b), X \cap Y \subseteq I\}.$$

Lemma 12 *Let $(L; \wedge, \vee)$ be a weakly idempotent lattice. Then the following identities are valid:*

$$f(a \wedge b) = f(a) \cap^* f(b); \tag{9}$$

$$f(a \vee b) = f(a) \cup^* f(b); \tag{10}$$

$$i(a \wedge b) = i(a) \cup^* i(b); \tag{11}$$

$$i(a \vee b) = i(a) \cap^* i(b). \tag{12}$$

Proof Let us prove the first equality. Suppose $F \in f(a \wedge b)$, then $a \wedge b \in F$. If $c \in [a \wedge b)$, then $c = a \wedge b$ and hence, $c \in F$, or $a \wedge b \leq c = K([c])$. Thus, since F is a filter, we obtain that: $c = K([c]) = K(inf([c],[c])) = c \wedge c \in F$. Hence, $[a \wedge b) \subset F$, and by Lemma 11, we have: $[a \wedge a) \cup [b \wedge b) \subseteq [a \wedge b)$. Thus, there exist $X = [a \wedge a)$ and $Y = [b \wedge b)$ such that $X \cup Y \subseteq F$, i.e. $F \in f(a) \cap^* f(b)$.

Conversely, if $F \in f(a) \cap^* f(b)$, then there exist $X \in f(a \wedge a)$ and $Y \in f(b \wedge b)$, such that $X \cup Y \subseteq F$ hence, $a \wedge a \in X$ and $b \wedge b \in Y$. Then $a \wedge a, b \wedge b \in X \cup Y \subseteq F$; hence, since F is a filter, we get $a \wedge b = (a \wedge a) \wedge (b \wedge b) \in F$. Thus, $F \in f(a \wedge b)$.

The rest of the identities are proved similarly.

4 A set-theoretical representation for arbitrary weakly idempotent lattices

From Lemma 12 it immediately follows that the algebras $f(L) = (f(L); \cap^*, \cup^*)$ and $i(L) = (i(L); \cup^*, \cap^*)$ are weakly idempotent lattices.

Theorem 1 *Every weakly idempotent lattice $L = (L; \wedge, \vee)$ is isomorphic to the weakly idempotent lattice $f(L) = (f(L); \cap^*, \cup^*)$, i.e.*

$$L \cong f(L).$$

Proof The mapping $\varphi : L \to (f(L); \cap^*, \cup^*)$, which is defined by the following rule:

$$\varphi : x \mapsto f(x)$$

is an isomorphism. Indeed:

1. Injectivity. Suppose $\varphi(a) = \varphi(b)$, i.e. $f(a) = f(b)$. Then, since $[a) \in f(a)$ and $[b) \in f(b)$, we obtain that $b \in [a)$ and $a \in [b)$. By Corollary 3 we have two cases: $b = a$ or $a \leq b = K([b])$ and $b \leq a = K([a])$. In the second case, by Lemma 5, we have: $a \wedge a = b \wedge b$; and by equality (6), we obtain that $a \wedge a = K(inf([a],[a])) = K([a]) = a$ and $b \wedge b = K(inf([b],[b])) = K([b]) = b$. Hence, $a = b$.
2. The proof of the surjectivity of φ is obvious.
3. The fact that φ is a homomorphism follows directly from the identities (9) and (10):

$\varphi(x \wedge y) = f(x \wedge y) = f(x) \cap^* f(y) = \varphi(x) \cap^* \varphi(y)$;
$\varphi(x \vee y) = f(x \vee y) = f(x) \cup^* f(y) = \varphi(x) \cup^* \varphi(y)$.

Corollary 5 *For every weakly idempotent lattice $L = (L; \wedge, \vee)$ we have:*

$$L \cong i(L).$$

Proof The mapping $\varphi_1 : L \to (i(L); \cup^*, \cap^*)$, which is defined by the following rule:

$$\varphi_1 : x \mapsto i(x)$$

is an isomorphism.
The proof of the injectivity of φ_1 is similar to Theorem 1, and the fact that φ_1 is a homomorphism follows immediately from equalities (11) and (12).

Note that the operation $\cup^*$, defined on $f(L)$, coincides with the operation $\cup^{**}$, defined in the following way:

$$f(a) \cup^{**} f(b) = f(a \wedge a) \cup f(b \wedge b)$$

$$\cup \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cap Y \subseteq F\}.$$

Indeed, let $F' \in f(a \wedge a) \cup f(b \wedge b)$, then $a \wedge a \in F'$ or $b \wedge b \in F'$. Hence $a \vee b \in F'$. From Lemma 11 we have $[a \vee b) = [a \wedge a) \cap [b \wedge b)$. Hence, there exist $X = [a \wedge a)$ and $Y = [b \wedge b)$ such that $X \cap Y \subseteq F'$, yield $F' \in \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cap Y \subseteq F\}$ and thus $f(a \wedge a) \cup f(b \wedge b) \subseteq \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cap Y \subseteq F\}$. So, $f(a) \cup^* f(b) = f(a) \cup^{**} f(b)$ on $f(L)$.

