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ON AN EQUIVALENCE FOR DIFFERENTIATION BASES OF DYADIC RECTANGLES

BY

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Abstract. The paper considers differentiation properties of rare bases of dyadic rectangles corresponding to increasing sequences $\{\nu_k\}$ of integers. We prove that the condition

$$\sup_k (\nu_{k+1} - \nu_k) < \infty$$

is necessary and sufficient for such a basis to be equivalent to the full basis of dyadic rectangles.

1. Introduction. Let $\mathcal{R}$ be the family of all half-closed rectangles $[a, b) \times [c, d)$ in $\mathbb{R}^2$ and $\mathcal{Q} \subset \mathcal{R}$ be the family of half-closed squares in $\mathbb{R}^2$. Moreover, let $\mathcal{R}^{\text{dyadic}}$ be the family of dyadic rectangles of the form

$$[(i-1)/2^n, i/2^n) \times [(j-1)/2^m, j/2^m), \qquad i, j, n, m \in \mathbb{Z}, \tag{1}$$

and $\mathcal{Q}^{\text{dyadic}}$ be the family of dyadic squares ($n = m$). We have $\mathcal{R}^{\text{dyadic}} \subset \mathcal{R}$ and $\mathcal{Q}^{\text{dyadic}} \subset \mathcal{Q}$. For a given rectangle $R \in \mathcal{R}$ we denote by $\operatorname{len}(R)$ the length of the larger side of R.

DEFINITION 1.1. A family $\mathcal{M} \subset \mathcal{R}$ is said to be a *differentiation basis* (or simply a *basis*) if for any $x \in \mathbb{R}^2$ there exists a sequence of rectangles $R_k \in \mathcal{M}$ such that $x \in R_k$, $k = 1, 2, \ldots$, and $\operatorname{len}(R_k) \to 0$ as $k \to \infty$.

Let $\mathcal{M} \subset \mathcal{R}$ be a differentiation basis. For any function $f \in L^1(\mathbb{R}^2)$ we define

$$\delta_{\mathcal{M}}(x, f) = \limsup_{\operatorname{len}(R) \to 0,\, x \in R \in \mathcal{M}} \left| \frac{1}{|R|} \int_R f(t)\, dt - f(x) \right|.$$

We say that *the integral of f is differentiable at $x \in \mathbb{R}^2$ with respect to the basis $\mathcal{M}$* if $\delta_{\mathcal{M}}(x, f) = 0$. Set

$$\mathcal{F}(\mathcal{M}) = \{f \in L^1(\mathbb{R}^2) : \delta_{\mathcal{M}}(x, f) = 0 \text{ almost everywhere}\},$$
$$\mathcal{F}^+(\mathcal{M}) = \{f \in L^1(\mathbb{R}^2) : f(x) \geq 0,\ \delta_{\mathcal{M}}(x, f) = 0 \text{ almost everywhere}\}.$$

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Thus $\mathcal{F}(\mathcal{M})$ is the family of functions having almost everywhere differentiable integrals with respect to the basis $\mathcal{M}$.

Let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a convex function. Denote by $\Phi(L)(\mathbb{R}^2)$ the class of measurable functions f on $\mathbb{R}^2$ such that $\Phi(|f|) \in L^1(\mathbb{R}^2)$. If Φ satisfies the Δ_2-condition $\Phi(2x) \le k\Phi(x)$, then $\Phi(L)$ is an Orlicz space with the norm

$$\|f\|_\Phi = \inf\Big\{c > 0 : \int_{\mathbb{R}^2} \Phi(|f|/c) \le 1\Big\}.$$

The following classical theorems state that the optimal Orlicz space whose functions have a.e. differentiable integrals with respect to the entire family of rectangles $\mathcal{R}$ is the space

$$L(1 + \log L)(\mathbb{R}^2) \subset L^1(\mathbb{R}^2),$$

corresponding to the case $\Phi(t) = t(1 + \log^+ t)$ ([1]).

Theorem A (Jessen–Marcinkiewicz–Zygmund [4]).

$$L(1 + \log L)(\mathbb{R}^2) \subset \mathcal{F}(\mathcal{R}).$$

Theorem B (Saks [7]). *If*

$$\Phi(t) = o(t \log t) \qquad \text{as } t \to \infty,$$

then $\Phi(L)(\mathbb{R}^2) \not\subset \mathcal{F}(\mathcal{R})$. Moreover, there exists a positive function $f \in \Phi(L)(\mathbb{R}^2)$ such that $\delta_{\mathcal{R}}(x, f) = \infty$ everywhere.

