ALGEBRAS WITH MEDIAL-LIKE FUNCTIONAL EQUATIONS ON QUASIGROUPS

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Abstract. We consider 14 medial-like balanced functional equations with four object variables for a pair (f, g) of binary quasigroup operations. Then, we prove that every algebra $(B; f, g)$ with quasigroup operations satisfying a medial-like balanced functional equation has a linear representation on an abelian group $(B; +)$.

1. Introduction

A binary algebra $\mathbf{B}$ is an ordered pair $(B; F)$, where B is a nonempty set and F is a family of binary operations $f : B^2 \to B$. The set B is called the universe (base, underlying set) of the algebra $\mathbf{B} = (B; F)$. If F is finite, say $F = \{f_1, \ldots, f_k\}$, we often write $(B; f_1, \ldots, f_k)$ for $(B; F)$, by [5]. The algebra $\mathbf{B}$ is a groupoid if it has only one binary operation.

A binary quasigroup is usually defined to be a groupoid $(B; f)$ such that for any $a, b \in B$ there are unique solutions x and y to the following equations:

$$f(a, x) = b \quad \text{and} \quad f(y, a) = b,$$

by [13]. If $(B; f)$ is quasigroup we say that f is a quasigroup operation. A loop is a quasigroup with unit (e) such that

$$f(e, x) = f(x, e) = x.$$

Groups are associative quasigroups, i.e. they satisfy:

$$f(f(x, y), z) = f(x, f(y, z))$$

and they necessarily contain a unit. A quasigrouup is commutative if

$$f(x, y) = f(y, x). \tag{1.1}$$

Commutative groups are known as abelian groups.

A triple (α, β, γ) of bijections from a set B onto a set C is called an isotopy of a groupoid $(B; f)$ onto a groupoid $(C; g)$ provided

$$\gamma f(x, y) = g(\alpha x, \beta y)$$

for all $x, y \in B$. $(C; g)$ is then called an isotope of $(B; f)$, and groupoids $(B; f)$ and $(C; g)$ are called isotopic to each other. An isotopy of $(B; f)$ onto $(B; f)$ is called an autotopy of $(B; f)$. Let α and β be permutations of B and let ι denote the identity map on B. Then (α, β, ι) is a principal isotopy of a groupoid $(B; f)$ onto a groupoid $(B; g)$ means that (α, β, ι) is an isotopy of $(B; f)$ onto $(B; g)$. Isotopy is a generalization of isomorphism. Isotopic image of a quasigroup is again a quasigroup. A loop isotopic to a group is isomorphic to it. Every quasigroup is isotopic to some loop i.e., it is a loop isotope.

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A binary quasigroup $(B; f)$ is linear over an abelian group if

$$f(x, y) = \varphi x + a + \psi y,$$

where $(B; +)$ is an abelian group, φ and ψ are automorphisms of $(B; +)$ and $a \in B$ is a fixed element. Quasigroup linear over an abelian group is also called a T-quasigroup.

Quasigroups are important algebraic (combinatorial, geometric) structures which arise in various areas of mathematics and other disciplines. We mention just a few of their applications: in combinatorics (as latin squares, see [6]), in geometry (as nets/webs, see [3]), in statistics (see [8]), in special theory of relativity (see [16]), in coding theory and cryptography ([14]).

2. Functional equations on quasigroups

We use (object) variables x, y, u, v, z and operation symbols (i.e. functional variables) f, g. We assume that all operation symbols represent quasigroup operations. A functional equation is an equality s = t, where s and t are terms with symbols of unknown operations occurring in at least one of them.

Definition 2.1. Functional equation $s = t$ is *balanced* if every (object) variable appears exactly once in s and once in t.