Also note that the operation $\cap^*$ defined on $f(L)$ can be replaced with the operation $\cap^{**}$, defined in the following way:

$$f(a) \cap^{**} f(b) = f(a \wedge a) \cap f(b \wedge b)$$

$$\cap \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cup Y \subseteq F\}.$$

Indeed, let $F \in \{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cup Y \subseteq F\}$; then there exist filters X and Y of L such that $a \wedge a \in X$, $b \wedge b \in Y$ and $X \cup Y \subseteq F$. So $F \in f(a \wedge a), F \in f(b \wedge b)$, hence $\{F \in F(L) | \exists X \in f(a \wedge a), \exists Y \in f(b \wedge b), X \cup Y \subseteq F\} \subseteq f(a \wedge a) \cap f(b \wedge b)$. Hence $f(a) \cap^* f(b) = f(a) \cap^{**} f(b)$.

Moreover, $f(a) \cap^{**} f(b) = f(a \wedge a) \cap f(b \wedge b)$.

In the same way, we can show that the operations $\cap^*$ and $\cup^*$ on the set $i(L)$ coincide with the operations $\cap^{**}$ and $\cup^{**}$ defined as follows:

$$i(a) \cup^{**} i(b) = i(a \wedge a) \cup i(b \wedge b) \cup \{I \in I(L) | \exists X \in i(a \wedge a), \exists Y \in i(b \wedge b), X \cap Y \subseteq I\},$$

$$i(a) \cap^{**} i(b) = i(a \wedge a) \cap i(b \wedge b) \cap \{I \in I(L) | \exists X \in i(a \wedge a), \exists Y \in i(b \wedge b), X \cup Y \subseteq I\}.$$

Moreover:

$$i(a) \cap^{**} i(b) = i(a \wedge a) \cap i(b \wedge b).$$

Definition 9 A filter F of a weakly idempotent lattice $(L; \wedge, \vee)$ is called prime if F is a proper filter and

$$x \vee y \in F \to x \in F \text{ or } y \in F,$$

for $x, y \in L$.

Definition 10 An ideal I of a weakly idempotent lattice $(L; \wedge, \vee)$ is called prime if I is a proper ideal and

$$x \wedge y \in I \to x \in I \text{ or } y \in I,$$

for $x, y \in L$.

Definition 11 The weakly idempotent lattice $(L; \wedge, \vee)$ is called distributive if it satisfies the identities of distributivity:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

Denote the set of all prime filters of a weakly idempotent lattice L by $pF(L)$, and the set of all prime ideals of a weakly idempotent lattice L by $pI(L)$. Let us make the following designations: $pf(a) = \{X \in pF(L)|a \in X\}$ and $pi(a) = \{X \in pI(L)|a \in X\}$, $pf(L) = \{pf(a)|a \in L\}$ and $pi(L) = \{pi(a)|a \in L\}$.

The operations $\cap^{**}$ and $\cup^{**}$ can be defined on $pf(L)$ and $pi(L)$ and then identities similar to those in Lemma 12 can be proven for prime filters and prime ideals too. Thus, the sets $pF(L)$ and $pI(L)$ are closed under the operations $\cap^{**}$ and $\cup^{**}$.

Corollary 6 *Each distributive weakly idempotent lattice* $L = (L; \wedge, \vee)$ *is isomorphic to the distributive weakly idempotent lattice* $pf(L) = (pf(L); \cap^{**}, \cup)$*, i.e.*

$$L \cong pf(L).$$

Proof Let us show that on the set $pf(L)$ the operation $\cup^{**}$ converts to the set-theoretical union. Namely, let us consider $Z \in pf(x) \cup^{**} pf(y)$. By the corresponding result for (8) from Lemma 12 we have: $x \vee y \in Z$. Since Z is a prime filter, then $x \in Z$ or $y \in Z$, hence $Z \in pf(x \wedge x) \cup pf(y \wedge y)$. Thus, $pf(x) \cup^{**} pf(y) \subseteq pf(x \wedge x) \cup pf(y \wedge y)$.
The converse, namely, $pf(x \wedge x) \cup pf(y \wedge y) \subseteq pf(x) \cup^{**} pf(y)$ is trivial.
Using the identities of Lemma 12 for the case of $pf(L)$ we immediately get that $pf(L)$ is a weakly idempotent lattice. On the set $pf(L)$, we redefine in the following way the operation $\cap^{**}$: $pf(a) \cap^{**} pf(b) = pf(a \wedge a) \cap pf(b \wedge b)$. Hence, the weakly idempotent lattice $pf(L)$ is distributive.

Let us make some designations: the set of all prime filters of the lattice L is denoted by $\tilde{F}(L)$, $\tilde{f}(a) = \{X \in \tilde{F}(L)|a \in X\}$, $\tilde{f}(L) = \{\tilde{f}(a)|a \in L\}$.

Corollary 7 [9] *Every lattice* $L = (L; \wedge, \vee)$ *is isomorphic to the lattice* $\tilde{f}(L) = (\tilde{f}(L); \cap, \cup^*)$*, i.e.*

$$L \cong \tilde{f}(L).$$

Proof Since L is a lattice, the operation $\cap^*$ is converted into the set-theoretical intersection and the operation $\cup^*$ is defined as follows:

$$\tilde{f}(a) \cup^* \tilde{f}(b) = \{F \in \tilde{F}(L)|(\exists X \in \tilde{f}(a))(\exists Y \in \tilde{f}(b))X \cap Y \subseteq F\}.$$

It is obvious that $(\tilde{f}(L); \cap, \cup^*)$ is a lattice. Indeed: $\tilde{f}(a) \cap \tilde{f}(a) = \tilde{f}(a \wedge a) = \tilde{f}(a)$, $\tilde{f}(a) \cup^* \tilde{f}(a) = \tilde{f}(a \vee a) = \tilde{f}(a)$.