Similar theorems are also valid for the basis $\mathcal{R}^{\text{dyadic}}$. The first one trivially follows from the embedding $L(1 + \log L)(\mathbb{R}^2) \subset \mathcal{F}(\mathcal{R}) \subset \mathcal{F}(\mathcal{R}^{\text{dyadic}})$. The second can be deduced from the relation

$$\mathcal{F}^+(\mathcal{R}^{\text{dyadic}}) = \mathcal{F}^+(\mathcal{R}),$$

due to Zerekidze [9] (see also [10, 11]).

Let $\Delta = \{\nu_k : k = 1, 2, \ldots\}$ be an increasing sequence of positive integers. This sequence generates the rare basis $\mathcal{R}^{\text{dyadic}}_\Delta$ of dyadic rectangles of the form (1) with $n, m \in \Delta$. This kind of bases was first considered in [8], [2], [3]. Stokolos [8] proved that the analogue of the Saks theorem holds for any basis $\mathcal{R}^{\text{dyadic}}_\Delta$ with an arbitrary Δ sequence. That means $L(1+\log L)(\mathbb{R}^2)$ is again the largest Orlicz space contained in $\mathcal{F}(\mathcal{R}^{\text{dyadic}}_\Delta)$. G. A. Karagulyan [5] proved equivalence of some convergence conditions for multiple martingale sequences, which in particular imply some results of [8], [2], [3].

In this paper we prove

Theorem. *Let $\Delta = \{\nu_k\} \subset \mathbb{N}$ be an increasing sequence of positive integers. Then the condition*

$$\sup_{k\in\mathbb{N}} (\nu_{k+1} - \nu_k) < \infty \tag{2}$$

is necessary and sufficient for the equality $\mathcal{F}(\mathcal{R}^{\text{dyadic}}_\Delta) = \mathcal{F}(\mathcal{R}^{\text{dyadic}})$ to hold.

2. Some definitions and key functions. Denote by $\overline{E}$ and $\mathring{E}$ the closure and the interior of a set $E \subset \mathbb{R}^2$ respectively; $\mathbb{I}_E$ denotes the indicator function of E. A set $E \subset \mathbb{R}^2$ is said to be *simple* if it can be written as a union of squares of the form

$$[(i-1)/2^n, i/2^n) \times [(j-1)/2^n, j/2^n).$$

If n is the minimal integer with this property, then we write $\mathrm{wd}(E) = 2^{-n}$. Note that if E is a dyadic rectangle, then $\mathrm{wd}(E)$ coincides with the length of the smaller side of E. If E is a square, then $\mathrm{len}(E) = \mathrm{wd}(E)$. Denote

$$E_{ij}(n) = \bigcup_{k=0}^{n-1} [i/2, i/2 + 1/2^{k+1}) \times [j/2, j/2 + 1/2^{n-k}), \tag{3}$$

$$\begin{aligned} F_{ij}(n) &= [i/2, i/2 + 1/2^n) \times [j/2, j/2 + 1/2^n) \\ &= \bigcap_{k=0}^{n-1} [i/2, i/2 + 1/2^{k+1}) \times [j/2, j/2 + 1/2^{n-k}) \\ &\subset E_{ij}(n), \quad i, j = 0, 1, \end{aligned} \tag{4}$$

and define

$$E(n) = E_{00}(n) \cup E_{01}(n) \cup E_{10}(n) \cup E_{11}(n), \tag{5}$$

$$F(n) = F_{00}(n) \cup F_{01}(n) \cup F_{10}(n) \cup F_{11}(n) \subset E(n). \tag{6}$$

We introduce the functions

$$\begin{aligned} u(x, n) &= (n+1)2^{n-2}\big(\mathbb{I}_{F_{00}(n)}(x) + \mathbb{I}_{F_{11}(n)}(x) \\ &\qquad - \mathbb{I}_{F_{10}(n)}(x) - \mathbb{I}_{F_{01}(n)}\big), \quad n \in \mathbb{N}, \\ v(x) &= \mathbb{I}_{[0,1/2)\times[0,1/2)}(x) + \mathbb{I}_{[1/2,1)\times[1/2,1)}(x) \\ &\qquad - \mathbb{I}_{[0,1/2)\times[1/2,1)}(x) - \mathbb{I}_{[1/2,1)\times[0,1/2)}(x). \end{aligned}$$