Example 2.2. The following are various functional equations:

$$f(f(x, y), z) = f(x, f(y, z)), \tag{2.1}$$

$$f(f(x, y), f(u, v)) = f(f(x, u), f(y, v)), \tag{2.2}$$

$$f(f(x, y), f(u, v)) = f(f(f(x, u), y), v), \tag{2.3}$$

$$f(x, f(y, z)) = f(f(x, y), f(x, z)), \tag{2.4}$$

$$f(f(x, y), f(y, z)) = f(x, z), \tag{2.5}$$

$$f(x, x) = x. \tag{2.6}$$

Associativity ($Eq.$(2.1)), mediality ($Eq.$(2.2)) and pseudomediality ($Eq.$(2.3)) are balanced, transitivity ($Eq.$(2.5)), left distributivity ($Eq.$(2.4)) and idempotency ($Eq.$(2.6)) are not.

We can define $4! = 24$ balanced functional equations with four object variables on a quasigroup $(B; f)$:

$$f(f(x, y), f(u, v)) = f(f(x, y), f(u, v)) \tag{2.7}$$

$$f(f(x, y), f(u, v)) = f(f(x, y), f(v, u)) \tag{2.8}$$

$$f(f(x, y), f(u, v)) = f(f(x, u), f(y, v)) \tag{2.9}$$

$$f(f(x, y), f(u, v)) = f(f(x, u), f(v, y)) \tag{2.10}$$

$$f(f(x, y), f(u, v)) = f(f(x, v), f(y, u)) \tag{2.11}$$

$$f(f(x, y), f(u, v)) = f(f(x, v), f(u, y)) \tag{2.12}$$

$$f(f(x, y), f(u, v)) = f(f(y, x), f(u, v)) \tag{2.13}$$

$$f(f(x, y), f(u, v)) = f(f(y, x), f(v, u)) \tag{2.14}$$

$$f(f(x, y), f(u, v)) = f(f(y, u), f(x, v)) \tag{2.15}$$

$$f(f(x, y), f(u, v)) = f(f(y, u), f(v, x)) \tag{2.16}$$

$$f(f(x, y), f(u, v)) = f(f(y, v), f(x, u)) \tag{2.17}$$

$$f(f(x, y), f(u, v)) = f(f(y, v), f(u, x)) \tag{2.18}$$

$$f(f(x, y), f(u, v)) = f(f(u, x), f(y, v)) \tag{2.19}$$

$$f(f(x, y), f(u, v)) = f(f(u, x), f(v, y)) \tag{2.20}$$

$$f(f(x,y),f(u,v)) = f(f(u,y),f(x,v)) \tag{2.21}$$

$$f(f(x,y),f(u,v)) = f(f(u,y),f(v,x)) \tag{2.22}$$

$$f(f(x,y),f(u,v)) = f(f(u,v),f(x,y)) \tag{2.23}$$

$$f(f(x,y),f(u,v)) = f(f(u,v),f(y,x)) \tag{2.24}$$

$$f(f(x,y),f(u,v)) = f(f(v,x),f(y,u)) \tag{2.25}$$

$$f(f(x,y),f(u,v)) = f(f(v,x),f(u,y)) \tag{2.26}$$

$$f(f(x,y),f(u,v)) = f(f(v,y),f(x,u)) \tag{2.27}$$

$$f(f(x,y),f(u,v)) = f(f(v,y),f(u,x)) \tag{2.28}$$

$$f(f(x,y),f(u,v)) = f(f(v,u),f(x,y)) \tag{2.29}$$

$$f(f(x,y),f(u,v)) = f(f(v,u),f(y,x)) \tag{2.30}$$

The equation (2.7) is trivial i.e., all quasigroups are solutions of this equation. The equations (2.8), (2.13), (2.14), (2.23), (2.24) and (2.29) are all equivalent to the equation (1.1); solutions are commutative quasigroups.

Definition 2.3. The functional equation (2.9) is called *medial identity* and every quasigroup satisfying medial identity is called *medial quasigroup.*

Theorem 2.4. *If* $(B; f)$ *is a medial quasigroup then there exists an abelian group* $(B; +)$*, such that*

$$f(x,y) = \varphi(x) + c + \psi(y),$$

where $\varphi, \psi \in Aut(B; +)$*,* $\varphi\psi = \psi\varphi$ *and* $c \in Q$*, by* [15].