5 A set-theoretical representation for interlaced weakly idempotent bilattices

Bilattices are algebras with two separate lattice structures and weakly idempotent bilattices are algebras with two separate weakly idempotent lattice structures. Bilattices have been used as the basis for a denotational semantics for systems of inference that arise in artificial intelligence and knowledge-based logic programming ([22], [30]). In particular, they have been used to provide a general framework for an efficient procedural semantics of logic programming languages that can deal with incomplete as well as contradictory information.

Originally, Ginsberg [30] suggested to use bilattices as the underlying framework for various AI inference systems including those based on default logics, truth maintenance systems, probabilistic logics and others. These ideas were later pursued in [22], [23], [40], in the context of logic programming semantics. Studies of the algebraic properties of bilattices have provided insight into their internal structures, and have led to practical results, especially in reducing the computational complexity of bilattice-based multi-valued logic programs.

Ginsberg's original definition of bilattice postulated a connection of two lattice structures through a negation operation. However the terminology for bilattices is not uniform. For this reason, we often speak of a bilattice and weakly idempotent bilattice without negation for emphasis.

Definition 12 The algebra $L = (L; \wedge, \vee, *, \triangle)$ with four binary operations is called a weakly idempotent bilattice, if the reducts $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \triangle)$ are weakly idempotent lattices and it also satisfies the following identity:

$$a \wedge a = a * a. \tag{13}$$

If $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \triangle)$ are lattices, then the algebra $L = (L; \wedge, \vee, *, \triangle)$ is called a bilattice.

Every bilattice is a weakly idempotent bilattice. A bilattice L is called bounded if the reducts L_1 and L_2 are bounded lattices. A weakly idempotent bilattice L is called bounded if the reducts L_1 and L_2 are bounded weakly idempotent lattices.

Definition 13 A (weakly idempotent) bilattice $(L; \wedge, \vee, *, \triangle)$ is called distributive if each pair of the operations from the set $\{\wedge, \vee, *, \triangle\}$ satisfies the identities of distributivity.

Let us denote the quasiorder of the first reduct $L_1 = (L; \wedge, \vee)$ by the following: $\leq_\wedge$ and that of the second reduct $L_2 = (L; *, \triangle)$ by: $\leq_*$ (see Corollary 2), i.e.

$$x \leq_\wedge y \leftrightarrow x \wedge y = x \wedge x,$$

$$x \leq_* y \leftrightarrow x * y = x * x.$$

Definition 14 The weakly idempotent bilattice $(L; \wedge, \vee, *, \triangle)$ is called interlaced if all the basic weakly idempotent bilattice operations are quasiorder-preserving with respect to the both quasiorders, i.e.

$$a \leq_\wedge b, c \leq_\wedge d \to a * c \leq_\wedge b * d, a \triangle c \leq_\wedge b \triangle d,$$
$$a \leq_* b, c \leq_* d \to a \wedge c \leq_* b \wedge d, a \vee c \leq_* b \vee d.$$

Example 5 Every distributive weakly idempotent bilattice $(L; \wedge, \vee, *, \triangle)$ is interlaced.

Namely, if $x \leq_\wedge y$, then $(z*x)\wedge(z*y) = z*(x\wedge y) = z*(x\wedge x) = (z*x)\wedge(z*x)$ and $(z\triangle x)\wedge(z\triangle y) = z\triangle(x\wedge y) = z\triangle(x\wedge x) = (z\triangle x)\wedge(z\triangle x)$. So, $z*x \leq_\wedge z*y$ and $z\triangle x \leq_\wedge z\triangle y$ for any $z \in L$. If $x \leq_* y$, then $(z \wedge x) * (z \wedge y) = z \wedge (x * y) = z \wedge (x * x) = (z \wedge x) * (z \wedge x)$, hence $z \wedge x \leq_* z \wedge y$ and $(z \vee x) * (z \vee y) = z \vee (x * y) = z \vee (x * x) = (z \vee x) * (z \vee x)$. Thus, $z \vee x \leq_* z \vee y$ and $z \wedge x \leq_* z \wedge y$ for any $z \in L$.

Example 6 If $(L; \wedge, \vee)$ is a weakly idempotent lattice, then $(L; \wedge, \vee, \wedge, \vee)$ is an interlaced weakly idempotent bilattice. If $(L; \wedge, \vee)$ is a distributive weakly idempotent lattice, then $(L; \wedge, \vee, \wedge, \vee)$ is a distributive weakly idempotent bilattice. If $(L; \wedge, \vee)$ is a lattice, then $(L; \wedge, \vee, \wedge, \vee)$ is an interlaced bilattice. If $(L; \wedge, \vee)$ is a distributive lattice, then $(L; \wedge, \vee, \wedge, \vee)$ is a distributive bilattice.