Let $\omega \in \mathcal{Q}$ be an arbitrary square and ϕ_ω be the linear transformation of $\mathbb{R}^2$ taking ω onto the unit square $[0,1)^2 \subset \mathbb{R}^2$. For an arbitrary function f defined on $[0,1)^2$ and for a set $E \subset [0,1)^2$ we define

$$f_\omega(x) = f(\phi_\omega(x)), \quad E_\omega = (\phi_\omega)^{-1}(E) \subset \omega.$$

We have

$$\operatorname{supp} u_\omega(\cdot, n) = F_\omega(n), \tag{7}$$

$$\operatorname{supp} v_\omega = \omega, \tag{8}$$

$$|E_\omega(n)| = (n+1)|\omega|/2^n, \qquad |F_\omega(n)| = |\omega|/4^{n-1}, \tag{9}$$

$$\mathrm{wd}(E_\omega(n)) = \mathrm{wd}(F_\omega(n)) = \mathrm{wd}(\omega) \cdot 2^{-n}. \tag{10}$$

Simple calculations show that

$$\|u_\omega(\cdot,n)\|_1 = |E_\omega(n)| = (n+1)|\omega|/2^n, \tag{11}$$

$$\|v_\omega\|_1 = |\omega|. \tag{12}$$

Then observe that if $\omega \in \mathcal{Q}^{\mathrm{dyadic}}$ is a dyadic square, then for any $x \in E_\omega(n)$ there exists a dyadic rectangle $R(x) \in \mathcal{R}^{\mathrm{dyadic}}$ with

$$\frac{1}{|R(x)|}\Big|\int_{R(x)} u_\omega(x,n)\,dx\Big| = \frac{n+1}{2}, \quad x \in R(x) \subset E_\omega(n), \tag{13}$$

$$\mathrm{wd}(R(x)) = \mathrm{wd}(\omega)\cdot 2^{-n}. \tag{14}$$

Moreover, this rectangle coincides with the $(\phi_\omega)^{-1}$-image of one of the rectangles from (3). Similarly, if $\omega \in \mathcal{D}$, then

$$\frac{1}{|R(x)|}\Big|\int_{R(x)} v_\omega(x)\,dx\Big| = 1, \quad x \in R(x) \subset \omega, \tag{15}$$

$$\mathrm{wd}(R(x)) = \mathrm{wd}(\omega)/2, \tag{16}$$

for some square $R(x)$ with $|R(x)| = |\omega|/4$. In this case $R(x)$ coincides with one of the four squares forming ω.

3. Auxiliary lemmas. The following simple lemma has been proved in [6].

LEMMA 1. *Let $Q \in \mathcal{Q}^{\mathrm{dyadic}}$ be an arbitrary dyadic square. Suppose that a function $f \in L^1(\mathbb{R}^2)$, $f(x) = f(x_1,x_2)$, satisfies the condition* $\operatorname{supp} f \subset Q$ *and*

$$\int_{\mathbb{R}} f(x_1,t)\,dt = \int_{\mathbb{R}} f(t,x_2)\,dt = 0, \quad x_1,x_2 \in \mathbb{R}. \tag{17}$$

Then for any dyadic rectangle $R \in \mathcal{R}^{\mathrm{dyadic}}$ with $R \not\subset \mathring{Q}$,

$$\int_R f(x)\,dx = 0. \tag{18}$$

Proof. We suppose

$$Q = [\alpha_1,\beta_1)\times[\alpha_2,\beta_2), \quad R = [a_1,b_1)\times[a_2,b_2).$$

If $R \cap Q = \emptyset$, then (18) is trivial. Otherwise either $[\alpha_1,\beta_1) \subset [a_1,b_1)$ or $[\alpha_2,\beta_2) \subset [a_2,b_2)$. In the first case, using (17), we get

$$\int_R f(x)\,dx = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2)\,dx_1\,dx_2 = \int_{a_2}^{b_2}\int_{\alpha_1}^{\beta_1} f(x_1,x_2)\,dx_1\,dx_2$$
$$= \int_{a_2}^{b_2}\Big(\int_{\mathbb{R}} f(x_1,x_2)\,dx_1\Big)\,dx_2 = 0.$$