Definition 2.5. The functional equation (2.28) is called *paramedial identity* and every quasigroup satisfying medial identity is called *paramedial quasigroup.*

Theorem 2.6. *If* $(B; f)$ *is a paramedial quasigroup then there exists an abelian group* $(B; +)$*, such that*

$$f(x,y) = \varphi(x) + c + \psi(y),$$

where $\varphi, \psi \in Aut(B; +)$*,* $\varphi\varphi = \psi\psi$ *and* $c \in Q$*, by* [12].

Definition 2.7. A balanced equation $s = t$ is *Belousov* if for every subterm p of s (t) there is a subterm q of t (s) such that p and q have exactly the same variables.

Example 2.8. Functional equations (1.1), (2.30) and the following are Belousov equations:

$$f(x,y) = f(x,y),$$
$$f(x,f(y,z)) = f(f(z,y),x).$$

The equations (2.1) – (2.3) are non-Belousov.

By [9], we have the following results:

Theorem 2.9. *The solutions of the equation* (2.30) *belong to the variety of* 4*-palindromic quasigroups.*

Theorem 2.10. *The equations* (2.10) – (2.12)*,* (2.15) – (2.22) *and* (2.25) – (2.27) *are equivalent to commutative (para)mediality; solutions constitute the variety of commutative* T*-quasigroups (i.e., with* $\varphi = \psi$*).*

3. Algebras with Medial-like functional equations

As a generalization of the functional equations (2.7) – (2.30), let us consider the following balanced functional equations:

$$f(g(x,y),g(u,v)) = g(f(x,y),f(u,v)) \tag{3.1}$$
$$f(g(x,y),g(u,v)) = g(f(x,y),f(v,u)) \tag{3.2}$$
$$f(g(x,y),g(u,v)) = g(f(x,u),f(y,v)) \tag{3.3}$$
$$f(g(x,y),g(u,v)) = g(f(x,u),f(v,y)) \tag{3.4}$$
$$f(g(x,y),g(u,v)) = g(f(x,v),f(y,u)) \tag{3.5}$$
$$f(g(x,y),g(u,v)) = g(f(x,v),f(u,y)) \tag{3.6}$$
$$f(g(x,y),g(u,v)) = g(f(y,x),f(u,v)) \tag{3.7}$$
$$f(g(x,y),g(u,v)) = g(f(y,x),f(v,u)) \tag{3.8}$$
$$f(g(x,y),g(u,v)) = g(f(y,u),f(x,v)) \tag{3.9}$$
$$f(g(x,y),g(u,v)) = g(f(y,u),f(v,x)) \tag{3.10}$$
$$f(g(x,y),g(u,v)) = g(f(y,v),f(x,u)) \tag{3.11}$$
$$f(g(x,y),g(u,v)) = g(f(y,v),f(u,x)) \tag{3.12}$$
$$f(g(x,y),g(u,v)) = g(f(u,x),f(y,v)) \tag{3.13}$$
$$f(g(x,y),g(u,v)) = g(f(u,x),f(v,y)) \tag{3.14}$$
$$f(g(x,y),g(u,v)) = g(f(u,y),f(x,v)) \tag{3.15}$$
$$f(g(x,y),g(u,v)) = g(f(u,y),f(v,x)) \tag{3.16}$$
$$f(g(x,y),g(u,v)) = g(f(u,v),f(x,y)) \tag{3.17}$$
$$f(g(x,y),g(u,v)) = g(f(u,v),f(y,x)) \tag{3.18}$$
$$f(g(x,y),g(u,v)) = g(f(v,x),f(y,u)) \tag{3.19}$$
$$f(g(x,y),g(u,v)) = g(f(v,x),f(u,y)) \tag{3.20}$$
$$f(g(x,y),g(u,v)) = g(f(v,y),f(x,u)) \tag{3.21}$$
$$f(g(x,y),g(u,v)) = g(f(v,y),f(u,x)) \tag{3.22}$$
$$f(g(x,y),g(u,v)) = g(f(v,u),f(x,y)) \tag{3.23}$$
$$f(g(x,y),g(u,v)) = g(f(v,u),f(y,x)) \tag{3.24}$$