For the applications and characterizations of bilattices in various varieties see [3], [11], [14], [15], [17], [21]-[23], [26]-[30], [33], [40], [41], [50], [52], [59], [65]–[67]. An important open problem in the study of varieties of bilattices is to characterize the finitely generated free algebras. The reference [69] provides further context for the importance of this problem in pure and applied algebra.

Note that until 2000 bilattices with bounds were investigated, but since 2006 bilattices and weakly idempotent bilattices without bounds have been studied (see [59]). In the current paper we also considered bilattices and weakly idempotent bilattices without bounds (i.e. the general case).

For more detail about the second order formulae and second order languages see [16], [36], [37]. According to [42], [43], [48], [49] a hyperidentity is a universal second-order formula of the following type:

$$\forall X_1, \dots, X_m \forall x_1, \dots, x_n (w_1 = w_2),$$

where $X_1, \dots, X_m$ are functional variables, and $x_1, \dots, x_n$ are object variables in the words (terms) of w_1, w_2. Hyperidentities are usually written without quantifiers: $w_1 = w_2$. We say that the hyperidentity $w_1 = w_2$ is satisfied in the algebra $(Q; F)$ if this equality is valid when every object variable and every functional variable in it is replaced by any element from Q and by any operation of the corresponding arity from F (supposing the possibility of such replacement) (see also [2], [4], [5], [7], [35], [45], [53], [68], [70], [72], [73], [74]).

On the characterization of hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean and De Morgan algebras see [44],

[48],[49],[54]-[56]. For more about hyperidentities in term (polynomial) algebras, see [6], [19], [32], [34], [44], [47], [60], [61], [64], [71], [76]. For application of hyperidentities in discrete mathematics see [38], [57], [58].

For example, a weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$ is distributive iff it satisfies the following hyperidentity:

$$X(Y(x, y), z) = Y(X(x, z), X(y, z));$$

One can prove that a weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$ is interlaced iff it satisfies the following hyperidentity:

$$X(Y(X(x, y), z), Y(y, z)) = Y(X(x, y), z).$$

For the categorical definition of the hyperidentity, in [42] the (bi)homomorphisms between two algebras $(Q; F)$ and $(Q'; F')$ are defined as the pairs $(\varphi; \tilde{\psi})$ of mappings:

$$\varphi : Q \to Q', \tilde{\psi} : F \to F', |A| = |\tilde{\psi}A|,$$

with the following condition:

$$\varphi(a_1, \ldots, a_n) = (\tilde{\psi}A)(\varphi a_1, \ldots, \varphi a_n)$$

for any $A \in F, a_1, \ldots, a_n \in Q, |A| = n$. (More about the application of such morphisms in the cryptography can be found in [1].)

Algebras with their (bi)homomorphisms $(\varphi; \tilde{\psi})$ (as morphisms) form a category. The product in this category is called a superproduct of algebras and is denoted by $Q \bowtie Q'$ for the two algebras Q and Q'. For example, the superproduct of the two weakly idempotent lattices $(Q; +, \cdot)$ and $(Q'; +, \cdot)$ is a binary algebra $(Q \times Q'; (+, +), (\cdot, \cdot), (+, \cdot), (\cdot, +))$ with four binary operations, where the pairs of the operations operate component-wise, i.e.

$$(A, B)((x, y), (u, v)) = (A(x, u), B(y, v)).$$

Example 7 The superproduct $Q \bowtie Q'$ is an interlaced weakly idempotent bilattice for every two weakly idempotent lattices $(Q; +, \cdot)$ and $(Q'; +, \cdot)$.

Indeed: denote the quasiorders of Q and Q' by $\leq_1$ and $\leq_2$, correspondingly, the quasiorders of $Q \bowtie Q'$ – by $\leq_I$ and $\leq_{II}$, which are defined in the following way: $(a, b) \leq_I (c, d) \iff a \leq_1 c, b \leq_2 d$ and $(a, b) \leq_{II} (c, d) \iff a \leq_1 c, d \leq_2 b$. Take $(a, b) \leq_I (c, d)$ and $(e, f) \in Q \times Q'$. Show that $(a, b)(+, \cdot)(e, f) \leq_I (c, d)(+, \cdot)(e, f)$. From $(a, b) \leq_I (c, d)$ we have $a \leq_1 c, b \leq_2 d$, hence for $e \in Q$ and $f \in Q'$ we get: $a+e \leq_1 c+e$ and $b \cdot f \leq_2 d \cdot f$, thus $(a + e, b \cdot f) \leq_I (c + e, d \cdot f)$ and $(a, b)(+, \cdot)(e, f) \leq_I (c, d)(+, \cdot)(e, f)$. The rest of the conditions are proved similarly.

Example 8 The superproduct $Q \bowtie Q'$ is a distributive weakly idempotent bilattice for every two weakly idempotent distributive lattices $(Q; +, \cdot)$ and $(Q'; +, \cdot)$. The superproduct $Q \bowtie Q'$ is a distributive bilattice for every two distributive lattices $(Q; +, \cdot)$ and $(Q'; +, \cdot)$. The superproduct $Q \bowtie Q'$ is an interlaced bilattice for every two lattices $(Q; +, \cdot)$ and $(Q'; +, \cdot)$.

