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COMBINED CONTROL OF GUARANTEED SEARCH FOR A MOVING OBJECT WITH GEOMETRIC CONSTRAINTS

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Բանալի բառեր. Երաշխավորված փնտրում, կոմբինացված ղեկավարում
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Ավետիսյան Վ.Վ., Ստեփանյան Վ.Ս.
Շարժական օբյեկտի երաշխավորած փնտրման կոմբինացված ղեկավարումը երկրաչափական սահմանափակումների դեպքում

Դիտարկվում է հորիզոնական հարթության վրա շարժվող որոնելի օբյեկտի երաշխավորված փնտրման խնդիրը, որի սկզբնական վիճակը հայտնի է տրված բազմության ճշտությամբ: Փնտրումն իրականացվում է եռաչափ տարածության մեջ արագացմամբ ղեկավարվող օբյեկտի կողմից, որի ուղղահայաց կոորդինատի վրա դրված երկրաչափական սահմանափակումն արգելում է փնտրող օբյեկտին բարձրանալ ավելի, քան տրված թույլատրելի բարձրությունը: Մշակվել է կոմբինացված ղեկավարման ալգորիթմ, որի դեպքում որոնելի օբյեկտի երաշխավորված փնտրումն իրագործվում է տարածական հետագծով՝ կազմված ուղղագիծ հատվածներից և կորագիծ հատվածներից՝ մոնոտոն նվազող շառավղներով շրջանագծերի տեսքով: Խնդրի երկրաչափական և ֆիզիկական պարամետրերի համար ստացվել է պայման, որի դեպքում ղեկավարման առաջարկված ալգորիթմը լուծում է երաշխավորված փնտրման խնդիրը:

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Комбинированное управление гарантированным поиском подвижного объекта при геометрических ограничениях

Рассматривается задача гарантированного поиска движущегося на горизонтальной плоскости искомого объекта, начальное состояние которого известно с точностью до заданного множества. Поиск осуществляется в трехмерном пространстве управляемым по ускорению ищущим объектом, на вертикальную координату которого наложено геометрическое ограничение, запрещающее ищущему объекту подниматься выше заданной допустимой высоты. Разработан алгоритм комбинированного управления, при котором гарантированный поиск искомого объекта реализуется по пространственной траектории, состоящей из прямолинейных участков и криволинейных участков в виде окружностей с монотонно убывающими радиусами. Для геометрических и физических параметров задачи получено условие, при котором предложенный алгоритм управления разрешает задачу гарантированного поиска.

In this paper we consider the problem of locating an object moving on a horizontal plane, whose initial position is known to be from a given subset of points of the plane. The search is carried out by the means of accelerating an object through space, which adheres to certain geometrical constraints on the vertical plane such as that the object cannot move past a certain maximum elevation. A combined control algorithm has been developed, that is guaranteed to locate the object by means of a varying linear and curvilinear trajectories, modeled as circles with varying radii. Geometrical and physical parameters have been calculated which allow to solve the problem of guaranteed positioning.

Introduction. The problem of a variation law development of searching object's (SO) controlling acceleration vector limited by absolute value is considered. SO starts three-dimensional motion from a given initial state of rest and has to detect the moving target object (TO) in a finite time. TO's motion is horizontal and controlled by acceleration. The initial state of TO is known to SO up to a given set of uncertainty. The absolute values of the acceleration and the velocity of TO are limited. SO is geometrically constrained, so that it cannot collide with known still obstacles (e.g. ground) or the maximum elevation is limited in case of the object being a flying device. TO is considered to be detected if it lies within the circular base of a cone whose apex's coordinates are the current coordinates of SO. In [1] the time-optimal guaranteed search problem is solved without elevation constraints. A minimax approach was developed which allows to reduce the problem of optimal guaranteed search, i.e. fastest absorption of the domain of uncertainty that is expanding with maximal speed, to the optimal control problem with free right end solved with the Pontryagin's Maximum Principle [2] in the class of control problems with constant acceleration. Nevertheless, the implementation of the approach mentioned in [1] depending on the initial parameters of the searching system seems impossible for the problem of guaranteed search with constraints on elevation (including optimal guaranteed search) due to limited possibilities for detection disk expansion necessary for absorption of TO's uncertainty domain expanding in time. For this reason this paper offers another approach based on development of a combined control algorithm for SO allowing a multi-step search of TO by means of linear and curvilinear regions with monotonically decreasing radii. Other approaches to the related problems see in [3-5].

1. Problem statement. Suppose there are two point objects X and Y, where X is the searching one and Y is the target. X performs three-dimensional motion in the gravitational field of the Earth and Y on the surface of the Earth. The motion equations of the objects can be given in the following form:

$$\begin{aligned} X:\ & \ddot{x}_1 = w_{X1}, \quad \ddot{x}_2 = w_{X2}, \quad \ddot{x}_3 = w_{X3} - g, \\ & x_1(0) = R_0, \ x_2(0) = 0, \quad x_2(0) = 0, \\ & \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = 0, \ \dot{x}_3(0) = 0, \\ & 0 \le x_3(t) \le h, \ |w_{X1}| \le W_X, \quad w_X = (w_{X1}, w_{X2}, w_{X3})^{\mathrm{T}}, \quad t \ge 0, \end{aligned} \tag{1.1}$$

$$\begin{aligned} Y:\ & \ddot{y}_i = w_{Yi}, \quad i = 1,2, \\ & y_i(0) = y_i^0, \quad \dot{y}_i(0) = \dot{y}_i^0, \quad i = 1,2, \\ & |\dot{y}(t)| \le V_Y, \quad |w_Y(t)| \le W_Y, \quad \dot{y} = (\dot{y}_1, \dot{y}_2)^{\mathrm{T}}, \quad w_Y = (w_{Y1}, w_{Y2})^{\mathrm{T}}, \ t \ge 0. \end{aligned} \tag{1.2}$$

In (1.1), (1.2) x_i, y_i – geometrical coordinates of the objects X, Y; w_{Xi}, w_{Yi} – coordinates of controlling accelerations of objects, which are piecewise continuous vector-functions of t; W_X, W_Y – maximal possible values of controlling accelerations w_X, w_Y respectively; V_Y – maximal possible speed of the object Y; h – maximal allowed value of coordinate x_3 of the object X during the motion; g – gravitational acceleration; R_0 – given positive number. The symbols $(\ )^{\mathrm{T}}$ and $|\cdot|$ are the operations of transposition and Euclidian norm of vectors, respectively.