The second case is proved similarly. ∎

Lemma 2. *Let m be a positive integer and Q be a dyadic square. Then for any simple set $E \subsetneq [0,1)^2$, there exists a finite family Ω of dyadic squares $\omega \subset Q$ such that*

$$E_\omega \cap E_{\omega'} = \emptyset \quad \text{if } \omega \neq \omega', \tag{19}$$

$$\min_{\omega\in\Omega} \operatorname{wd}(\omega) = \operatorname{wd}(Q)\cdot(\operatorname{wd}(E))^m, \tag{20}$$

$$\Big|Q \setminus \bigcup_{\omega\in\Omega} E_\omega\Big| = |Q|(1-|E|)^m. \tag{21}$$

Proof. We will define a sequence of sets G_k, $k = 1,\dots,m$, with

$$Q = G_1 \supset \cdots \supset G_m, \tag{22}$$

and finite families of dyadic squares $\Omega_k \subset \mathcal{D}$, $k = 1,\dots,m+1$, such that

$$\operatorname{wd}(\omega) = \operatorname{wd}(Q)\cdot(\operatorname{wd}(E))^{k-1}, \quad \omega\in\Omega_k,\ k=1,\dots,m+1, \tag{23}$$

$$G_k = \bigcup_{\omega\in\Omega_k} \omega, \quad k=1,\dots,m+1, \tag{24}$$

$$G_k = G_{k-1} \setminus \bigcup_{\omega\in\Omega_{k-1}} E_\omega = \bigcup_{\omega\in\Omega_{k-1}} (\omega\setminus E_\omega), \quad k=2,\dots,m+1. \tag{25}$$

We use induction. For the base step we just take $G_1 = Q$ and let Ω_1 consist of a single rectangle Q. Suppose we have already chosen the sets G_k and the families Ω_k for $k = 1,\dots,p$ satisfying (22)–(25). Set

$$G_{p+1} = G_p \setminus \bigcup_{\omega\in\Omega_p} E_\omega = \bigcup_{\omega\in\Omega_p} (\omega\setminus E_\omega).$$

From the induction hypothesis of (23) it follows that

$$\operatorname{wd}(\omega\setminus E_\omega) = \operatorname{wd}(\omega)\cdot \operatorname{wd}(E) = \operatorname{wd}(Q)\cdot(\operatorname{wd}(E))^p.$$

Hence G_{p+1} is a union of dyadic squares with side lengths $\operatorname{wd}(Q)\cdot(\operatorname{wd}(E))^p$ and we define Ω_{p+1} as the collection of those squares. Thus we get G_{p+1} and Ω_{p+1} satisfying (22)–(25) for $k = p+1$, which completes the inductive definition.

Applying (11), (24) and (25) we obtain

$$|G_k| = |G_{k-1}| - \Big|\bigcup_{\omega\in\Omega_{k-1}} E_\omega\Big| = |G_{k-1}| - |E|\,|G_{k-1}| = (1-|E|)|G_{k-1}|,$$

and therefore

$$|G_{m+1}| = (1-|E|)^m|Q|. \tag{26}$$

The family $\Omega = \bigcup_{k=1}^{m+1} \Omega_k$ is as desired. Indeed, suppose $\omega,\omega' \in \Omega$. If $\omega,\omega' \in \Omega_k$ for some k, then according to (23) we have $\omega\cap\omega' = \emptyset$, and so (19) holds. If $\omega \in \Omega_k$, $\omega' \in \Omega_{k'}$ and $k < k'$, then

$$E_{\omega'} \subset \omega' \subset G_{k'},$$
$$E_\omega \subset G_k \setminus G_{k+1} \Rightarrow E_\omega \cap G_{k'} = \emptyset.$$

Thus we again get (19). Condition (20) immediately follows from (23), and (21) follows from (26) and from the relation

$$\Big|\bigcup_{\omega\in\Omega} E_\omega\Big| = \Big|\bigcup_{k=1}^{m+1}\bigcup_{\omega\in\Omega_k} E_\omega\Big| = \Big|\bigcup_{k=1}^{m+1} G_k \setminus G_{k+1}\Big|$$
$$= |Q \setminus G_{m+1}| = |Q|(1-(1-|E|)^m). \ \blacksquare$$