Lemma 13 *Let* $L = (L; \wedge, \vee, *, \triangle)$ *be an interlaced weakly idempotent bilattice. Then for any* $x, y \in L$*, the following inequalities are valid:*

$$x \wedge y \leq_\wedge x * y \leq_\wedge x \vee y, x \wedge y \leq_\wedge x \triangle y \leq_\wedge x \vee y,$$
$$x * y \leq_* x \wedge y \leq_* x \triangle y, x * y \leq_* x \vee y \leq_* x \triangle y.$$

Proof From Lemma 6 we have: $x \wedge y \leq_\wedge x$ and $x \wedge y \leq_\wedge y$; then, since L is an interlaced weakly idempotent bilattice, we have: $(x \wedge y) * (x \wedge y) \leq_\wedge x * y$. From the following identity: $x * x = x \wedge x$, of a weakly idempotent bilattice we obtain: $(x \wedge y) * (x \wedge y) = (x \wedge y) \wedge (x \wedge y) = x \wedge y$, i.e. $x \wedge y \leq_\wedge x * y$. Similarly, from Lemma 6, we have: $x \leq_\wedge x \vee y$ and $y \leq_\wedge x \vee y$, then, since L is an interlaced weakly idempotent bilattice, we obtain: $x * y \leq_\wedge x \vee y$. Thus, $x \wedge y \leq_\wedge x * y \leq_\wedge x \vee y$.

Definition 15 The nonempty subset $F \subseteq L$ is called a filter of a weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$ if F is a filter for both reducts: $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \triangle)$, i.e. if it satisfies the following conditions:
(ff1) if $x, y \in F$, then $x \wedge y \in F$;
(ff2) if $x, y \in F$, then $x * y \in F$;
(ff3) if $x \in F, y \in L$ and $x \leq_\wedge y$, then $y \vee y \in F$;
(ff4) if $x \in F, y \in L$ and $x \leq_* y$, then $y \triangle y \in F$.

Denote the set of all the filters of a weakly idempotent bilattice L by $FF(L)$.

Definition 16 The nonempty subset $I \subseteq L$ is called an ideal of a weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$, if I is a filter for the reduct $L_1 = (L; \wedge, \vee)$ and the ideal for the reduct $L_2 = (L; *, \triangle)$, i.e. if it satisfies the following conditions:
(fi1) if $x, y \in I$, then $x \wedge y \in I$;
(fi2) if $x, y \in I$, then $x \triangle y \in I$;
(fi3) if $x \in I, y \in L$ and $x \leq_\wedge y$, then $y \vee y \in I$;
(fi4) if $y \in I, x \in L$ and $x \leq_* y$, then $x * x \in I$.

Denote the set of all the ideals of a weakly idempotent bilattice L by $FI(L)$.

Lemma 14 *Let* X *be a non-empty subset of the interlaced weakly idempotent bilattice* $L = (L; \wedge, \vee, *, \triangle)$*. Then:*

$$[X]_f = X \cup \{x \in L | \exists x_1, \ldots, x_n \in X, \exists y \in L, x_1 \wedge \ldots \wedge x_n \leq_\wedge y \leq_* x \ \&\ x = K([x])\}$$

is the smallest filter of the interlaced weakly idempotent bilattice L *that contains* X*.*

Proof First of all it is straightforward to prove that

$$[X]_f = X \cup \{x \in L | \exists x_1, \ldots, x_n \in X, \exists y \in L, x_1 * \ldots * x_n \leq_* y \leq_\wedge x \ \&\ x = K([x])\}.$$

Then, similar to Lemma 9, we show that $[X]_f$ is a filter; moreover, it is the smallest filter that contains X.

Lemma 15 *Let X be a non-empty subset of the interlaced weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$. Then:*

$$[X]_i = X \cup \{x \in L | \exists x_1, \ldots, x_n \in X, \exists y \in L, x_1 \wedge \ldots \wedge x_n \leq_\wedge y \geq_* x \;\&\; x = K([x])\}$$

is the smallest ideal of the interlaced weakly idempotent bilattice L that contains X.

Corollary 8 *Let $L = (L; \wedge, \vee, *, \triangle)$ be an interlaced weakly idempotent bilattice. Then*

$$[a]_f = \{a\} \cup \{x \in L | \exists y \in L, a \leq_\wedge y \leq_* x \;\&\; x = K([x])\}$$

is the smallest filter of the interlaced weakly idempotent bilattice L that contains $a \in L$.

Proof Take $X = \{a\}$ in Lemma 14.

Corollary 9 *Let $L = (L; \wedge, \vee, *, \triangle)$ be an interlaced weakly idempotent bilattice. Then*

$$[a]_i = \{a\} \cup \{x \in L | \exists y \in L, a \leq_\wedge y \geq_* x \;\&\; x = K([x])\}$$

is the smallest ideal of the interlaced weakly idempotent bilattice L that contains $a \in L$.

Proof Take $X = \{a\}$ in Lemma 15.