Let us suppose that the only information about the phase coordinates of Y known to X is a given uncertainty set Y belongs to at the initial moment.

$$(y^0, \dot{y}^0) \in D_0 \times \dot{D}_0, \ D_0 = \{y^0 \in R^2 : \ |y^0| \le r_0\}, \ \dot{D}_0 = \{\dot{y}^0 \in R^2 : \ |\dot{y}^0| \le V_Y\}. \tag{1.3}$$

TO is considered to be detected at the very first moment $t = t^*$, when the following statement is true

$$y(t^*) \in G(x(t^*)), \text{ т.е. } |y(t^*) - x_c(t^*)| \le l(t^*), \ x_c = (x_{c1}, x_{c2}) \tag{1.4}$$

– i.e. it belongs to the moving circular base of the following cone:

$$G(x(t),C)=\begin{Bmatrix} \xi \in R^2 : \ |\xi(t)-x_c(t)| \le l(t)=Cx_3(t) \\ C=|\text{tg}\alpha|,\ 0<|\alpha|<\pi/2 \end{Bmatrix},\quad t\ge 0, \tag{1.5}$$

$$G(x(0),C)=G(x_c(0),x_3(0),C)=G_0,\qquad x(0)=(R_0,0,0).$$

The detecting disk (1.5) of SO (1.1) at the moment t while using all possible piecewise continuous controlling accelerations (admissible controls) $w_X(\tau)$, $|w_X(\tau)|\le W_X$, $0\le\tau\le t$, on the plane (x_1,x_2), represents a disk with a moving center (by means of control $(w_{X1}(t),w_{X2}(t))$ with center $x_c(t)=(x_{c1}(t),x_{c2}(t))$ and varied by using $w_{X3}(t)$ scalar control with $l(t)=Cx_3(t)$: $l(t)>0$ radius when $w_{X3}(t)>0$ and $l(t)<0$ when $w_{X3}(t)<0$.

According to (1.5), at the initial moment the detection disk $G_0=(R_0,0)$ is a point on the axis Ox_1. We assume, that $R_0>r_0$, i.e. initially the uncertainty disk has no intersection with the detection disk:

$$D_0\cap G_0=\varnothing. \tag{1.6}$$

The primary problem. For a given initial state $(x^0,\dot{x}^0)$ and given initial disk of uncertainty D_0 (1.3) and disk of detection G_0 (1.5), satisfying (1.6), find a number $T>0$ and admissible control $w_X(t)$ of object X on the $[t_0,\,T]$ interval, so that for any initial state $(y^0,\dot{y}^0)$ (1.3) of Y and any admissible control $w_Y(t)$ on the $[t_0,\,T]$ interval, the detection condition (1.4) is satisfied at some moment t^* not later than T: $t^*\le T$.

We will call the number $T>0$ and the admissible control $w_X(t)$, $0\le t\le T$ of the X guaranteed search time and guaranteeing control, respectively.

For the system (1.1) – (1.6) when solving the time-optimal guaranteed search problem without the constraint on axis x_3 in [1] we introduce the concept of uncertainty domain at moment T: $D(T)$ on the plane (y_1,y_2) is consisting of end points $y(T)=(y_1(T),y_2(T))$ of all trajectories of TO (1.2) for all possible initial states $(y^0,\dot{y}^0)\in D_0\times\dot{D}_0$ and constructed with all kind of piecewise continuous admissible controlling accelerations $w_Y(t)=(w_{Y1}(t),w_{Y2}(t))$, $|w_Y(t)|\le W_Y$, $0\le t\le T$ with a constraint on speed $|\dot{y}(t)|\le V_Y$, $0\le t\le T$.

Considering the above, in [1] an approach is suggested consisting of constructing an admissible control on motion of X, such that the detection disk of SO absorbs the disk of uncertainty (expanding in time) within a minimal guaranteed time T:

$$D(T)\subseteq G(x(T)), \tag{1.7}$$

which ensures the fulfillment of the condition (1.4) at some point $t^*\le T$ in time.

Based on the minimax approach, it was found that for guaranteed detection it is sufficient to consider the case when TO is initially on the boundary of the uncertainty domain (1.3), has no acceleration and has a vector-speed directed radially away from the center of the disk

$$\dot{y}^0=(\dot{y}_1^0,\dot{y}_2^0)=\left(V_Y y_1^0 r_0^{-1},\ V_Y y_2^0 r_0^{-1}\right),\qquad w_Y(t)\equiv 0,\ \ 0\le t\le T, \tag{1.8}$$

i.e. the radius of the disk $D(t)$ is increasing linearly:

$$r(t) = r_0 + tV_Y\,. \tag{1.9}$$

Thereby the optimal guaranteed search problem was reduced to the problem of optimal control with free right end, which was solved with the Pontryagin's Maximum Principle [2] in the class of problems with constant accelerations. However, in presence of the constraint on the axes x_3 (1.1) in some cases depending on the initial state of the searching system, the implementation of this method of absorption (1.7) seems impossible in the problem of guaranteed (including optimal guaranteed) search, as the possibilities are limited for the detection disk to expand which is necessary when having the conditions (1.8), (1.9). For this reason, this paper offers another approach based on development of a hybrid control algorithm for SO allowing a multi-step search of TO.

