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M a t h e m a t i c s

ON A CONNECTION BETWEEN A CLASS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS AND INTEGRAL OPERATORS WITH SEMI-SEPARABLE KERNEL

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In the present paper the connection between a class of systems of differential equations and integral operators with semi-separable kernel is established. Using a matrix of the system the inverse to a given integral operator is constructed. Moreover by putting some additional conditions on the kernel of integral operator and by the help of inverse of integral operator a fundamental matrix of the system is constructed.

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Introduction. Denote by $L_{p,loc}(\alpha,\beta)$, $1 \leqslant p < \infty$, $-\infty \leqslant \alpha < \beta \leqslant \infty$, the set of all real valued function f such that $f \in L_p(\alpha',\beta')$ for every $[\alpha',\beta'] \subset (\alpha,\beta)$.

For an arbitrary linear space X we denote by X^n ($X^{n\times m}$) the set of n-dimensional columns (matrices of order $n \times m$) with elements from X.

Suppose $c,d \in L^n_{1,loc}(\alpha,\beta)$ and $dc^T \in L^{n\times n}_{1,loc}(\alpha,\beta)$. Consider the system of ordinary differential equations of the form

$$\frac{dz}{dx} = d(x)c^T(x)z. \tag{1}$$

As a solution of this system we mean (see [1, § 16]) an absolutely continuous on each boundary subinterval $[\alpha',\beta'] \subset (\alpha,\beta)$ function, which almost everywhere satisfies the system of equations (1).

Let $\mathcal{D}$ is a set of functions $y \in L_{1,loc}(\alpha,\beta)$ such that $d \cdot y \in L^n_{1,loc}(\alpha,\beta)$.

Along with the system (1) the integral operator $\Gamma_\xi : (L_{1,loc}(\alpha,\beta) \to L_{1,loc}(\alpha,\beta))$, ($\xi \in R \cap (\alpha,\beta)$) with a semi-separable kernel and domain $\mathcal{D}$

$$(\Gamma_\xi y)(x) = y(x) - c^T(x)\int_\xi^x d(t)y(t)\,dt \tag{2}$$

is considered.

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It is evident, that if $-\infty < \alpha < \beta < \infty$ and $d \in L_q^n(\alpha,\beta)$, $c \in L_p^n(\alpha,\beta)$, $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, then the operator $\Gamma_\xi : L_p(\alpha,\beta) \to L_q(\alpha,\beta)$ is bounded.

In [2, 3] the matrix analogue of the operator Γ_0 in $L_2^n(0,\alpha)$, $\alpha < +\infty$, with $c \in L_2^{n\times m}(0,\alpha)$, $d \in L_2^{m\times n}(0,\alpha)$ has been investigated. In [2], using the coupling method, the properties of the indicated operator in terms of fundamental matrix of some system of differential equations like (1) is considered.

In the present paper under some sufficiently general conditions on the kernel of operator Γ_ξ using the fundamental matrix of the system (1), the inverse operator to $\Gamma_\xi : \mathcal{D} \to \mathcal{D}$ is constructed. Moreover, by putting some additional conditions on the kernel of integral operator and using the inverse of Γ_α, we construct the fundamental matrix of (1).

Formula for the Inverse Operator. Let Φ be a fundamental matrix of system (1), where the components of the matrix are absolutely continuous functions on every bounded subinterval of (α,β). Then the formula

$$(L_\xi y) = y(x) + c^T \Phi(x) \int_\xi^x \Phi^{-1}(t) d(t) y(t)\, dt$$

correctly defines the integral operator L_ξ with a semi-separable kernel and $L_\xi : \mathcal{D} \to \mathcal{D}$.

Theorem 1. There hold the identities

$$\Gamma_\xi (L_\xi y)(x) = y(x), \qquad L_\xi (\Gamma_\xi y)(x) = y(x),\ y \in \mathcal{D}.$$

Proof. Using the equality $dc^T\Phi = \Phi'$, for $y \in \mathcal{D}$ we obtain

$$\Gamma_\xi (L_\xi y)(x) = (L_\xi y)(x) + (\Gamma_\xi y)(x) - y(x) - c^T(x) \int_\xi^x \Phi'(t) \int_\xi^t \Phi^{-1}(\tau) d(\tau) y(\tau)\, d\tau\, dt.$$

Changing the order of integration in the last term, we get

$$c^T(x) \int_\xi^x \Phi'(t) \int_\xi^t \Phi^{-1}(\tau) d(\tau) y(\tau)\, d\tau\, dt =$$
$$= c^T(x) \int_\xi^x \int_\tau^x \Phi'(t)\, dt\, \Phi^{-1}(\tau) d(\tau) y(\tau)\, d\tau = (L_\xi y)(x) + (\Gamma_\xi)(x) - 2y(x).$$

Then substituting the last term into the above expression, we get $\Gamma_\xi (L_\xi y)(x) = y(x)$.