Lemma 16 *Let $L = (L; \wedge, \vee, *, \triangle)$ be an interlaced weakly idempotent bilattice. The following assertions are valid on L:*

$$[a \wedge a]_f \cup [b \wedge b]_f \subseteq [a \wedge b]_f; \;\; [a \vee b]_f = [a \wedge a]_f \cap [b \wedge b]_f;$$
$$[a \wedge a]_f \cup [b \wedge b]_f \subseteq [a * b]_f; \;\; [a \triangle b]_f = [a \wedge a]_f \cap [b \wedge b]_f;$$
$$[a \wedge a]_i \cup [b \wedge b]_i \subseteq [a \wedge b]_i; \;\; [a \vee b]_i = [a \wedge a]_i \cap [b \wedge b]_i;$$
$$[a \wedge a]_i \cup [b \wedge b]_i \subseteq [a \triangle b]_i; \;\; [a * b]_i = [a \wedge a]_i \cap [b \wedge b]_i.$$

Proof Let us prove the equality: $[a \vee b]_f = [a \wedge a]_f \cap [b \wedge b]_f$. If $x \in [a \vee b]_f$, then, by Corollary 8 we have: $a \vee b \leq_\wedge c \leq_* x = K([x])$, then $a \leq_\wedge c \leq_* x$ and $b \leq_\wedge c \leq_* x$; thus, $a \vee a \leq_\wedge a \leq_\wedge c \leq_* x$ and $a \vee b \leq_\wedge b \leq_\wedge c \leq_* x$. Thus, $x \in [a \vee a]_f$ and $x \in [b \vee b]_f$, i.e. $x \in [a \vee a]_f \cap [b \wedge b]_f$.

Conversely, let $x \in [a \vee a]_f \cap [b \vee b]_f$, then: $x \in [a \vee a]_f$ and $x \in [b \vee b]_f$. From $x \in [a \wedge a]_f$ by Corollary 8 follows that there exists $c \in L$ such that: $a \vee a \leq_\wedge c \leq_* x$ and from Lemma 6 we have: $a \vee b \leq_\wedge a \leq_\wedge a \vee a$ and it follows that $a \vee b \leq_\wedge c \leq_* x$, i.e. $x \in [a \vee b]_f$. If $x \in [b \wedge b]_f$, then we similarly obtain that $x \in [a \vee b]_f$. Thus, $[a \vee b]_f = [a \vee a]_f \cup [b \vee b]_f$. The rest of the assertions are proved similarly.

For every $a \in L$, we denote: $Bf(a) = \{X \in FF(L) | a \in X\}$ and $Bi(a) = \{X \in FI(L) | a \in X\}$.
Let us define on the sets $Bf(L) = \{Bf(a) | a \in L\}$ and $Bi(L) = \{Bi(a) | a \in L\}$ the binary operations $\cap^*$ and $\cup^*$ in the following manner:

$$Bf(a) \cap^* Bf(b) = \{F \in FF(L) | (\exists X \in Bf(a \wedge a))(\exists Y \in Bf(b \wedge b)), X \cup Y \subseteq F\};$$

$$Bf(a) \cup^* Bf(b) = \{F \in FF(L) | (\exists X \in Bf(a \wedge a))(\exists Y \in Bf(b \wedge b)), X \cap Y \subseteq F\};$$

$$Bi(a) \cap^* Bi(b) = \{I \in FI(L) | (\exists X \in Bi(a \wedge a))(\exists Y \in Bi(b \wedge b)), X \cup Y \subseteq I\};$$

$$Bi(a) \cup^* Bi(b) = \{I \in FI(L) | (\exists X \in Bi(a \wedge a))(\exists Y \in Bi(b \wedge b)), X \cap Y \subseteq I\}.$$

Lemma 17 *Let $L = (L; \wedge, \vee, *, \triangle)$ be an interlaced weakly idempotent bilattice. Then the following equalities are valid on $(Bf(L); \cap^*, \cup^*)$ and $(Bi(L); \cap^*, \cup^*)$:*

$$\begin{aligned} Bf(a \wedge b) = Bf(a) \cap^* Bf(b);\ \ Bf(a \vee b) = Bf(a) \cup^* Bf(b);\\ Bf(a * b) = Bf(a) \cap^* Bf(b);\ \ Bf(a \triangle b) = Bf(a) \cup^* Bf(b);\\ Bi(a \wedge b) = Bi(a) \cap^* Bi(b);\ \ Bi(a \vee b) = Bi(a) \cup^* Bi(b);\\ Bi(a * b) = Bi(a) \cup^* Bi(b); Bi(a \triangle b) \quad = Bi(a) \cap^* Bi(b). \end{aligned}$$

Proof Let us show the first equality. Suppose that $F \in Bf(a \wedge b)$, then $a \wedge b \in F$. Let $c \in [a \wedge b]_f$, then by Corollary 8, there exists such $c \in L$ that: $a \wedge b \leq_\wedge z \leq_* c = K([c])$, so $[a \wedge b]_f \subseteq F$. By Lemma 16 we have: $[a \wedge a]_f \cup [b \wedge b]_f \subseteq [a \wedge b]_f$. Thus, there exist $X = [a \wedge a]_f$ and $Y = [b \wedge b]_f$ such that $X \cap Y \in F$, i.e. $F \in Bf(a) \cap^* Bf(b)$. Conversely, let $F \in Bf(a) \cap^* Bf(b)$, then there exist $X \in Bf(a \wedge a)$ and $Y \in Bf(b \wedge b)$ such that $X \cup Y \in F$; hence, $a \wedge a \in X$ and $b \wedge b \in Y$. Then we see that $a \wedge a, b \wedge b \in X \cup Y \in F$ and $a \wedge b = (a \wedge a) \wedge (b \wedge b) \in F$. Thus, $F \in Bf(a \wedge b)$ and $Bf(a \wedge b) = Bf(a) \cap^* Bf(b)$. The rest of the equalities are proved similarly.

