CALOGERO MODEL AND ITS GENERALIZATION

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In this proceedings we review the rational Calogero model bound by external oscillator and Coulomb field using the Dunkl operator approach.

1. INTRODUCTION

The rational *Calogero* model is an integrable system of N one-dimensional identical particles interacting with each other by the inverse-square potential. The Hamiltonian has the following form (see [1] for the review):

$$\mathcal{H}_0 = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j} \frac{g(g \mp 1)}{(x_i - x_j)^2}, \qquad g \geq 0. \tag{1.1}$$

An external harmonic oscillator potential does not affect on its integrability [2]:

$$\mathcal{H}_\omega = \frac{\boldsymbol{p}^2}{2} + \frac{\omega^2 \boldsymbol{x}^2}{2} + \sum_{i<j} \frac{g(g \mp 1)}{(x_i - x_j)^2}, \tag{1.2}$$

were the two vectors describe the momenta and coordinates of the particles with unit mass, $m = 1$,

$$\boldsymbol{p} = (p_1, p_2, \dots, p_N), \qquad \boldsymbol{x} = (x_1, p_2, \dots, x_N).$$

In our units we also set $\hbar = 1$ for the Plank constant. Otherwise, it will change the coupling coefficient by $g(g \mp \hbar)$. The particles are supposed to be identical, and the minus sign in the hamiltonian (1.2) is chosen for bosons while the plus is applied for fermions.

We refer to the system (1.2) as the *Calogero-oscillator* model. Actually, in the literature this system is referred to as the Calogero model, while the unbound system $\mathcal{H}_0$ is referred as the Calogero-Moser system due to Moser who established the Liouville integrability [3]. Our notations are more proper for reflecting the structure of underlying models.

The *Calogero-Coulomb* system is defined in the same way by including the Calogero interaction potential into the N-dimensional Coulomb system with $1/r$ potential [4],

$$\mathcal{H}_\gamma = \frac{\boldsymbol{p}^2}{2} - \frac{\gamma}{r} + \sum_{i<j} \frac{g(g \mp 1)}{(x_i - x_j)^2}, \qquad r = \sqrt{\boldsymbol{x}^2}. \tag{1.3}$$

Apart from the above two models, it has no clear interpretation as a N-particle system since the Coulomb potential mixes the coordinates of different particles in a strange way. Alternatively, one can consider the Hamiltonian $\mathcal{H}_\gamma$ as a single N-dimensional Coulomb particle in the field formed by the Calogero potential.

The three systems $\mathcal{H}_0$, $\mathcal{H}_\omega$ and $\mathcal{H}_\gamma$ are superintegrable [5–7] like their $g = 0$ analogs without the Calogero potential. The superintegrability means that apart from the N (commutative) Liouville integrals they have additional $N - 1$ integrals.

In this proceedings we well shortly review the Calogero, Calogero-oscillatir and Calogero-Coulomb systems pointing out especially on their analogy, respectively, with the free-particle, oscillator and Coulomb systems. In short, the Calogero interaction potential does not change drastically the common properties of these well-known systems. To archive this goal, we will use the Dunkl operator approach [8], which hides the Calogero potential into the derivative providing a structure similar to the covariant derivative.

2. CALOGERO-OSCILLATOR SYSTEM

The similarity between the Calogero model and a free particle, as well as between the Calogero-oscillator model and an oscillator, is clearly elucidated from the perspective of the matrix model reduction and the exchange operator formalism (see [1] for the review). Let us briefly outline the second approach, elaborated independently by Polychronakos [9] and by Brink, Hansson, and Vasiliev [10], which then has been found to be related with seminal work by Dunkl [8]. Following these authors, we can take into account the Calogero interaction, replacing the momenta $p_i = -\imath\partial_i$ by the deformed momenta

$$\pi_i = -\imath\nabla_i, \tag{2.1}$$

defined in terms of the Dunkl operators

$$\nabla_i = \partial_i - \sum_{j\neq i} \frac{g}{x_i - x_j} s_{ij}. \tag{2.2}$$

Here s_{ij} is the exchange operator between the ith and jth coordinates:

$$s_{ij}\psi(\ldots, x_i, \ldots, x_j, \ldots) = \psi(\ldots, x_j, \ldots, x_i, \ldots). \tag{2.3}$$

Amazingly, such operators commute like usual partial derivatives:

$$[\nabla_i, \nabla_j] = 0. \tag{2.4}$$

Meanwhile, their commutations with the coordinates are more involved and expressed through the permutations:

$$[\nabla_i, x_j] = S_{ij} = \begin{cases} -g s_{ij} & \text{for } i \neq j, \\ 1 + g\sum_{k\neq i} s_{ik} & \text{for } i = j. \end{cases} \tag{2.5}$$

In the absence of the inverse-square potential ($g = 0$), the above relations define the usual Heisenberg algebra.