2. Fastest maximum elevation reaching step. In this step, SO performs a vertical motion on purpose of reaching the maximum elevation with zero speed at the end of the motion. Such a motion is implemented with the solution of the following optimal performance problem.

Problem 1. Find a controlling acceleration $w^*_{Xi}(t)$, $t \in [0, t_1]$, $i = 1,2,3$ (1.1), that ensures the movement of X (1.1) from a given initial state of rest (1.1) to a given terminal state of rest

$$x_1(t_1) = R_0, \;\; \dot{x}_1(t_1) = 0, \;\; x_2(t_1) = 0, \;\; \dot{x}_2(t_1) = 0, \;\; x_3(t_1) = h, \;\; \dot{x}_3(t_1) = 0, \tag{2.1}$$

within the minimal time t_1.

This is a two-point optimal control problem. According to the Pontryagin's Maximum Principle of optimal control [2], providing the quickest transition from one point to another in the phase space is the vector function $w^*_X = (w^*_{X1}(t), w^*_{X2}(t), w^*_{X3}(t))$ with the following components:

$$\begin{aligned} &w^*_{X1} = w^*_{X2} = 0, \qquad 0 \le t \le t_1, \\ &w^*_{X3} = W_X \operatorname{sign}\left[\left(t_1/2 - t\right)h\right], \qquad t_1 = 2\sqrt{hW_X\left[\left(W_X + g\right)\left(W_X - g\right)\right]^{-1}}. \end{aligned} \tag{2.2}$$

Thus, the control (2.2) ensures SO to reach maximum elevation $x_3 = h$ with zero terminal speed within minimal time t_1 (2.2). At the time $t = t_1$ the radius of the detection disk reaches a maximum value $l(t_1) = Cx_3(t_1) = Ch$, and the detection disk on the plane Ox_1x_2 takes the following form:

$$G(t_1) = \left\{(x_1, x_2):\; (x_1 - R_0)^2 + x_2^2 = l^2(t_1),\;\; l(t_1) = Cx_3(t_1) = Ch\;\right\}. \tag{2.3}$$

Since Y in the time interval $0 \le t \le t_1$ can be at a maximum distance from the center of the initial disk of uncertainty, if and only if at the initial time it is on the boundary of the uncertainty set (1.3) and has an initial velocity (1.8), which provides expansion of the disk of uncertainty with the highest rate V_Y (1.9) then at the moment $t = t_1$ it can be located in any point of the circle

$$D(t_1) = \left\{(y_1, y_2):\; y_1^2 + y_2^2 = r_1^2\right\}, \qquad r_1 = r(t_1) = r_0 + V_Y t_1\,. \tag{2.4}$$

3. Step of fastest contiguation of the detection and uncertainty circles. While X is on the maximum permissible height with the zero-speed state (2.1) at the moment $t = t_1$, it performs a linear horizontal movement along the axis Ox_1 in the direction toward the center of the circle of uncertainty within the time interval $t_1 \le t \le t_2$ until the first contact at $t = t_2$ of the detecting circle $G(t_2)$ and the circle of uncertainty $D(t_2)$.

Since TO within the time interval $t_1 \le t \le t_2$ continues its motion having maximum absolute value of the velocity vector (1.8), then at the time $t = t_2$ it can be anywhere on the boundaries of the uncertainty circle

$$D(t_2) = \left\{(y_1, y_2):\ y_1^2 + y_2^2 = r_2^2\right\}, \tag{3.1}$$

where given (1.9), (2.4) r_2 is calculated as follows:

$$r_2 = r(t_2) = r_1 + V_Y(t_2 - t_1) = r_0 + V_Y t_2 . \tag{3.2}$$

This means that the required motion of SO can be implemented by the control vector $w_X^* = (w_{X1}^*(t), w_{X2}^*(t), w_{X3}^*(t))$, wherein the second and third components are specified as

$$w_{X2}^*(t) = 0\,,\ w_{X3}^*(t) = g\,,\quad t_1 \le t \le t_2 \tag{3.3}$$

and the first component is determined by solving the following optimal control problem.

Problem 2. Find an optimal control $w_{X1}^*(t)$, $t_1 \le t \le t_2$ which satisfies the constraint

$$\left|w_{X1}(t)\right| \le \sqrt{W_X^2 - g^2} \tag{3.4}$$

(given (1.1), (3.3)) and along with given control constants (3.3) provides the displacement of the X from a rest state (2.1) to the terminal rest state

$$\begin{aligned} &x_1(t_2) = r_0 + V_Y t_2 + Ch, \quad x_2(t_2) = 0, \quad x_3(t_2) = h, \\ &\dot{x}_1(t_2) = 0, \quad \dot{x}_2(t_2) = 0, \quad \dot{x}_3(t_2) = 0 \end{aligned} \tag{3.5}$$

within a minimal time $t_2 - t_1$.

Note that the first two boundary conditions in (3.5) can also be written in the form $x_1(t_2) = r_2 + Ch = y_1(t_2) + Ch$, $x_2(t_2) = y_2(t_2) = 0$ in view of (3.1), (3.2), express the situation of outside contact between the circles G and D at time t_2.