Definition 3.1. A pair (f, g) of binary operations is called:

- *medial pair of operations*, if the algebra $(B; f, g)$ satisfies the equation (3.3).
- *paramedial pair of operations*, if the algebra $(B; f, g)$ satisfies the equation (3.22).

Definition 3.2. A binary algebra $\mathbf{B} = (B; F)$ is called:

- *medial algebra*, if every pair of operations of the algebra $\mathbf{B}$ is medial (or, the algebra $\mathbf{B}$ satisfying medial hyperidentity).
- *paramedial algebra*, if every pair of operations of the algebra $\mathbf{B}$ is paramedial (or, the algebra $\mathbf{B}$ satisfying paramedial hyperidentity).

The following results are obtained in [11] and [7] respectively.

Theorem 3.3. *Let the set B, forms a quasigroup under the binary operations f and g. If the pair of binary operations (f, g) is medial, then there exists a binary operation,$+$, under which B forms an abelian group and for arbitrary elements $x, y \in B$ we have:*

$$f(x, y) = \varphi_1(x) + \psi_1(y) + c_1,$$

$$g(x,y) = \varphi_2(x) + \psi_2(y) + c_2,$$

where c_1, c_2 *are fixed elements of* B*, and* $\varphi_i, \psi_i \in Aut(B;+)$ *for* $i = 1,2$*, such that:* $\varphi_1\psi_2 = \psi_2\varphi_1$*,* $\varphi_2\psi_1 = \psi_1\varphi_2$*,* $\psi_1\psi_2 = \psi_2\psi_1$ *and* $\varphi_1\varphi_2 = \varphi_2\varphi_1$*. The group* $(B;+)$*, is unique up to isomorphisms.*

Theorem 3.4. *Let the set* B*, forms a quasigroup under the binary operations* f *and* g*. If the pair of binary operations* (f,g) *is paramedial, then there exists a binary operation,+, under which* B *forms an abelian group and for arbitrary elements* $x, y \in B$ *we have:*

$$f(x,y) = \varphi_1(x) + \psi_1(y) + c_1,$$
$$g(x,y) = \varphi_2(x) + \psi_2(y) + c_2,$$

where c_1, c_2 *are fixed elements of* B*, and* φ_i *and* ψ_i *are automorphisms on the abelian group* $(B;+)$ *for* $i = 1,2$*, such that:* $\varphi_1\varphi_2 = \psi_2\psi_1$*,* $\varphi_2\varphi_1 = \psi_1\psi_2$*,* $\varphi_1\psi_2 = \varphi_2\psi_1$ *and* $\psi_1\varphi_2 = \psi_2\varphi_1$*. The group* $(B;+)$*, is unique up to isomorphisms.*

Example 3.5. Let $B = Z_2 \times Z_2$. We denote the elements of the group B as follows:

$$\{(0;0),(1;0),(0;1),(1;1)\}$$

and the set of all automorphisms of B as follow:

$$AutB = \{\varepsilon, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6\},$$

where $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\varphi_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\varphi_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\varphi_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\varphi_5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $\varphi_6 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

If $\varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x_1 & x_2 \end{pmatrix}$ then

$$\varphi(X) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Put

$$f_1(x,y) = \varphi_2(x) + \varphi_3(y) + c,$$
$$f_2(x,y) = \varphi_3(x) + \varphi_2(y) + c,$$
$$f_3(x,y) = \varepsilon(x) + \varphi_5(y),$$
$$f_4(x,y) = \varepsilon(x) + \varphi_6(y),$$

where $c \in B$, then the algebra $(B; f_1, f_2)$ is a paramedial algebra with quasigroup operations which is not medial and $(B; f_3, f_4)$ is a medial algebra with quasigroup operations which is not paramedial.