LEMMA 3. *Let $L > 1$ be an integer and let $Q \in \mathcal{Q}^{\mathrm{dyadic}}$. Then there exist $f \in L^\infty(\mathbb{R}^2)$ and $\alpha(L) \in \mathbb{N}$, $\beta(L) > 0$ such that*

$$\operatorname{supp} f \subset Q, \tag{27}$$
$$\|f\|_\infty \le \beta(L), \tag{28}$$
$$|\operatorname{supp} f| \le \frac{2|Q|}{\beta(L)}, \tag{29}$$
$$\operatorname{wd}(\operatorname{supp} f) \ge \operatorname{wd}(Q)\cdot 2^{-\alpha(L)}, \tag{30}$$
$$\int_R f(x)\,dx = 0, \qquad R \in \mathcal{R}^{\mathrm{dyadic}},\ R \not\subset \mathring{Q}, \tag{31}$$

and for any $x \in Q$ there exists a rectangle $R(x) \subset Q$ satisfying

$$\operatorname{wd}(R(x)) \ge \operatorname{wd}(Q)\cdot 2^{-\alpha(L)}, \tag{32}$$
$$\frac{1}{|R(x)|}\Big|\int_{R(x)} f(t)\,dt\Big| \ge L. \tag{33}$$

Proof. Let $n = 2L$ and denote

$$\alpha(L) = n(2^n+1), \quad \beta(L) = (n+1)2^{n-2}, \tag{34}$$
$$m = m(L) = \left[\frac{2^n(\ln(n+1)+(n-2)\ln 2)}{n+1}\right] + 1 < 2^n. \tag{35}$$

Let $E = E(n)$ be the set defined in (5). We have $|E(n)| = (n+1)/2^n$ and $\operatorname{wd}(E(n)) = 2^{-n}$. Applying Lemma 2, we find a family Ω of dyadic squares $\omega \subset Q$ with properties (19)–(21). Set

$$G = \bigcup_{\omega\in\Omega} E_\omega(n), \qquad G_1 = Q \setminus G. \tag{36}$$

According to (21), (34) and (35), we have

$$|G_1| = (1-|E(n)|)^m|Q| = \Big(1 - \frac{n+1}{2^n}\Big)^m |Q| < \frac{|Q|}{\beta(L)}.$$

From (20) and (35) it follows that

$$G_1 = \bigcup_{\omega\in\Omega_1} \omega,$$

where Ω_1 is a family of squares with

$$\min_{\omega\in\Omega_1} \operatorname{wd}(\omega) = \min_{\omega\in\Omega} \operatorname{wd}(\omega) = \operatorname{wd}(Q)\cdot(\operatorname{wd}(E(n)))^m \ge \operatorname{wd}(Q)\cdot 2^{-n\cdot 2^n}. \tag{37}$$

Define

$$f(x) = \sum_{\omega\in\Omega} u_\omega(x,n) + \beta(L)\sum_{\omega\in\Omega_1} v_\omega(x) =: g(x) + g_1(x).$$

Clearly this function satisfies (27) and (28). Moreover,

$$\operatorname{supp} g = \bigcup_{\omega\in\Omega} F_\omega(n) \subset G, \qquad \operatorname{supp} g_1 = G_1,$$
$$\operatorname{supp} f = \operatorname{supp} g \cup \operatorname{supp} g_1.$$

This together with (9) and (36) implies

$$|\operatorname{supp} f| = \bigcup_{\omega\in\Omega} |F_\omega(n)| + |G_1| = \frac{1}{(n+1)2^{n-2}}\sum_{\omega\in\Omega} |E_\omega(n)| + |G_1|$$
$$= \frac{1}{(n+1)2^{n-2}}|G| + |G_1| \le \frac{2|Q|}{\beta(L)},$$

and therefore we get (29). Using (37), we obtain

$$\operatorname{wd}(\operatorname{supp} g) \ge \min_{\omega\in\Omega} \operatorname{wd}(\omega)\cdot\operatorname{wd}(F(n)) = \operatorname{wd}(Q)\cdot 2^{-n(2^n+1)} = \operatorname{wd}(Q)\cdot 2^{-\alpha(L)},$$
$$\operatorname{wd}(\operatorname{supp} g_1) \ge \min_{\omega\in\Omega_1} \operatorname{wd}(\omega) \ge \operatorname{wd}(Q)\cdot 2^{-n\cdot 2^n} > \operatorname{wd}(Q)\cdot 2^{-\alpha(L)},$$

and hence we get (30). Condition (31) follows from Lemma 1, since f satisfies (17) according to the definitions of $u_\omega(x,n)$ and $v_\omega(x)$. To prove (33), take $x \in Q$. We have either $x \in G$ or $x \in G_1$. In the first case, $x \in E_\omega(n)$ for some $\omega \in \Omega$. By (13) there exists a dyadic rectangle $R = R(x)$ with $x \in R \subset E_\omega(n)$ such that

$$\frac{1}{|R|}\Big|\int_R f(t)\,dt\Big| = \frac{1}{|R|}\Big|\int_R u_\omega(t,n)\,dt\Big| = \frac{n+1}{2} > L.$$