Since with the given controls (3.3) the system (1.1) does not move by the coordinates x_2, x_3 (with boundary conditions (2.1) and (3.5)), then the problem 2 is reduced to one-dimensional (regarding the coordinate x_1) problem of optimal control with free right end:

$$\ddot{x}_1 = w_{X1}, \quad x_1(t_1) = R_0, \quad \dot{x}_1(t_1) = 0, \quad x_1(t_2) = r_0 + V_Y t_2 + Ch, \quad \dot{x}_1(t_2) = 0\,. \tag{3.6}$$

Solving (3.6) as a two-point optimal problem, analytical expressions for the optimal control and the corresponding minimum travel time is found:

$$w_{X1}^* = \sqrt{W_X^2 - g^2}\,\mathrm{sign}\left\{\left[\left(t_2 + t_1\right)/2 - t\right]\left(r_0 + V_Y t_2 + Ch - R_0\right)\right\}, \tag{3.7}$$

$$t_2 = t_1 + 2\left[-\frac{V_Y}{\sqrt{W_X^2 - g^2}} + \sqrt{\frac{V_Y^2}{W_X^2 - g^2} + \frac{R_0 - Ch - r_0 - V_Y t_1}{\sqrt{W_X^2 - g^2}}}\right], \tag{3.8}$$

where t_1 is calculated according to (2.2).

Thus, at the moment $t = t_2$, in the state (3.5) of the object X, the circle of the detecting disk

$$G(t_2) = \left\{(x_1, x_2):\ [x_1 - x_1(t_2)]^2 + x_2^2 = l^2(t_2),\ \ l(t_2) = Cx_3(t_2) = Ch\ \right\} \tag{3.9}$$

contacts the circle of uncertainty (3.1), (3.2) inside.

4. Helper problem. Starting at moment $t = t_2$ (3.8), when SO is in the rest state (3.5) and the detection disk (3.9) and the uncertainty disk (3.1) are in contact, SO performs the search via flat motion $x_3 = h$ and the detection disk has a constant radius $l = Cx_3(t) \equiv Ch$, $t \geq t_2$. Taking into account the equations of motion (1.1) and (3.5) for the x_3 coordinate and the velocity $\dot{x}_3$ at the moment $t = t_2$, X carries such a motion with

$$w_{X3}^*(t) \equiv g, \quad t \geq t_2. \tag{4.1}$$

It follows that only the flat movement of the X which is defined by the first two equations (1.1) is a subject to review. We introduce the polar coordinate system (ρ, φ, O) so that the pole O is in the center of the disk of uncertainty, and the polar axis runs through the center of the detection disk having coordinates (3.5) at the moment $t = t_2$. In the first two equations we switch to polar coordinates ρ, φ associated with the original Cartesian coordinates x_1, x_2 with the following relations:

$$x_1 = \rho\cos\varphi, \quad x_2 = \rho\sin\varphi. \tag{4.2}$$

The equations of the plane motion of SO (1.1), represented in polar coordinates are as follows:

$$\ddot{\rho} - \rho\dot{\varphi}^2 = w_\rho, \quad \frac{d}{dt}(\rho\dot{\varphi}) = w_\varphi, \tag{4.3}$$

where w_ρ and w_φ are radial and tangential components of acceleration of the X, respectively. They are associated with the first two components of the vector of controlling acceleration $w_X = (w_{X1}, w_{X2}, w_{X3})$ as follows:

$$w_{X1} = w_\rho\cos\varphi - w_\varphi\sin\varphi, \quad w_{X2} = w_\rho\sin\varphi + w_\varphi\cos\varphi. \tag{4.4}$$

In view of (4.1), (4.4) the constraint on the absolute value of controlling acceleration (1.1) takes the form

$$\sqrt{w_{X1}^2 + w_{X2}^2} = \sqrt{w_\rho^2 + w_\varphi^2} \leq \sqrt{W_X^2 - g^2}, \quad t \geq t_2, \tag{4.5}$$

and the initial conditions (3.5) will be

$$\rho(t_2) = r_2 + Ch, \quad \dot{\rho}(t_2) = 0, \quad \varphi_2(t_2) = 0, \quad \dot{\varphi}(t_2) = 0. \tag{4.6}$$

For the detection of TO, SO carries a circular motion around the center O with a radius $\rho(t_2) = \mathrm{const}$ by choosing a direction for the encircling maneuver, which actually is the positive direction of the reference polar angle.

Then, motion controls (4.2) will be

$$-\rho(t_2)\dot{\varphi}^2 = w_\rho, \quad \rho(t_2)\ddot{\varphi} = w_\varphi, \tag{4.7}$$

Let us find the variation laws for the controlling accelerations $w_\rho(t)$ and $w_\varphi(t)$ which satisfy the constraint (4.5), while moving circumferentially with the constant radius

$\rho(t_2)$ according to the equations (4.7), the center of the detection disk goes from the state of rest (3.6) to the state of rest

$$\rho(t_*) = r_2 + Ch, \quad \dot{\rho}(t_*) = 0, \quad \varphi_2(t_*) = 0, \quad \dot{\varphi}(t_*) = 0. \tag{4.8}$$

within a minimal time $t_* - t_2$.

The required controls $w_\rho(t)$ and $w_\varphi(t)$ are as follows. Constraining the tangential control acceleration:

$$\left| w_\varphi(t) \right| \le \varepsilon, \quad t \in [t_2, t_*], \tag{4.9}$$

where $\varepsilon > 0$ and t_* – are unknown constant and time respectively.