The balanced functional equations (3.1), (3.2), (3.7), (3.8), (3.17), (3.18), (3.23) and (3.24) are Belousov, while the balanced functional equations (3.3) – (3.6), (3.9) – (3.16) and (3.19) – (3.22) are non-Belousov.

Definition 3.6. If $(B;\cdot)$ is a group, then the bijection, $\alpha : B \to B$, is called a *holomorphism* of $(B;\cdot)$ if

$$\alpha(x \cdot y^{-1} \cdot z) = \alpha x \cdot (\alpha y)^{-1} \cdot \alpha z,$$

for every $x, y, z \in B$.

Note that this concept is equivalent to the concept of quasiautomorphism of groups, by [2]. The set of all holomorphisms of $(B;\cdot)$ is denoted by $Hol(B;\cdot)$ and it is a group under the superposition of mappings: $(\alpha \cdot \beta)x = \beta(\alpha x)$, for every $x \in B$.

The following properties of holomorphisms were proved for Muofang loops in [10].

Lemma 3.7. *Let for bijections $\alpha_1, \alpha_2, \alpha_3$ on the group $(B;\cdot)$, the following identity is satisfied:*

$$\alpha_1(x \cdot y) = \alpha_2(x) \cdot \alpha_3(y).$$

Then $\alpha_1, \alpha_2, \alpha_3 \in Hol(B;\cdot)$.

Lemma 3.8. *Every holomorphism α of the group $(B;\cdot)$ has the following form:*

$$\alpha x = \varphi x \cdot k$$

where, $\varphi \in Aut(B;\cdot)$ and $k \in B$.

Theorem 3.9. *Let the set B forms a quasigroup under the binary operations f and g. If the pair (f, g) of binary operations satisfies one of the balanced non-Belousov functional equations ((3.4) – (3.6), (3.9) – (3.16) or (3.19) – (3.21)), then there exists a binary operation '+' under which B forms an abelian group and for arbitrary elements $x, y \in B$ we have:*

$$f(x, y) = \varphi_1(x) + \psi_1(y) + c_1,$$
$$g(x, y) = \varphi_2(x) + \psi_2(y) + c_2,$$

where c_1, c_2 are fixed elements of B and $\varphi_i, \psi_i \in Aut(B;+)$ for $i = 1, 2$, such that:

- **–:** $\varphi_1\varphi_2 = \varphi_2\varphi_1$, $\varphi_1\psi_2 = \psi_2\psi_1$, $\psi_1\varphi_2 = \varphi_2\psi_1$ *and* $\psi_1\psi_2 = \psi_2\varphi_1$ *for the equation* (3.4).
- **–:** $\varphi_1\varphi_2 = \varphi_2\varphi_1$, $\varphi_1\psi_2 = \psi_2\varphi_1$, $\psi_1\varphi_2 = \psi_2\psi_1$ *and* $\psi_1\psi_2 = \varphi_2\psi_1$ *for the equation* (3.5).
- **–:** $\varphi_1\varphi_2 = \varphi_2\varphi_1$, $\varphi_1\psi_2 = \psi_2\psi_1$, $\psi_1\varphi_2 = \psi_2\varphi_1$ *and* $\psi_1\psi_2 = \varphi_2\psi_1$ *for the equation* (3.6).
- **–:** $\varphi_1\varphi_2 = \psi_2\varphi_1$, $\varphi_1\psi_2 = \varphi_2\varphi_1$, $\psi_1\varphi_2 = \varphi_2\psi_1$ *and* $\psi_1\psi_2 = \psi_2\psi_1$ *for the equation* (3.9).
- **–:** $\varphi_1\varphi_2 = \psi_2\psi_1$, $\varphi_1\psi_2 = \varphi_2\varphi_1$, $\psi_1\varphi_2 = \varphi_2\psi_1$ *and* $\psi_1\psi_2 = \psi_2\varphi_1$ *for the equation* (3.10).
- **–:** $\varphi_1\varphi_2 = \psi_2\varphi_1$, $\varphi_1\psi_2 = \varphi_2\varphi_1$, $\psi_1\varphi_2 = \psi_2\psi_1$ *and* $\psi_1\psi_2 = \varphi_2\psi_1$ *for the equation* (3.11).
- **–:** $\varphi_1\varphi_2 = \psi_2\psi_1$, $\varphi_1\psi_2 = \varphi_2\varphi_1$, $\psi_1\varphi_2 = \psi_2\varphi_1$ *and* $\psi_1\psi_2 = \varphi_2\psi_1$ *for the equation* (3.12).
- **–:** $\varphi_1\varphi_2 = \varphi_2\psi_1$, $\varphi_1\psi_2 = \psi_2\varphi_1$, $\psi_1\varphi_2 = \varphi_2\varphi_1$ *and* $\psi_1\psi_2 = \psi_2\psi_1$ *for the equation* (3.13).
- **–:** $\varphi_1\varphi_2 = \varphi_2\psi_1$, $\varphi_1\psi_2 = \psi_2\psi_1$, $\psi_1\varphi_2 = \varphi_2\varphi_1$ *and* $\psi_1\psi_2 = \psi_2\varphi_1$ *for the equation* (3.14).
- **–:** $\varphi_1\varphi_2 = \psi_2\varphi_1$, $\varphi_1\psi_2 = \varphi_2\psi_1$, $\psi_1\varphi_2 = \varphi_2\varphi_1$ *and* $\psi_1\psi_2 = \psi_2\psi_1$ *for the equation* (3.15).
- **–:** $\varphi_1\varphi_2 = \psi_2\psi_1$, $\varphi_1\psi_2 = \varphi_2\psi_1$, $\psi_1\varphi_2 = \varphi_2\varphi_1$ *and* $\psi_1\psi_2 = \psi_2\varphi_1$ *for the equation* (3.16).
- **–:** $\varphi_1\varphi_2 = \varphi_2\psi_1$, $\varphi_1\psi_2 = \psi_2\varphi_1$, $\psi_1\varphi_2 = \psi_2\psi_1$ *and* $\psi_1\psi_2 = \varphi_2\varphi_1$ *for the equation* (3.19).
- **–:** $\varphi_1\varphi_2 = \varphi_2\psi_1$, $\varphi_1\psi_2 = \psi_2\psi_1$, $\psi_1\varphi_2 = \psi_2\varphi_1$ *and* $\psi_1\psi_2 = \varphi_2\varphi_1$ *for the equation* (3.20).
- **–:** $\varphi_1\varphi_2 = \psi_2\varphi_1$, $\varphi_1\psi_2 = \varphi_2\psi_1$, $\psi_1\varphi_2 = \psi_2\psi_1$ *and* $\psi_1\psi_2 = \varphi_2\varphi_1$ *for the equation* (3.21).

The group $(B;+)$ is unique up to isomorphisms.

Proof. Let $(B; f, g)$ be an algebra with the pair (f, g) of binary quasigroup operations satisfies the non-Belousov functional equation (3.4). We use the main result of [1]. Let the set B forms a quasigroup under six operations $A_i(x, y)$ (for $i = 1, \dots, 6$). If these operations satisfy the following equation:

$$A_1(A_2(x, y), A_3(u, v)) = A_4(A_5(x, u), A_6(y, v)), \tag{3.25}$$

for all elements x, y, u and v of the set B then there exists an operation '+' under which B forms an abelian group isotopic to all these six quasigroups. And there exists eight one-to-one mappings; α, β, γ, δ, ϵ, ψ, φ and χ of B onto itself such that:

$$\begin{aligned} A_1(x,y) &= \delta x + \varphi y, \\ A_2(x,y) &= \delta^{-1}(\alpha x + \beta y), \\ A_3(x,y) &= \varphi^{-1}(\chi x + \gamma y), \\ A_4(x,y) &= \psi x + \epsilon y, \\ A_5(x,y) &= \psi^{-1}(\alpha x + \chi y), \\ A_6(x,y) &= \epsilon^{-1}(\beta x + \gamma y). \end{aligned}$$

