On Functional Equations with Division and Regular Operations

S. Davidov, A. Krapež and Yu. Movsisyan

Yerevan State University, Yerevan;
Mathematical Institute of the Serbian Academy of Science and Arts;
Yerevan State University, Yerevan;
E-mail:*davidov@ysu.am, sasa@mi.sanu.ac.rs, yurimovsisyan@yahoo.com*

Binary algebra is an algebra with binary operations. Let $(Q;\cdot)$ be a groupoid and $a \in Q$. Denote by L_a (R_a) the map of Q to Q such that $L_a(x) = ax$ $(R_a(x) = xa)$ for all $x \in Q$.

A groupoid $(Q;\cdot)$ is said to be a division (cancellation) groupoid if L_a and R_a are surjective (injective) for every $a \in Q$. If the groupoid $(Q;\cdot)$ is a division (cancellation) groupoid then the operation $(\cdot)$ is called division (cancellation) operation. A quasigroup is a cancellation and division groupoid. A loop is a quasigroup $(Q;\cdot)$ with the unit e such that $ex = xe = x$ for any $x \in Q$.

A binary algebra $(Q;\Sigma)$ is called division (cancellation) algebra if $(Q;A)$ is a division (cancellation) groupoid for any operation $A \in \Sigma$.

A groupoid $(Q;\cdot)$ is called left quasigroup-regular or left regular, in short, if:

$$ca = cb \to R_a = R_b,$$

where $a, b, c \in Q$. Right quasigroup-regularity of groupoids or right regularity of groupoids, in short, are defined dually. A groupoid is quasigroup-regular or just regular, in short, if it is both left and right regular. A binary algebra $(Q;\Sigma)$ is called quasigroup-regular or just regular, in short, if the groupoid $(Q;A)$ is regular for any operation $A \in \Sigma$ (in contrast to semigroup-regularity (i.e regularity in the semigroup theory) and ring-regularity.

The groupoid $(Q;\cdot)$ is called homotopic to the groupoid $(Q;\circ)$ if there exist three maps α, β, γ of Q to Q such that $\gamma(x \cdot y) = \alpha x \circ \beta y$ for all $x, y \in Q$. The homotopy of the form $T = (\alpha, \beta, id_Q)$ (where id_Q is the identity map of Q) is called principal homotopy. We say that the groupoid $(Q;\cdot)$ is epitopic to the groupoid $(Q;\circ)$ if the mappings α, β, γ are surjections. So we have the concept of principal epitopicity as well.

The following Lemmas are general versions of the corresponding Albert theorems for quasigroups [3, 4, 5, 6].

Lemma 1. *Every division and regular groupoid* $(Q;\cdot)$ *is principally epitopic to a certain loop* $(Q;+)$.

Lemma 2. *If the loop* $(Q;\circ)$ *is principally homotopic to the group* $(Q;\cdot)$, *then they are isomorphic. Hence, if the group* $(Q;\circ)$ *is principally homotopic to the group* $(Q;\cdot)$, *then they are isomorphic.*

Functional equations are the equations in which the unknown (or unknowns) are functions. In this talk we give solutions of: associative, medial, paramedial, transitive and Kolmogoroff equations [1, 2] in algebras with division and regular operations, i.e. we give solutions of various classical equations in algebras with division and regular operations, which frequently appear in algebra (even in classical algebra) and its applications.

References

[1] J. Aczel. Remarques algebriques sur la solution donnee par M.Frechet a l'equation de Kolmogoroff. *Pupl. Math. Debrecen*, 4:33-42, 1955.

[2] J. Aczel and J. Dhombres. Functional Equations in Several Variables. *Cambridge University Press*, 1991.

[3] A. A. Albert. Quasigroups I. *Trans. Amer. Math. Soc.*, 54:507-519, 1943.

[4] A. A. Albert. Quasigroups II. *Trans. Amer. Math. Soc.*, 55:401-419, 1944.

[5] H. O. Pflugfelder. Quasigroups and loops: Introduction. *Sigma Series in Pure Mathematics*, 1990.

[6] J. D. H. Smith. An Introduction to Quasigroups and Their Representations. *Chapman and Hall/CRC*, 2007.