First, from the two-point optimal control problem (4.6) - (4.9) we determine the optimal controlling tangential acceleration w_φ. The maximum principle implies that the desired control is an on-off control with the switching point $t = \tau$:

$$w_\varphi = \varepsilon \operatorname{sign}\left[2\pi\left(\tau - t\right)\right], \quad \tau = \left(t_* + t_2\right)/2, \quad t_* = t_2 + 2\sqrt{2\pi\rho(t_2)\varepsilon^{-1}}. \tag{4.10}$$

Then, by integrating the second equation (4.7) with the control (4.10) and the boundary conditions (4.6), (4.8), we find the function of the angular velocity of the time $\dot{\varphi}(t)$, and after applying it in the first equation (4.7), we find the variation of the radial acceleration of the time

$$w_\rho(t) = \begin{cases} \varepsilon^2 \left(t - t_2\right)^2 \rho^{-1}(t_2), & t_2 \le t \le \tau, \\ \varepsilon^2 \left(-t + 2\tau - t_2\right)^2 \rho^{-1}(t_2), & \tau \le t \le t_*. \end{cases} \tag{4.11}$$

The concave and continuous function (4.11) produces zeros on the ends of the interval $t_2 \le t \le t_*$: $w_\rho(t_2) = w_\rho(t_*) = 0$. In the interval $t_2 \le t \le t_*$ it monotonically increases and in the interval $\tau \le t \le t_*$ it monotonically decreases, producing the maximal value on the middle $\tau = \left(t_* + t_2\right)/2$ of the interval:

$$\max_{t_2 \le t \le t_*} w_\rho(t) = w_\rho(\tau)\Big|_{\tau = (t_* + t_2)/2} = \varepsilon^2 (t_* - t_2)^2 \rho^{-1}(t_2)/4 \Big|_{t_* - t_2 = 2\sqrt{2\pi\rho(t_2)\varepsilon^{-1}}} = 2\pi\varepsilon. \tag{4.12}$$

From (4.9) and (4.12) follows

$$w_\rho^2(t) + w_\varphi^2(t) < 4\pi^2\varepsilon^2 + \varepsilon^2, \quad t_2 \le t \le t_*. \tag{4.13}$$

By virtue of (4.13), in $t_2 \le t \le t_*$ the constraint (4.5) is ensured, if the following inequality is satisfied for $\varepsilon > 0$:

$$4\pi^2\varepsilon^2 + \varepsilon^2 \le W_X^2 - g^2. \tag{4.14}$$

The solution of the inequality (4.14) is

$$\varepsilon \in \left(0, \varepsilon_0\right], \quad \varepsilon_0 = \sqrt{\left(W_X^2 - g^2\right)\left(4\pi^2 + 1\right)^{-1}}. \tag{4.15}$$

Thus, $\varepsilon = \varepsilon_0$ (4.15) is the maximal value when the constraint (4.14) is not violated where the time t_* and control (4.10) are optimal:

$$t_* = t_2 + 2\sqrt{2\pi\rho(t_2)\varepsilon_0^{-1}}, \quad w_\varphi = \varepsilon_0 \operatorname{sign}\left\{ 2\pi\left[\left(t_* + t_2\right)/2 - t\right]\right\}. \tag{4.16}$$

Moving with maximal velocity V_Y, SO being on some point of circle $D(t_2)$ (3.1) at the moment $t = t_2$, within the time $t_* - t_2$ will pass a distance $V_Y(t_* - t_2)$ and at the moment $t = t_*$ can be maximally displaced from the origin, i.e. on some point of circle $D(t_*) = \left\{(y_1, y_2):\ y_1^2 + y_2^2 = r_*^2\right\}$ with the radius $r_* = r_2 + V_Y(t_* - t_2)$. From this and (4.16) follows, that if the condition

$$r_* - r_2 = 2V_Y\sqrt{2\pi\rho(t_2)\varepsilon_0^{-1}} < 2Ch, \tag{4.17}$$

is satisfied, then within the optimal time of one full rotation of SO around the center O, TO does not have enough time to leave the circular ring with width $2l = 2Ch$ and stay undetected.

5. Combined control algorithm. Suppose that for the given initial parameters $r_0, R_0, C, h, V_Y, W_X, g$ at time $t = t_2$ the condition (4.17), written in the form

$$\sqrt{2\pi\rho(t_2)\varepsilon_0^{-1}} < ChV_Y^{-1} \tag{5.1}$$

is satisfied.

Starting from $t = t_2$, with the tangential acceleration equal to zero $w_\varphi = 0$, SO moves as fast as possible along the axis Ox_1 in the direction of the pole O from the state (4.6)((3.5)) performing a displacement $\Delta x_1^{(1)} > 0$ (defined below). Using (4.4) and the following controls

$$w_{X1}^*(t) = -\sqrt{W_X^2 - g^2}\,\mathrm{sign}\left\{\left[\left(t_3 + t_2\right)/2 - t\right]\Delta x_1^{(1)}\right\}, \tag{5.2}$$

$$w_{X2}^*(t) = 0,\quad w_{X3}^*(t) = g,\quad t_2 \le t \le t_3,\quad t_3 = t_2 + 2\sqrt{\Delta x_1^{(1)}\left(W_X^2 - g^2\right)^{-1/2}}$$

X will make the below transition to the state of rest within a minimum time $t_3 - t_2$:

$$x_1(t_3) = \rho(t_3) = \rho(t_2) - \Delta x_1^{(1)},\quad x_2(t_3) = 0,\quad \dot{x}_1(t_3) = 0,\quad \dot{x}_2(t_3) = 0. \tag{5.3}$$

Meanwhile, during the time $t_3 - t_2$ the uncertainty disk of TO (moving with V_Y velocity) will expand and its radius reaches the value

$$r(t_3) = r_3 = r_2 + V_Y(t_3 - t_2) \tag{5.4}$$

at the moment $t = t_3$.