On the other hand, it follows from the equality $dc^T = \Phi'\Phi^{-1}$ that

$$L(\xi)(\Gamma_\xi y)(x) = (\Gamma_\xi y)(x) + (L_\xi y)(x) - y(x) -$$
$$- c^T(x)\Phi(x) \int_x^\xi \Phi^{-1}(t)\Phi'(t)\Phi^{-1}(t) \int_\xi^t d(\tau) y(\tau)\, d\tau\, dt =$$
$$= (\Gamma_\xi y)(x) + (L_\xi y)(x) - y(x) + c^T(x)\Phi(x) \int_x^\xi \int_\tau^t (\Phi^{-1}(t))' dt d(\tau) y(\tau) d\tau =$$
$$= (\Gamma_\xi y)(x) + (L_\xi y)(x) - y(x) + c^T(x)\Phi(x) \int_x^\xi (\Phi^{-1}(x) - \Phi^{-1}(\tau)) d(\tau) y(\tau) d\tau = y(x).$$

□

Corollary 1. Let $c \in L_p(\alpha,\beta)$, $d \in L_q(\alpha,\beta)$. Then the operator Γ_ξ is bounded and invertible in $L_p(\alpha,\beta)$ and $\Gamma_\xi^{-1} = L_\xi$.

Nondegenerate Representations of an Integral Operator. Further it is assumed that $-\infty < \alpha = \xi < \beta < \infty$, $c \in L_p^n(\alpha,\beta)$, $d \in L_q^n(\alpha,\beta)$. Vector-function $a = (a_1,\dots,a_n)^T \in L_q^n(\alpha,\beta)$ is called nondegenerate at the point $x = \alpha$, if there is $\varepsilon > 0$ such that $a_1,\dots,a_n$ are linearly independent in $L_q(\alpha,\alpha+\varepsilon')$ for any $0 < \varepsilon' < \varepsilon$. Vector-function $b = (b_1,\dots,b_n)^T \in L_p^n(\alpha,\beta)$ is called nondegenerate at the point $x = \beta$, if there is $\varepsilon > 0$ such that $b_1,\dots,b_n$ are linearly independent in $L_p(\beta-\varepsilon',\beta)$ for each $0 < \varepsilon' < \varepsilon$.

It is clear that the representation (2) of operator Γ_α is not unique. We say that representation (2) is nondegenerate at the point $x = \alpha$, if the vector-function $d(x)$ is nondegenerate at this point and $c_1,...,c_n$ are linearly independent in $L_p(\alpha,\beta)$ $(c = c_1,...c_n)^T$. Similarly, we say that representation (2) is nondegenerate at the point $x = \beta$, if the vector-function $c^T(x)$ is nondegenerate at this point and $d_1,...,d_n$ are linearly independent in $L_p(\alpha,\beta)$ $(d = d_1,...d_n)^T$. The representation (2) is called nondegenerated, if it is nondegenerated at point $x = \alpha$ as well as at $x = \beta$.

Lemma 1. Let (2) ($\xi = \alpha$) and

$$(\Gamma_\alpha y)(x) = y(x) - \tilde{c}^T(x)\int_\alpha^x \tilde{d}(t)y(t)\,dt, \quad x \in (\alpha,\beta), \tag{3}$$

where $\tilde{c} \in L_p^m(\alpha,\beta)$, $\tilde{d} \in L_q^m(\alpha,\beta)$, are two different representations of the operator Γ_α. Then

a) if representation (2) is nondegenerate at the point $x = \alpha$, then $m \geqslant n$ and there exists a $m \times n$ matrix A with constant components such that

$$c^T(x) = \tilde{c}^T A; \tag{4}$$

b) if representation (2) is nondegenerate at the point $x = \beta$, then $m \geqslant n$ and there exists a matrix B with constant components such that

$$d(x) = B\tilde{d}(x). \tag{5}$$

Proof. Suppose representation (2) is nondegenerate at the point $x = \alpha$, $d = (d_1,\dots,d_n)^T$ and there is a $\varepsilon > 0$ such that for all ε', $0 < \varepsilon' < \varepsilon$, $d_1,\dots,d_n$ are linearly independent in $L_q(\alpha,\alpha+\varepsilon')$. Fix $\varepsilon' > 0$. Suppose $y_1,\dots,y_n \in L_p(\alpha,\alpha+\varepsilon')$ satisfy to system of equalities