Now, let $A_i^*(x,y) = A_i(y,x)$ then, using this equality in (3.25) we have:

$$A_1(A_2(x,y), A_3(u,v)) = A_4(A_5(x,u), A_6^*(y,v)), \tag{3.26}$$

and so,

$$A_6^*(x,y) = A_6(y,x) = \epsilon^{-1}(\beta y + \gamma x) = \epsilon^{-1}(\gamma x + \beta y),$$

as $(B;+)$ is an abelian group. So, let

$$\begin{aligned} A_1 = A_5 = A_6^* = f, \\ A_2 = A_3 = A_4 = g. \end{aligned}$$

With these assumptions, from the equation (3.26) we can reach to the equation (3.4). Since $A_1 = A_5$, we have:

$$\begin{aligned} &\delta x + \varphi y = \psi^{-1}(\chi x + \alpha y) \\ &\Rightarrow \psi(\delta x + \varphi y) = \chi x + \alpha y \\ &\Rightarrow \psi(x + y) = \chi(\delta^{-1} x) + \alpha(\varphi^{-1} y) \\ &\Rightarrow \psi \in Hol(B;+) \end{aligned}$$

by, Lemma 3.7. Similarly, since $A_1 = A_6^*$, we have: $\epsilon \in Hol(A;+)$. Therefore, by Lemma 3.8 there exist $\varphi_2, \psi_2 \in Aut(B;+)$ such that:

$$\begin{aligned} \psi x &= \varphi_2 x + a, \\ \epsilon x &= b + \psi_2 x, \end{aligned}$$

where a, b are fixed elements in B. Hence,

$$\begin{aligned} &g(x,y) = A_4(x,y) = \psi x + \epsilon y = \\ &\varphi_2 x + a + b + \psi_2 x = \varphi_2 x + c_2 + \psi_2 x, \end{aligned}$$

where $c_2 = a + b$ is a fixed element in B. By the same manner, we can show that: $\delta, \varphi \in Hol(B;+)$, since $A_2 = A_4^*$ and $A_3 = A_4^*$. So, there exist $\varphi_1, \psi_1 \in Aut(B;+)$ such that:

$$\begin{aligned} \delta x &= \varphi_1 + d, \\ \varphi x &= e + \psi_1, \end{aligned}$$

where d and e are fixed elements in B. Hence,

$$f(x,y) = A_1(x,y) = \delta x + \varphi y = \varphi_1 x + c_1 + \psi_1 y,$$

where $c_1 = d + e$ is a fixed element in B. Now, put:

$$\begin{aligned} f(x,y) &= \varphi_1(x) + \psi_1(y) + c_1, \\ g(x,y) &= \varphi_2(x) + \psi_2(y) + c_2, \end{aligned}$$

in the equation (3.4) then, it is easy to check that $\varphi_1\varphi_2 = \varphi_2\varphi_1$, $\varphi_1\psi_2 = \psi_2\psi_1$, $\psi_1\varphi_2 = \varphi_2\psi_1$ and $\psi_1\psi_2 = \psi_2\varphi_1$. The uniqueness of the group $(B;+)$ follows from the Albert theorem: if two groups are isotopic, then they are isomorphic, by [4].