In the second case from (15) we obtain

$$\frac{1}{|R|}\Big|\int_R f(t)\,dt\Big| = \frac{\beta(L)}{|R|}\Big|\int_R v_\omega(t)\,dt\Big| \ge 2^n > L$$

for some square $R = R(x)$ with $x \in R \subset \omega$. Obviously in either case $R(x)$ satisfies (32). The lemma is proved. ∎

Proof of Theorem. Necessity. Let $\varDelta = \{\nu_k\}$ be a sequence with

(38) $$\gamma = \sup_{k\in\mathbb{N}}(\nu_{k+1} - \nu_k) < \infty,$$

and suppose

$$\mathcal{F}(\mathcal{R}_\varDelta^{\text{dyadic}}) \setminus \mathcal{F}(\mathcal{R}^{\text{dyadic}}) \neq \emptyset.$$

That means there exists $f \in L^1(\mathbb{R}^2)$ such that

(39) $$\delta_{\mathcal{R}_\varDelta^{\text{dyadic}}}(x, f) = 0 \quad \text{a.e.},$$

(40) $$\delta_{\mathcal{R}^{\text{dyadic}}}(x, f) > \alpha, \quad x \in E,$$

where $\alpha > 0$ and $|E| > 0$. According to (39) for any $x \in \mathbb{R}^2$ one can choose $\delta(x) > 0$ such that the conditions

$$x \in R \in \mathcal{R}_\varDelta^{\text{dyadic}}, \quad \text{len}(R) < \delta(x),$$

imply

(41) $$\left|\frac{1}{|R|}\int_R f - f(x)\right| < \alpha/2.$$

For some $\delta > 0$ the set $F = \{x \in E\colon \delta(x) \geq \delta\} \subset E$ has positive measure. Then, using the representation

$$F = \bigcup_{j\in\mathbb{Z}}\{x \in F\colon j\alpha/2 \leq f(x) < (j+1)\alpha/2\},$$

we find a set

(42) $$G = \{x \in F\colon j_0\alpha/2 \leq f(x) < (j_0+1)\alpha/2\} \subset F$$

having positive measure. Combining (40)–(42), we will have

(43) $$\delta_{\mathcal{R}^{\text{dyadic}}}(x, f) > \alpha, \quad x \in G,$$

(44) $$\left|\frac{1}{|R|}\int_R f - f(x)\right| < \alpha/2 \text{ if } x \in R\cap G,\ R \in \mathcal{R}_\varDelta^{\text{dyadic}},\ \text{len}(R) < \delta,$$

(45) $$\sup_{x,y\in G}|f(x) - f(y)| \leq \alpha/2.$$

Since almost all points of G are density points, we may fix $x_0 \in G$ with

$$\lim_{\text{len}(R)\to 0,\, x_0\in R\in\mathcal{R}^{\text{dyadic}}} \frac{|R\cap G|}{|R|} = 1.$$

Using this relation and (43), we find a rectangle

$$R' = [(p-1)/2^n, p/2^n) \times [(q-1)/2^m, q/2^m),$$

such that

$$x_0 \in R' \in \mathcal{R}^{\text{dyadic}}, \quad \operatorname{len}(R') < \delta, \tag{46}$$

$$\left| \frac{1}{|R'|} \int_{R'} f - f(x_0) \right| > \alpha, \tag{47}$$

$$|R' \cap G| > (1 - 4^{-\gamma})|R'|, \tag{48}$$

where γ is the number (38). Moreover, we may suppose

$$\nu_{k_t - 1} < n \le \nu_{k_t}, \quad \nu_{k_s - 1} < m \le \nu_{k_s}, \tag{49}$$

for some integers t, s. This and (38) imply that R' is a union of rectangles of the form

$$[(i-1)/2^{\nu_{k_t}}, i/2^{\nu_{k_t}}) \times [(j-1)/2^{\nu_{k_s}}, j/2^{\nu_{k_s}}) \in \mathcal{R}_{\Delta}^{\text{dyadic}},$$