$$\int_\alpha^{\alpha+\varepsilon'} d_i(t)y_j(t)\,dt = \delta_{ij}, \quad i,j = 1,\dots,n,$$

where δ_{ij} is the Kroneker's symbol. Denote by $\tilde{y}_j$ ($j = 1,\dots,n$) the extension of y_j on (α,β) such that $\tilde{y}_j \equiv 0$ in $(\alpha+\varepsilon',\beta)$. By comparing the values of $\Gamma_\alpha\tilde{y}_j$, $j = 1,\dots,n$, in (2) and (3) we get

$$c_j(x) = \sum_{i=1}^m a_{ij}\tilde{c}_i(x), \quad j = 1,\dots,n, \tag{6}$$

almost everywhere on $(\alpha+\varepsilon',\beta)$, where

$$a_{ij} = \int_\alpha^{\alpha+\varepsilon'} \tilde{d}_i(t)y_j(t)\,dt.$$

Taking into account that ε' is arbitrary positive number, we can say that equality (6) holds for almost all $x \in (\alpha, \beta)$. If we get

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix},$$

then we obtain (4). Moreover, since $c_1, \dots, c_n$ are linearly independent and $span\{c_1, \dots, c_n\} \subset span\{\tilde{c}_1, \dots, \tilde{c}_n\}$, then $m \geqslant n$ and, proposition a) is proved.

Let the conjugate operator $\Gamma^*_\alpha : L_q(\alpha, \beta) \to L_q(\alpha, \beta)$ is written in the forms

$$(\Gamma^*_\alpha y)(x) = y(x) - d^T(x) \int_x^\beta c(t)y(t)\,dt \text{ and}$$

$$(\Gamma^*_\alpha y)(x) = y(x) - \tilde{d}^T(x) \int_x^\beta \tilde{c}(t)y(t)\,dt.$$

Now, assume that the functions $z_1, \dots, z_n \in L_q(\beta - \varepsilon', \beta)$, where $\varepsilon' > 0$ is sufficiently small, are defined by equalities

$$\int_{\beta-\varepsilon'}^\beta c_i(t)z_i(t)\,dt = \delta_{ij}, \quad i, j = 1, \dots, n.$$

By the extension of these functions on $(\alpha, \beta - \varepsilon')$ such that $\tilde{c}_i(t) = 0$ on this interval, as in the case a), the equalities

$$d_i(x) = \sum_{j=1}^m b_{ij}\tilde{d}_j, \quad i = 1, \dots, n, \tag{7}$$

can be proved. Denoting

$$b_{ij} = \int_{\beta-\varepsilon'}^\beta \tilde{c}_j(t)y_i(t)\,dt$$

and

$$B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{n1} & \cdots & b_{nn} \end{pmatrix}$$

it is easy to see that (7) is equivalent to (5). Moreover, from the embedding $span\{d_1, \dots, d_n\} \subset span\{\tilde{d}_1, \dots, \tilde{d}_n\}$ is follows, that $n \leqslant m$. □

This Lemma gives us the following description of the nondegenerate representation of the operators Γ_α and Γ_α^{-1}.

Theorem 2. Suppose (2) is a nondegenerate representation of the operators Γ_α. Then

a) representation (3) is nondegenerate if only if $m = n$ and there exists a nondegenerate matrix B of order $n \times n$ such that

$$c^T(x) = \tilde{c}^T(x)B^{-1}, \qquad \tilde{d}(x) = Bd(x); \tag{8}$$

b) every nondegenerate representation of the operator Γ_α^{-1} has the form

$$(\Gamma_\alpha^{-1}y)(x) = y(x) - c(x)\Phi(x) \int_\alpha^x \Phi^{-1}(t)y(t)\,dt, \tag{9}$$

where Φ is any fundamental matrix of system (1).