The proof is similar for the rest of non-Belousov functional equations ((3.5), (3.6), (3.9) – (3.16) and (3.19) – (3.21)). □

Corollary 3.10. *Let $(B; F)$ be a binary algebra with quasigroup operations which satisfies one of the non-Belousov functional equations ((3.4) – (3.6), (3.9) – (3.16) or (3.19) – (3.21)) then there exists an abelian group $(B; +)$ such that every operation $f_i \in F$ is represented by the following rule:*

$$f_i(x, y) = \varphi_i(x) + c_i + \varphi_i(y),$$

where $c_i \in B$ and $\varphi_i \in Aut(B; +)$ such that $\varphi_i\varphi_j = \varphi_j\varphi_i$, for every $1 \leqslant i, j \leqslant |F|$. The group $(B; +)$ is unique up to isomorphisms.

Proof. Let $(B; F)$ satisfies the non-Belousov functional equation (3.4). If $f_0 \in F$ is a fixed operation then, by Theorem 2.10, f is principally isotopic to the abelian group operation '+' on B. Now, if $f_i \in F$ is any operation, then the pair of binary operations (f_0, f_i) satisfies the non-Belousov functional equation (3.4). Hence, f_0 and f_i are principally isotopic to another abelian group operation '$*$' on B. Thus, by transitivity of isotopy, any operation f_i is principally isotopic to the same abelian group operation '+'. Hence, according to the proof of previous theorem we have:

$$f_i(x, y) = \varphi_i(x) + \psi_i(y) + c_i,$$

where $c_i \in B$ and $\varphi_i, \psi_i \in Aut(B; +)$ such that $\varphi_i\varphi_j = \varphi_j\varphi_i$, $\varphi_i\psi_j = \psi_j\psi_i$, $\psi_i\varphi_j = \varphi_j\psi_i$ and $\psi_i\psi_j = \psi_j\varphi_i$, for every $1 \leqslant i, j \leqslant |F|$. If $i = j$ then

$$\psi_i^2 = \psi_i\varphi_i \Rightarrow \psi_i = \varphi_i.$$

So,

$$f_i(x, y) = \varphi_i(x) + c_i + \varphi_i(y),$$

for $1 \leqslant i \leqslant |F|$. By putting

$$f_j(x, y) = \varphi_j(x) + c_j + \varphi_j(y),$$

$$f_i(x, y) = \varphi_i(x) + c_i + \varphi_i(y),$$

in (3.4) we can obtain $\varphi_i\varphi_j = \varphi_j\varphi_i$, for every $1 \leqslant i, j \leqslant |F|$. The proof is similar for the rest of non-Belousov functional equations (3.5), (3.6), (3.9) – (3.16) and (3.19) – (3.21). □

The Theorems 3.3 – 3.9 enable us to give a short proof of Theorems 2.4, 2.6 and 2.10:

Corollary 3.11. *Every quasigroup $(B; f)$ satisfying one of the balanced non-Belousov functional equations (2.9) – (2.12), (2.15) – (2.22) or (2.25) – (2.28) has a linear representation on an abelian group $(B; +)$:*

$$f(x, y) = \varphi x + c + \psi y,$$

where $c \in B$ is fixed element and φ and ψ are automorphisms on the abelian group $(B; +)$ such that:

- **–:** *$\varphi\psi = \psi\varphi$ for the equation (2.9).*
- **–:** *$\varphi = \psi$ for the equations (2.10) – (2.12), (2.15) – (2.22) and (2.25) – (2.27).*
- **–:** *$\varphi^2 = \psi^2$ for the equation (2.28).*

Proof. If $(B; f)$ satisfy the equation (2.9) then by putting $f = g$ and using the Theorem 3.3, the proof is obvious. If $(B; f)$ satisfy one of the equations (2.10) – (2.12), (2.15) – (2.22) or (2.25) – (2.27), then by putting $f = g$ and using the Theorem 3.9, the proof is obvious. If $(B; f)$ satisfy the equation (2.28) then by putting $f = g$ and using the Theorem 3.4, the proof is obvious. □

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