The Calogero-oscillator Hamiltonian (1.2) can be obtained by the restriction of the generalized Hamiltonian

$$\mathcal{H}_\omega^{\text{gen}} = \frac{\boldsymbol{\pi}^2}{2} + \frac{\omega^2 \boldsymbol{x}^2}{2} \tag{2.6}$$

to the symmetric wavefunctions [10]:

$$s_{ij}\psi_s(x) = \psi_s(x). \tag{2.7}$$

In these terms there is a remarkable similarity between the the Calogero-oscillator and ordinary oscillator systems. To show this correspondence, let us introduce an analog of spectrum generated operators,

$$a_i^\pm = \frac{x_i \mp \nabla_i}{\sqrt{2}}, \tag{2.8}$$

where, for simplicity, we set $\omega = 1$ here and in the following. It is easy to see that they obey the same commutation rule as Dunkl operators and coordinates:

$$[a_i, a_j^+] = S_{ij}, \qquad [a_i, a_j] = [a_i^+, a_j^+] = 0. \tag{2.9}$$

Then the generalized Calogero model in harmonic potential potential is expressed via the introduced operators like in the absence of the Calogero potential, $g = 0$ when the system is reduced to the usual oscillator model [9, 10],

$$\mathcal{H}_\omega^{\text{gen}} = \frac{1}{2}\sum_{i=1}^{N}(a_i^+ a_i + a_i a_i^+) = \sum_{i=1}^{N} a_i^+ a_i + \frac{N}{2} - S. \tag{2.10}$$

In the derivation of the second equation we have used the commutation rule (2.9) with the notation

$$S = \sum_{i<j} S_{ij}. \tag{2.11}$$

Note that this element commuted with all permutations,

$$[s_{ij}, S] = 0. \tag{2.12}$$

The restriction of the Hamiltonian to bosonic (fermionic) wave functions where the pair exchange take the values ± 1, is given by the expression

$$\mathcal{H}_\omega = \sum_{i=1}^{N} a_i^+ a_i + E_0^\pm$$

with the ground state energy

$$E_0^\pm = \frac{N}{2} \pm \frac{gN(N-1)}{2}. \tag{2.13}$$

The plus stands for the bosonic wave function while the minus is for the fermions,

$$s_{ij}\psi_\pm(\boldsymbol{x}) = \pm\psi_\pm(\boldsymbol{x}). \tag{2.14}$$

The analogy with oscillator goes further if we look at the commutation rules of these operators with the Hamiltonian

$$[\mathcal{H}_\omega^{\text{gen}}, a_i^\pm] = \pm a_i^\pm. \tag{2.15}$$

The bosonic ground state wave function $\psi_0 = \psi_{0+}$ with the ground state energy E_0^+ must be annihilated under the action of all lowering operators [10],

$$a_i\psi_0(\boldsymbol{x}) = 0 \qquad \text{or} \qquad \frac{\partial_i\psi_0(\boldsymbol{x})}{\psi_0(\boldsymbol{x})} = -x_i + \sum_{j\neq i} \frac{g}{x_i - x_j}.$$

Using the logarithmic derivative, the solution can be easily found,

$$\psi_0(\boldsymbol{x}) = \prod_{i<j} |x_i - x_j|^g \exp\left(-\tfrac{1}{2}\boldsymbol{x}^2\right).$$

Similarly to the isotropic oscillator case, the excited states are generated by the rising operators acting on the ground state. So, up to a normalisation factor, they are

$$\psi_{n_1\dots n_N}(\boldsymbol{x}) = (a_1^+)^{n_1}(a_2^+)^{n_2}\dots(a_N^+)^{n_N}\psi_0(\boldsymbol{x}), \tag{2.16}$$

where n_i are nonnegative integer numbers, $n_i \geq 0$. Using the relation (2.15), it can be checked that these wave functions are eigenstates of the nonlocal Hamiltinian

$$\mathcal{H}_\omega^{\text{gen}}\psi_{n_1\dots n_N} = E_{n_1\dots n_N}\psi_{n_1\dots n_N}, \qquad E_{n_1\dots n_N} = E_0 + \frac{N}{2} + \sum_{i=1}^{N} n_i. \tag{2.17}$$