From Lemma 17, it immediately follows that $Bf(L) = (Bf(L); \cap^*, \cup^*)$ and $Bi(L) = (Bi(L); \cap^*, \cup^*)$ satisfy the identities (1)–(5).Hence, $Bf(L)$ and $Bi(L)$ are weakly idempotent lattices.

Theorem 2 *Every interlaced weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$ is isomorphic to the superproduct of the two weakly idempotent lattices $Bf(L) = (Bf(L); \cap^*, \cup^*)$ and $Bi(L) = (Bi(L); \cap^*, \cup^*)$, i.e.*

$$L \cong Bf(L) \bowtie Bi(L).$$

Proof Let us show that the mapping:

$$\varphi : L \to (Bf(L); \cap^*, \cup^*) \bowtie (Bi(L); \cap^*, \cup^*)$$

which is defined by:

$$\varphi : x \mapsto (Bf(x), Bi(x)) \tag{14}$$

is an isomorphism. The proof of the injectivity of φ is similar to the proof of Theorem 1, while the surjectivity is obvious.
The fact that φ is a homomorphism follows from Lemma 17:
$\varphi(x \wedge y) = (Bf(x \wedge y), Bi(x \wedge y)) = (Bf(x) \cap^* Bf(y), Bi(x) \cap^* Bi(y)) = (Bf(x), Bi(x))(\cap^*, \cap^*)(Bf(y), Bi(y)) = \varphi(x)(\cap^*, \cap^*)\varphi(y)$;
$\varphi(x \vee y) = (Bf(x \vee y), Bi(x \vee y)) = (Bf(x) \cup^* Bf(y), Bi(x) \cup^* Bi(y)) = (Bf(x), Bi(x))(\cup^*, \cup^*)(Bf(y), Bi(y)) = \varphi(x)(\cup^*, \cup^*)\varphi(y)$;
$\varphi(x * y) = (Bf(x * y), Bi(x * y)) = (Bf(x) \cap^* Bf(y), Bi(x) \cup^* Bi(y)) = (Bf(x), Bi(x))(\cap^*, \cup^*)(Bf(y), Bi(y)) = \varphi(x)(\cap^*, \cup^*)\varphi(y)$;
$\varphi(x \triangle y) = (Bf(x \triangle y), Bi(x \triangle y)) =$
$(Bf(x) \cup^* Bf(y), Bi(x) \cap^* Bi(y)) = (Bf(x), Bi(x))(\cup^*, \cap^*)(Bf(y), Bi(y)) = \varphi(x)(\cup^*, \cap^*)\varphi(y)$.

Note that the operations $\cup^*$ and $\cap^*$ on the sets $Bf(L)$ and $Bi(L)$ coincide with the operations $\cup^{**}$ and $\cap^{**}$ respectively, which are defined in the following way:

$$Bf(a) \cup^{**} Bf(b) = Bf(a \wedge a) \cup Bf(b \wedge b) \cup \{F \in FF(L) | \exists X \in Bf(a \wedge a), \exists Y \in Bf(b \wedge b), X \cap Y \subseteq F\},$$
$$Bf(a) \cap^{**} Bf(b) = Bf(a \wedge a) \cap Bf(b \wedge b) \cap \{F \in FF(L) | \exists X \in Bf(a \wedge a), \exists Y \in Bf(b \wedge b), X \cup Y \subseteq F\},$$
$$Bi(a) \cup^{**} Bi(b) = Bi(a \wedge a) \cup Bi(b \wedge b) \cup \{I \in FI(L) | \exists X \in i(a \wedge a), \exists Y \in i(b \wedge b), X \cap Y \subseteq I\},$$
$$Bi(a) \cap^{**} Bi(b) = Bi(a \wedge a) \cap Bi(b \wedge b) \cap \{I \in FI(L) | \exists X \in i(a \wedge a), \exists Y \in i(b \wedge b), X \cup Y \subseteq I\}.$$

Moreover, $Bf(a) \cap^{**} Bf(b) = Bf(a \wedge a) \cap Bf(b \wedge b)$ and $Bi(a) \cap^{**} Bi(b) = Bi(a \wedge a) \cap Bi(b \wedge b)$.

Now, let us introduce the concepts of prime filters and prime ideals for a weakly idempotent bilattice.

Definition 17 The filter F of a weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$ is called prime, if it is prime filter for both reducts $L_1 = (L; \wedge, \vee)$ and $L_2 = (L; *, \triangle)$.

Definition 18 The ideal I of a weakly idempotent bilattice $L = (L; \wedge, \vee, *, \triangle)$ is called prime, if it is a prime filter for the reduct $L_1 = (L; \wedge, \vee)$ and prime ideal for the reduct $L_2 = (L; *, \triangle)$.

Denote the set of all prime filters of a weakly idempotent bilattice L by $pBF(L)$, and the set of all prime ideals of a weakly idempotent bilattice L by $pBI(L)$. Let us make the following designations: $pBf(a) = \{X \in pBF(L) | a \in X\}$ and $pBi(a) = \{X \in pBI(L) | a \in X\}$, $pBf(L) = \{pBf(a) | a \in L\}$ and $pBi(L) = \{pBi(a) | a \in L\}$.