We require that at time $t = t_3$ the right point $(r(t_3), 0)$ of intersection of the circle of uncertainty with the axis Ox_1 be more left than the right point $(x(t_3) + Ch, 0)$ of intersection of the circle of detection with the axis Ox_1: $r(t_3) < x_1(t_3) + Ch$, i.e. with (5.3), (5.4) the following condition to be satisfied

$$V_Y(t_3 - t_2) < 2Ch - \Delta x_1^{(1)}. \tag{5.5}$$

The inequality (5.5) sets a constraint on desired value for $\Delta x_1^{(1)} > 0$ and after replacement of $t_3 - t_2$ with (5.2) it is reduced to the following inequality:

$$\Delta x_1^{(1)} + 2V_Y\sqrt{\left(W_X^2 - g^2\right)^{-1/2}}\sqrt{\Delta x_1^{(1)}} - 2Ch < 0, \tag{5.6}$$

and the solution is

$$\Delta x_1^{(1)} \in (0,\ a),\quad a = \left[-V_Y \sqrt{\left(W_X^2 - g^2\right)^{-1/2}} + \sqrt{V_Y^2 \left(W_X^2 - g^2\right)^{-1/2} + 2Ch} \right]^2 . \quad (5.7)$$

If during the time $t_3 - t_2$ the detection of Y is not happening, then at $t = t_3$ being in the state (5.3), for which $\Delta x_1^{(1)}$ is from the interval (5.7), SO performs a fastest possible rotation with the constant radius $\rho(t_3) = \rho(t_2) - \Delta x_1^{(1)}$ in the way described in the section 4 from the rest state

$$\rho(t_3) = \rho(t_2) - \Delta x_1^{(1)},\quad \dot\rho(t_3) = 0,\quad \varphi(t_3) = 0,\quad \dot\varphi(t_3) = 0 \quad (5.8)$$

to the terminal state of rest

$$\rho(t_4) = \rho(t_3) = \rho(t_2) - \Delta x_1^{(1)},\quad \dot\rho(t_4) = 0,\quad \varphi(t_4) = 2\pi,\quad \dot\varphi(t_4) = 0 \quad (5.9)$$

within a minimal possible time $t_4 - t_3$.

SO performs such a relocation with the tangential (4.16) and radial (4.11) control accelerations, also with the control (4.1), related to the time interval $t_3 \le t \le t_4$:

$$w_\varphi = \varepsilon_0 \operatorname{sign}\left\{2\pi\left[\left(t_4 + t_3\right)/2 - t\right]\right\},\qquad t_4 = t_3 + 2\sqrt{2\pi\rho(t_3)\varepsilon_0^{-1}}, \quad (5.10)$$

$$w_\rho(t) = \begin{cases} \varepsilon_0^2 \left(t - t_3\right)^2 \rho^{-1}(t_3), & t_3 \le t \le \tau_4, \\ \varepsilon_0^2 \left(-t + 2\tau_4 - t_3\right)^2 \rho^{-1}(t_3), & \tau_4 \le t \le t_4, \end{cases} \qquad \tau_4 = \left(t_3 + t_4\right)/2.$$

$$w_{X3}^*(t) = g,\quad t_3 \le t \le t_4,$$

where $\rho(t_3)$ and ε_0 are deduced from (5.8) and (4.15), respectively.

Given the constraint (5.7), the desired value for $\Delta x_1^{(1)}$ will be determined from the following equation:

$$r_3 + (t_4 - t_3)V_Y = r_2 + 2Ch - \Delta x_1^{(1)}, \quad (5.11)$$

which, using (5.2), (5.4) and (5.10) is transformed to

$$2V_Y \sqrt{2\pi(\rho(t_2) - \Delta x_1^{(1)})\varepsilon_0^{-1}} = 2Ch - 2V_Y \sqrt{\Delta x_1^{(1)} \left(W_X^2 - g^2\right)^{-1/2}} - \Delta x_1^{(1)}. \quad (5.12)$$

Here, the value of $\rho(t_2)$ using (2.2) (3.2) (3.8) (4.6) is expressed in terms of the given known parameters $r_0, R_0, C, h, V_Y, W_X, g$ of the problem.

With the condition (5.1), the equation (5.12) is solvable against $\Delta x_1^{(1)}$ on the interval (5.7). Solving it, we find the value for $\Delta x_1^{(1)}$ wherein during the full rotation time $t_4 - t_3$ around O, TO moving with constant velocity from the boundary of the uncertainty disk, will be detected. The radius of the disk will be:

$$r(t_3) = r_3 = r_2 + 2V_Y \sqrt{\Delta x_1^{(1)} \left(W_X^2 - g^2\right)^{-1/2}}. \quad (5.13)$$

Thus, if no detection of TO (1.4) occurs at any moment $t^* \in [t_2, t_4]$ of the time interval $t_2 \le t \le t_4$, then the execution of the combined control (5.2), (5.10) during the time $t_4 - t_2$ results to reduction of the diameter $r_2 = r(t_2)$ of the uncertainty domain by $\Delta x_1^{(1)}$; $0 < \Delta x_1^{(1)} < Ch < r_2$, i.e. the uncertainty domain of TO at time $t = t_4$ is contained in a circle with a radius

$r_4 = r(t_4) = r_2 - \Delta x_1^{(1)} > 0$, $\qquad 0 < r_4 < r_2$. (5.14)

Here, the following cases are possible:

a) $r(t_4) = r_4 < Ch$, $\qquad$ b) $r(t_4) = r_4 \geq Ch$. (5.15)

In case of (5.15)(a), let us find a condition, for which the controls

$$w_{X1}^* = -\sqrt{W_X^2 - g^2}, \quad w_{X2}^* = 0, \quad w_{X3}^* = g, \qquad t \geq t_4, \tag{5.16}$$

of linear motion along the axis Ox_1 from the state (5.9)(recorded in Cartesian coordinates (4.2)), ensure satisfaction of the absorption condition (1.7) not later than some finite time T.