In order to construct the excitations of the local Calogero-oscillator Hamiltonian, we must restrict ourself to the symmetric wave functions and hence symmetrized rising-lowering operators

$$B_l^\pm = \sum_{i=1}^{N} (a_i^\pm)^l. \tag{2.18}$$

Then the excitations are described by the states [11]

$$|k_1, \dots, k_N\rangle = (B_1^+)^{k_1}(B_2^+)^{k_2}\dots(B_N^+)^{k_N}\psi_0(\boldsymbol{x}) \tag{2.19}$$

with being again some nonnegative integers, $k_i = 0, 1, 2\dots$. Taking into account the fact that the operator B_l rises the spectrum level on the value l due to the relation

$$[\mathcal{H}_\omega, B_l^+] = [\mathcal{H}_\omega^{\text{gen}}, B_l^+] = \pm l B_l^+, \tag{2.20}$$

we immediately get the explicit spectrum for the Calogero-oscillator Hamiltonian with the bosons

$$E_{k_1\ldots k_N} - E_0 + k_1 + 2k_2 + \cdots + Nk_N.$$

The Liouville integrals of motion of the unbound nonlocal Calogeo model $\mathcal{H}_0^{\text{gen}}$ are just the commuting Dunkl momenta. For the local Hamiltonian $\mathcal{H}_0$ they must be symmetrised in order to act correctly on the bosonic or fermionic wave functions [1],

$$I_k = \sum_i \pi_i^k, \qquad k = 1, 2, \ldots, N. \tag{2.21}$$

For the Calogero system bound by the oscillator potential $\mathcal{H}_\omega$ they are given by [9]

$$I_k = \sum_i (a_i^+ a_i)^k. \tag{2.22}$$

Apart from the Liouville integrals, these models possess additional $N - 1$ integrals of motion. Such systems are called superintegrable. Such additional constants can be described using the deformed angular momentum operator. Define the Dunkl angular momentum tensor by substituting the Dunkl momentum in place of the usual momentum [12, 13]:

$$L_{ij} = x_i \pi_j - x_j \pi_i : \tag{2.23}$$

Using the commutation relations between coordinated and Dunkl momentum, it is easy to show that the deformed momentum are integrals for generalised Calogero and Calogero-oscillator Hamiltonians, which are distinct from the Liouville ones:

$$[L_{ij}, \mathcal{H}_0^{\text{gen}}] = [L_{ij}, \mathcal{H}_\omega^{\text{gen}}] = 0.$$

The commutator between L_{ij} are deformations of usual $so(N)$ angular momentum computational rule

$$[L_{ij}, L_{kl}] = -\imath L_{il} S_{jk} - \imath L_{jk} S_{il} + \imath L_{ik} S_{lj} + \imath L_{jl} S_{ik}.$$

The constants of motion of the Calogero-oscillator model $\mathcal{H}_\omega$ can be associated with the symmetric polynomials ensuring their valid action on the wave functions (2.7). 13

$$\mathcal{L}_{2k} = \sum_{i<j} L_{ij}^{2k}. \tag{2.24}$$

3. CALOGERO-COULOMB SYSTEM AND OTHER EXTENSIONS

Define generalized Calogero-Coulomb Hamiltonian in terms of Dunkl operators [6]:

$$\mathcal{H}_\gamma^{\text{gen}} = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma}{r}. \tag{3.1}$$

After the reduction to the bosonic or fermionic states it reduced to the Calogero-Coulomb model (1.3), first introduced by Khare [ ], The last system is maximally superintegrable [6, 7]. The nonlocal Hamiltonian preserves the Dunkl angular momentum and Runge-Lenz vector:

$$[\mathcal{H}_\gamma^{\text{gen}}, L_{ij}] = 0, \qquad [\mathcal{H}_\gamma^{\text{gen}}, A_i] = 0. \tag{3.2}$$

Here the deformed expression for the Runge-Lenz vector has the following form [7]:

$$A_i = \frac{1}{2} \sum_j \{L_{ij}, \pi_j\} - \frac{\gamma x_i}{r} + \frac{\imath}{2}[\pi_i, S], \tag{3.3}$$

where the permutation-invariant element S is defined in equation (2.11). It vanished at the $g = 0$ point so that the expression of A_i reduces to the standard N-dimensional Runge-Lenz vector in this case. The deformed angular momentum and Runge-Lenz vector form with the Hamiltonian a quadratic algebra,

$$[A_i, L_{kl}] = \imath A_k S_{li} - \imath A_l S_{ki}, \qquad [A_i, A_j] = -2\imath \mathcal{H}_\gamma^{\text{gen}} L_{ij}. \tag{3.4}$$

The integrals of motion for the local Calogero-Coulomb model (1.3) are obtained by taking symetric polynomials on L_{ij} (2.24) and A_k,

$$\mathcal{A}_k = \sum_i A_i^k.$$

These invariants produce $2N-1$ independent set of integrals for the Calogero-Coulomb system ensuring its maximal superintegrability.

So far, the Calogero, Calogero-oscillator and Calogero-Coulomb systems are defined on the flat Euclidean space. These systems have been extended also to the surfaces with constant curvature: N-dimensional sphere and hyperboloid [6, 14]. The matter id that the Dunkl operator provides an appropriate deformation of the Laplace-Beltrami operator on sphere which incorporates the Calogero interaction terms [14]. The algebra of constants of motion for nonlocal Hamiltonians acquire now the additional deformation parameter related to the curvature. The superintegrability property is preserved.

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