The operations $\cap^{**}$ and $\cup^{**}$ can be defined on $pBf(L)$ and $pBi(L)$ and an analogue of Lemma 17 can be proven. So, the sets $pBF(L)$ and $pBI(L)$ are closed under the operations $\cap^{**}$ and $\cup^{**}$.

Corollary 10 *Every distributive weakly idempotent bilattice* $L = (L; \wedge, \vee, *, \triangle)$ *is isomorphic to the superproduct of the two distributive weakly idempotent lattices:* $pBf(L) = (pBf(L); \cap^{**}, \cup)$ *and* $pBi(L) = (pBi(L); \cap^{**}, \cup)$*, i.e.*

$$L \cong pBf(L) \bowtie pBi(L).$$

Proof Similarly to the proof of Corollary 6, we show that on the sets $pBf(L)$ and $pBi(L)$ the operations $\cup^{**}$ and $\cup^{**}$ convert to the set-theoretical union. Using the identities of Lemma 17 for prime filters and prime ideals we get that $pBf(L)$ and $pBi(L)$ are weakly idempotent lattices. On the set $pBf(L)$ the operation $\cap^{**}$, we redefine in the following way: $pBf(a) \cap^{**} pBf(b) = pBf(a \wedge a) \cap pBf(b \wedge b)$ and $pBi(a) \cap^{**} pBi(b) = pBi(a \wedge a) \cap pBi(b \wedge b)$. Hence, the weakly idempotent lattices $pBf(L)$ and $pBi(L)$ are distributive.

Denote the set of all filters of the bilattice L by $\tilde{BF}(L)$ and the set of all ideals by $\tilde{BI}(L)$. Let us make the following designations: $\tilde{Bf}(a) = \{X \in \tilde{BF}(L) | a \in X\}$, $\tilde{Bi}(a) = \{X \in \tilde{Bi}(L) | a \in X\}$, $\tilde{Bf}(L) = \{\tilde{Bf}(a) | a \in L\}$, $\tilde{Bi}(L) = \{\tilde{Bi}(a) | a \in L\}$.

Corollary 11 *Every interlaced bilattice* $L = (L; \wedge, \vee, *, \triangle)$ *is isomorphic to the superproduct of the two lattices* $\tilde{Bf}(L) = (\tilde{Bf}(L); \cap, \cup^{**})$ *and* $\tilde{Bi}(L) = (\tilde{Bi}(L); \cap, \cup^{**})$*, i.e.*

$$L \cong \tilde{Bf}(L) \bowtie \tilde{Bi}(L).$$

Proof Since L is a bilattice, the operation $\cap^{**}$ is converted into the set-theoretical intersection, and the operation $\cup^{**}$ is defined as follows:

$$\tilde{f}(a) \cup^{**} \tilde{f}(b) = \tilde{f}(a) \cup \tilde{f}(b) \cup \{F \in \tilde{F}(L) | (\exists X \in \tilde{f}(a))(\exists Y \in \tilde{f}(b)) X \cap Y \subseteq F\},$$

$$\tilde{i}(a) \cup^{**} \tilde{i}(b) = \tilde{i}(a) \cup \tilde{i}(b) \{F \in \tilde{F}(L) | (\exists X \in \tilde{i}(a))(\exists Y \in \tilde{i}(b)) X \cap Y \subseteq F\}.$$

It is obvious that $\tilde{Bf}(L)$ and $\tilde{Bi}(L)$ are lattices. Indeed: $\tilde{Bf}(a) \cap \tilde{Bf}(a) = \tilde{Bf}(a \wedge a) = \tilde{Bf}(a)$, $\tilde{Bi}(a) \cup^{**} \tilde{Bi}(a) = \tilde{Bi}(a \vee a) = \tilde{Bi}(a)$.

Denote the set of all prime filters of a bilattice L by $p\tilde{B}F(L)$ and the set of all prime ideals of L by $p\tilde{B}I(L)$. Let us make the following designations: $p\tilde{B}f(a) = \{X \in p\tilde{B}F(L) | a \in X\}$, $p\tilde{B}i(a) = \{X \in p\tilde{B}I(L) | a \in X\}$, $p\tilde{B}f(L) = \{p\tilde{B}f(a) | a \in L\}$, $p\tilde{B}i(L) = \{p\tilde{B}i(a) | a \in L\}$

Corollary 12 *Every distributive bilattice* $L = (L; \wedge, \vee, *, \Delta)$ *is isomorphic to the superproduct of the two distributive lattices* $p\tilde{B}f(L) = (p\tilde{B}f(L); \cap, \cup)$ *and* $p\tilde{B}i(L) = (p\tilde{B}i(L); \cap, \cup)$*, i.e.*

$$L \cong pB\tilde{f}(L) \bowtie pB\tilde{i}(L).$$

Proof Similarly to the proof of Corollary 6, we get that the operations $\cup^{**}$ and $\cup^{**}$ convert into the set-theoretical union on the sets $p\tilde{B}f(L)$ and $p\tilde{B}i(L)$, respectively.

For the bounded case last result was proved by Gargov [26].

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