Proof. Assume there is a nondegenerate matrix B as in (8). Then is clear that the representation (3) is nondegenerate. Let the representation (3) is nondegenerate too. Then from Lemma 1 it follows that $m = n$ and $span\{d_1,\dots,d_n\} = span\{\tilde{d}_1,\dots,\tilde{d}_n\}$. Since the systems of the functions $d_1,\dots,d_n$ and $\tilde{d}_1,\dots,\tilde{d}_n$ are linearly independent, there exists a nondegenerate matrix B such that $\tilde{d} = Bd$. Substituting it in (3), we obtain

$$(\Gamma_\alpha y)(x) = y(x) - \tilde{c}(x)B\int_\alpha^x d(t)y(t)\,dt.$$

Choosing the functions $y_1,\dots,y_n$ as in part a) of Lemma 1 and comparing the values $\Gamma_\alpha y_i,\ i = 1,\dots,n$, in the last representation and in (2), we get $c = \tilde{c}B$, which proves (8). Proposition b) and equality (9) follow from Corollary 1 and proposition a). □

Fundamental Matrix. For a matrix-function $G_1 \in L_1^{n\times n}[\alpha,\beta]$ and for a matrix B of the order $n\times n$ with constant components, we denote $U(G,B)$ by thew matrix-function such that $(U(G,B))' = G$, $U(G,B)(\alpha) = B$.

Lemma 2. If there exists a vector-function $p \in L_p^n[\alpha,\beta]$ such that $c^T U(dp^T,B) = p^T$ for some matrix B and if $U(dp^T,B)$ is nondegenerate at some point of $[\alpha,\beta]$, then $U(dp^T,B)$ is a fundamental matrix of the system (1) in $[\alpha,\beta]$.

The proof follows from the definition of matrix $U(dp^T,B)$ and equality $c^T U(dp^T,B) = p^T$:

$$U'(dp^T,B) = dp^T = dc^T U(dp^T,B),$$

which means that $U(dp^T,B)$ is a fundamental matrix of the system (1).

Lemma 3. If there exists a vector-function $q \in L_q^n[\alpha,\beta]$ such that $U(-qc^T,A)d = q$ for some matrix A and if $U(-qc^T,A)$ is nondegenerate at some point of $[\alpha,\beta]$, then $U(-qc^T,A)^{-1}$ is a fundamental matrix of the system (1).

Proof. The proof immediately follows from the properties of the matrix $U(-qc^T,A)$. Namely,

$$\begin{aligned}(U(-qc^T,A)^{-1})' &= -U(-qc^T,A)^{-1}U'(-qc^T,A)U(-qc^T,A)^{-1} = \\ &= U(-qc^T,A)^{-1}qc^T U(-qc^T,A)^{-1} = dc^T U(-qc^T,A)^{-1}.\end{aligned}$$

□

Theorem 3. Let (2) be a nondegenerate representation of the operator Γ_α and

$$(\Gamma_\alpha^{-1}y)(x) = y(x) + p^T(x)\int_\alpha^x q(t)y(t)\,dt \tag{10}$$

is a nondegenerate representation of an operator Γ_α^{-1}. Then for any nondegenerate matrices B and A the matrix-functions $U(dp^T,B)$ and $(U(-qc^T,A))^{-1}$ are the fundamental matrices for the system (1) in $[\alpha,\beta]$.

Proof. Let $U_0 = U(dp^T,0)$. Using integration by part, it is easy to see that

$$\begin{aligned}&c^T(x)\int_\alpha^x (dp^T)(t)\int_\alpha^t q(\tau)y(\tau)\,d\tau\,dt = \\ &= c^T(x)U_0(x)\int_\alpha^x q(t)y(t)\,dt - c^T(x)\int_\alpha^x U_0(t)q(t)y(t)\,dt, \quad y \in L_p(\alpha,\beta).\end{aligned}$$

Hence, taking into account (2) and (10), we get

$$(\Gamma_\alpha \Gamma_\alpha^{-1} y)(x) = y(x) + \\ + (p^T(x) - c^T(x)U_0(x)) \int_\alpha^x q(t)y(t)\,dt - c^T(x) \int_\alpha^x (d(t) - U_0(t)q(t))y(t)\,dt. \quad (11)$$

Since representation (10) is nondegenerate, from theorem 2, point b) it is follows that there exists fundamental matrix Φ of the system (1) such that

$$p^T = c^T \Phi, \qquad q^T = \Phi^{-1} d. \quad (12)$$

So $p^T - c^T U_0 = c^T(\Phi - U_0)$. The columns of the matrix-function $\Phi - U_0$ are the solutions of the system (1) (see Lemma 2) and moreover $\det(\Phi(\alpha) - U_0(\alpha)) = = \det(\Phi(\alpha)) \neq 0$. From this we obtain that $\Phi - U_0$ is the fundamental matrix of the system (1) and, therefore, the vector function $(c^T(\Phi - U_0))^T$ is nondegenerate at the point $x = \beta$. Similarly from (12) it is follows that $d - U_0 q = (p - U_0)\Phi^{-1} d$ and, therefore, $d - U_0 q$ is nondegenerate at the point $x = \alpha$.