First, we integrate the equation (1.1) given the controls (5.16) and initial conditions (5.9)((4.2)). Then, the resulting expressions for $t = T$ we put in the final terms

$$x_1(T) - Ch = y_1(T), \qquad x_1(T) \geq 0, \qquad y_1(T) = -r_4 - V_Y(T - t_4), \tag{5.17}$$

describing the relative position of the disks $G(x(T))$ and $D(T)$ [1], corresponding to the absorption condition (1.7) [1].

The relations (5.17) can also be represented as the following system against the parameter $T > 0$:

$$\begin{aligned} &w_{X1}^* T^2 + V_Y T + R_4 = 0, \quad w_{X1}^* T^2 - w_{X1}^* t_4^2 - 2w_{X1}^* t_4 + 2r(t_4) \geq 0, \\ &R_4 = -w_{X1}^* t_4^2 - 2w_{X1}^* t_4 + 4r(t_4) - 2_Y t_4. \end{aligned} \tag{5.18}$$

If (5.18) is solvable against $T > 0$, then the controls (5.16) are guaranteeing on the interval $t_4 \leq t \leq T$, and the time T (minimal positive root of the equation (5.18)) is the guaranteed search time, since at this point the boundary condition (5.17) is satisfied, which is equivalent to the absorption condition (1.7) and detection condition (1.4).

If (5.18) is not solvable, then in both (5.15)(a) and (b) cases, starting from the moment $t = t_4$ secondarily applying the controls (5.2) and (5.10) related to intervals $t_4 \leq t \leq t_5$ and $t_5 \leq t \leq t_6$, respectively, will give

$$r_6 = r(t_6) = r_4 - \Delta x_1^{(2)}, \tag{5.19}$$

where $\Delta x_1^{(2)}$ is determined from the following equation:

$$r_5 + (t_6 - t_5)V_Y = r_4 + 2Ch - \Delta x_1^{(2)}, \qquad \Delta x_1^{(2)} \in (0, a). \tag{5.20}$$

The equation (5.20) with the help of similar forumals (5.2), (5.4), (5.10), related to differences ($t_5 - t_2$), ($r_5 - r_4$), ($t_6 - t_5$), respectively, is recorded as

$$2V_Y\sqrt{2\pi\left(\rho(t_2) - \Delta x_1^{(1)} - \Delta x_1^{(2)}\right)\varepsilon_0^{-1}} = 2Ch - 2V_Y\sqrt{\Delta x_1^{(2)}\left(W_X^2 - g^2\right)^{-1/2}} - \Delta x_1^{(2)}, \tag{5.21}$$

$$\Delta x_1^{(2)} \in (0, a).$$

where $\Delta x_1^{(1)}$ is determined in the previous step on the interval $t_2 \leq t \leq t_4$.

According to (5.15), considering the following possible cases. In case of (5.15)(a), if the solution $\Delta x_1^{(2)}$ of the equation (5.21) satisfies the inequality

$$r_4 \leq \Delta x_1^{(2)} < Ch, \tag{5.22}$$

then from (5.19) follows, that $r_6 = r(t_6) = r_4 - \Delta x_1^{(2)} \leq 0$, i.e. the absorption of the uncertainty domain by the detection domain happens at the moment $t = t_6$.

In case of (5.15)(a), if the solution $\Delta x_1^{(2)}$ of the equation (5.21) satisfies the inequality

$$0 < \Delta x_1^{(2)} < r_4 < Ch\,, \tag{5.23}$$

i.e.

$$r_6 = r(t_6) = r_4 - \Delta x_1^{(2)} > 0 \tag{5.24}$$

on the interval $t_4 \le t \le t_6$ no detection occurs, then the uncertainty domain at moment $t = t_6$ is contained in a disk with diameter $r(t_6) = r_6 < r_4$, moreover, as it follows from (5.19), (5.24) and equations (5.12), (5.21) with the condition (5.1)

$$\Delta x_1^{(1)} = r_2 - r_4 < r_4 - r_6 = \Delta x_1^{(2)}\,. \tag{5.25}$$

Similarly to the case (5.15)(a), we can use the control (4.1) when $t \ge t_6$ and from the equation (5.18), where t_4 is replaced with t_6, we can find the guaranteed absorption time, if (5.18) is solvable against $T > 0$. Otherwise, and also in case of (5.15)(b), if on the interval $t_4 \le t \le t_6$ no detection happens and

$$0 < \Delta x_1^{(2)} < Ch \le r_4\,, \tag{5.26}$$

then at $t = t_6$ the relations (5.25) are relevant again. Then we move to the next step of the combined control and so on, until one of the conditions (5.22) or (5.23) related to the current step are satisfied.

Suppose that at moment $t = t_{2n}$, $n \ge 3$ before the n-th step, no detection has happened during the time $t_{2n-2} \le t \le t_{2n}$ and therefore, the execution of the combined control on the interval $t_{2n-2} \le t \le t_{2n}$ resulted to reduction of the radius $r_{2n-2} = r(t_{2n-2})$ of the uncertainty circle by $\Delta x_1^{(n-1)}$, $0 < \Delta x_1^{(n-1)} < Ch$, i.e. the uncertainty domain of TO at moment $t = t_{2n}$ is contained in a disk with the

$$r_{2n} = r(t_{2n}) = r_{2n-2} - \Delta x_1^{(n-1)}, \quad r_{2n-2} > r_{2n}, \quad n \ge 3\,. \tag{5.27}$$