So from (11) and identity $\Gamma_\alpha(\Gamma_\alpha^{-1} y) = y$ the equalities

$$(\tilde{\Gamma}_\alpha y)(x) = y(x) - (p^T(x) - c^T(x)U_0(x)) \int_\alpha^x q(t)y(t)\,dt,$$

$$(\tilde{\Gamma}_\alpha y)(x) = y(x) - c^T(x) \int_\alpha^x (d(t) - U_0(x)q(t))y(t)\,dt$$

are two different nondegenerate representation of the same operator $\tilde{\Gamma}_\alpha$ and, therefore, from Theorem 2 it follows that there exists nondegenerate matrix B of order $n \times n$ such that

$$p^T - c^T U_0 = c^T B, \qquad d - U_0 q = Bq,$$

and so

$$c^T U(dp^T, B) = p^T, \qquad U(dp^T, B)q = d. \quad (13)$$

From the first equality of (13) and Lemma 2 it follows that $U(dp^T, B)$ is the fundamental matrix of the system (1).

It is evident that the same argument can be applied changing the places of the operators Γ_α and Γ_α^{-1}. While in (13) we need to change the vector-functions c, d, p, q to $-p$, q, $-c$, d respectively. It is easy to see (by replacing B to A) that the second equality (13) takes the from $U(-qc^T, A)d = q$. It remains to apply Lemma 3. □

Corollary 2. Let (2) and (10) be nondegenerate representation of the operators $\tilde{\Gamma}_\alpha$, Γ_α^{-1} and B be a nondegenerate matrix of order $n \times n$. Then

$$(U(-qc^T, B^{-1}))^{-1} = U(dp^T, B).$$

One Application. Let $0 < p_1 < \cdots < p_n$, $m_k > 0$ $(k = 1, \ldots, n)$ and consider the functions $c_k(x) = m_k e^{-p_k x}$, $x > 0$. Suppose the functions $d_k(x)$, $k = 1, \ldots, n$, are defined from the system of linear equations

$$d_i(x) + \sum_{j=1}^n m_i m_j (p_i + p_j)^{-1} e^{-(p_i + p_j)x} d_j(x) = c_i(x). \quad (14)$$

Denote $c = (c_1, \dots, c_n)^T$, $d = (d_1, \dots, d_n)^T$ and

$$A = \left(\delta_{ij} + \frac{m_i m_j}{p_i + p_j}\right)_{i,j=1}^{n}$$, where $\delta_{i,j}$ is Kroneker's symbol.

System (14) can be written in the form

$$U(-cc^T, A)d = c. \tag{15}$$

It is known [4] that the kernel $k(x,t)$ of the transformation operator $K : L_p(0,\infty) \to L_p(0,\infty)$ $(1 \leqslant p \leqslant \infty)$, which acts by the formula

$$(Ky)(x) = y(x) + \int_x^{\infty} k(x,t)y(t)\,dt$$

in the case of reflectionless potential with discreet spectrum, which consists of eigenvalues $(ip_k)^2$, and with right normalization coefficients, is defined by an equation $k(x,t) = -c^T(t)d(x)$. In the theory of L-convolution operators [5] the operators of the form

$$(\Gamma_0 y)(x) = (K^* y)(x) = y(x) - c^T(x)\int_0^x d(t)y(t)\,dt$$

play an important role. Note that $U(-cc^T, A)(x)$ tends to unit matrix when $x \to \infty$. Therefore for sufficiently large β, $\det U(-cc^T, A)(\beta) \neq 0$. From Lemma 3 it follows that the matrix-function $\Phi = [U(-cc^T, A)]^{-1}$ is a fundamental matrix of the system (1). Taking into account the relation $\Phi^T = \Phi$, it is easy to see that $c^T\Phi = c^T\Phi^T = d^T$ and $\Phi^{-1}d = c$. So from Theorem 1 in the space $L_p(0,\infty)$, Γ_0^{-1} acts as follows

$$(\Gamma_0^{-1} y)(x) = y(x) + d^T(x)\int_0^x c(t)y(t)\,dt.$$

Note, that from Theorem 3 it follows that as a fundamental matrix of the system (1) it can be taken the matrix-function

$$U(dd^T, E_n) = \left(\delta_{ij} + \int_0^x d_i(t)d_j(x)\,dt\right)_{i,j=1}^{n}.$$

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