Starting from the time $t = t_{2n}$ by applying the controls (5.2), (5.10) successively on time intervals $t_{2n} \le t \le t_{2n+1}$ и $t_{2n+1} \le t \le t_{2n+2}$, respectively, we get

$$r_{2n+2} = r(t_{2n+2}) = r_{2n} - \Delta x_1^{(n)}, \quad r_{2n} > r_{2n+2}, \quad n \ge 3\,, \tag{5.28}$$

where $\Delta x_1^{(n)}$ is determined from equation

$$r_{2n+1} + (t_{2n+2} - t_{2n+1})V_Y = r_{2n} + 2Ch - \Delta x_1^{(n)}, \quad n \ge 3\,,$$

recorded as

$$2V_Y\sqrt{2\pi\left(r_2 + Ch - \sum_{i=1}^{n}\Delta x_1^{(i)}\right)\varepsilon_0^{-1}} = 2Ch - 2V_Y\sqrt{\Delta x_1^{(n)}\left(W_X^2 - g^2\right)^{-1/2}} - \Delta x_1^{(n)}, \tag{5.29}$$

$$\Delta x_1^{(n)} \in (0, a), \quad n \ge 3,$$

with the help of similar to (5.10), (5.2), (5.8) recurrence formulas:

$$t_{2n+2} = t_{2n+1} + 2\sqrt{2\pi\rho(t_{2n+1})\varepsilon_0^{-1}}\,, \quad t_{2n+1} = t_{2n} + 2\sqrt{\Delta x_1^{(n)}\left(W_X^2 - g^2\right)^{-1/2}}\,, \tag{5.30}$$

$$\rho(t_{2n+1}) = \rho(t_{2n}) - \Delta x_1^{(1)} = r_2 + Ch - \sum_{i=1}^{n}\Delta x_1^{(i)}$$

Since, as if follows from (5.25), we have the recurrence relations

$$0 < r_{2n-2} - r_{2n} = \Delta x_1^{(n-1)} < \Delta x_1^{(n)} = r_{2n} - r_{2n+2} < Ch, \qquad n \ge 3, \tag{5.31}$$

then $0 < \Delta x_1^{(1)} < ... < \Delta x_1^{(n)} < Ch$ and the search process ends at such n, when $r_{2n} \le \Delta x_1^{(n)} < Ch$ or $\Delta x_1^{(n)} < r_{2n} \le Ch$. In the first case, $r_{2n+2} = r_{2n} - \Delta x_1^{(n)} \le 0$, i.e the absorption of the uncertainty domain by the detection disk is happening at time $t = t_{2n+2}$, and the detection of TO is occurring at some point of time on the interval $t_{2n} \le t \le t_{2n+2}$. In the second case, the uncertainty domain at time $t = t_{2n+2}$ is contained in a circle with radius $r(t_{2n+4}) = r_{2n+4} < r_{2n+2}$ and the controls (5.16) related to current time interval $t_{2n} \le t \le T$ (where T is the minimal positive root of the equation (5.18) written for the time t_{2n}) lead to the achievement of the absorption condition.

Here is an example of numerical implementation of the search control algorithm (5.27) - (5.31) for the system (1.1) - (1.6) with the following parameters

$$W_X = 100\,\text{ms}^{-2},\; V_Y = 5\,\text{ms}^{-1},\; r_0 = 25\,\text{m},\; h = 50\,\text{m}, \tag{5.32}$$

$$R_0 = 5000\,\text{m}, \quad g = 9.8\,\text{ms}^{-2}, \qquad C = 1.$$

First, for the parameters (5.32), with formulas (2.2), (2.4), (3.8), (3.2), (4.6), (5.18) the values $t_1 = 1,42$ s, $r_1 = 32,10$ m, $t_2 = 15,38$ s, $r_2 = 101,90$ m, $\rho(t_2) = 126,90$ m, $\varepsilon_0 = 15.64$ ms^{-2} were calculated, respectively and it has been established the feasibility of the condition (5.1), which by the proposed control algorithm ensures the detection of TO within a finite time. The calculation results showed that starting from time t_2 the search process, followed by detection of TO is carried out in three phases with combined control in the form of controls (5.2) (5.10) related to each step:

Step-1

$$15,38\text{s} = t_2 \le t \le t_4 = 30,72\text{s},\;\; \Delta x_1^{(1)} = 23.29\text{m},\;\; r_4 = r_2 - \Delta x_1^{(1)},\;\; r_2 = 101,90\text{m} \to r_4 = 78,61\text{m},$$

Step-2

$$30,72\text{s} = t_4 \le t \le t_6 = 44,31\text{s},\;\; \Delta x_1^{(2)} = 32.04\text{m},\;\; r_6 = r_4 - \Delta x_1^{(2)},\;\; r_4 = 78,62\text{m} \to r_6 = 46,58\text{m},$$

Step-3

$$44,31\text{s} = t_6 \le t \le t_8 = 54,43\text{s},\;\; r_8 = r_6 - \Delta x_1^{(3)},\;\; \Delta x_1^{(3)} = 49,44\text{m},\;\; r_6 = 46,57\text{m} \to r(t_8) = 0.$$

The table shows that the uncertainty domain of TO narrows after each step (the radius of the circle containing this region is reduced) and at the end of the third step it disappears. Consequently, at time $t_8 = 54,43$s the uncertainty domain is absorbed by the detection circle, i.e. the detection of TO happens not later than the guaranteed search time $T = t_8$.

Conclusion. For the problem of a guaranteed search of a moving object, a constructive combined control algorithm is developed, which allows the searching object moving in a fixed-height horizontal plane of a three-dimensional space to perform the search of the target object within a finite time. Using the produced guaranteeing controls in an explicit form, the search, followed by the detection of the target object is carried out on the spatial trajectory consisting of linear sections and curvilinear sections in a form of circles with monotonically decreasing radii.

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