and from (47) it follows that at least for one of these rectangles, say R'', we have

$$\left| \frac{1}{|R''|} \int_{R''} f - f(x_0) \right| > \alpha. \tag{50}$$

From (38) and (49) we get

$$|R''| = \frac{1}{2^{\nu_{k_t} + \nu_{k_s}}} \ge \frac{1}{2^{\nu_{k_t} + \nu_{k_s} - \nu_{k_t - 1} - \nu_{k_s - 1}}} \cdot \frac{1}{2^{n+m}} \ge |R'| \cdot 4^{-\gamma}.$$

From this and (48) we obtain $R'' \cap G \neq \emptyset$. Pick $x_1 \in R'' \cap G$. From (45) and (50) we get

$$\left| \frac{1}{|R''|} \int_{R''} f - f(x_1) \right| > \left| \frac{1}{|R''|} \int_{R''} f - f(x_0) \right| - |f(x_1) - f(x_0)| > \alpha/2. \tag{51}$$

On the other hand, $x_1 \in R'' \cap G$, $R'' \in \mathcal{R}_{\Delta}^{\text{dyadic}}$, $\operatorname{len}(R'') \le \operatorname{len}(R') < \delta_0$, and therefore by (44) we obtain

$$\left| \frac{1}{|R''|} \int_{R''} f - f(x_1) \right| < \alpha/2.$$

This contradiction completes the proof of the first part of our theorem.

Sufficiency. Now we suppose (2) does not hold, so there exists a sequence of integers $p_k \nearrow \infty$ such that

$$\lim_{k \to \infty} (\nu_{p_k + 1} - \nu_{p_k}) = \infty. \tag{52}$$

We may then find sequences of integers L_k and l_k, $k = 1, 2, \dots$, such that

$$l_{k+1} > l_k + \alpha(L_k), \quad k = 1, 2, \dots, \tag{53}$$

$$\nu_{p_k} < l_k < l_k + \alpha(L_k) < \nu_{p_k + 1}, \quad k = 1, 2, \dots, \tag{54}$$

$$L_{k+1} > 2^k \cdot (\beta(L_k) + k), \quad k = 1, 2, \dots, \tag{55}$$

where $\alpha(L)$ and $\beta(L)$ are the constants of Lemma 3. Applying Lemma 3 for $L = L_k$, $l = l_k$ and for the square

$$Q = Q_{ij}^k = [(i-1)/2^{l_k}, i/2^{l_k}) \times [(j-1)/2^{l_k}, j/2^{l_k}), \quad 1 \le i, j \le 2^{l_k},$$

we get functions $f_{ij}^k \in L^\infty(\mathbb{R}^2)$ satisfying

$$\operatorname{supp} f_{ij}^k \subset Q_{ij}^k, \tag{56}$$

$$\|f_{ij}^k\|_\infty \le \beta(L_k), \tag{57}$$

$$|\operatorname{supp} f_{ij}^k| \le 2|Q_{ij}^k|/\beta(L_k), \tag{58}$$

$$\operatorname{wd}(\operatorname{supp} f_{ij}^k) \ge 2^{-l_k-\alpha(L_k)}, \tag{59}$$

$$\int_R f_{ij}^k(x)\,dx = 0, \quad R \in \mathcal{R}^{\text{dyadic}},\ R \not\subset \mathring{Q}_{ij}^k, \tag{60}$$

and for any $x \in Q_{ij}^k$ there exists a dyadic rectangle $R_k(x) \subset Q_{ij}^k$ with

$$\operatorname{wd}(R_k(x)) \ge 2^{-l_k-\alpha(L_k)}, \tag{61}$$

$$\frac{1}{|R_k(x)|}\Big|\int_{R_k(x)} f_{ij}^k(t)\,dt\Big| \ge L_k. \tag{62}$$

Define

$$F_k(x) = \sum_{i,j=1}^{2^{l_k}} f_{ij}^k(x).$$

From (56)–(62) we conclude that

$$|\operatorname{supp} F_k| \le \frac{2}{\beta(L_k)}, \tag{63}$$

$$\operatorname{wd}(\operatorname{supp} F_k) \ge 2^{-l_k-\alpha(L_k)}, \tag{64}$$

$$\|F_k\|_\infty \le \beta(L_k), \tag{65}$$

$$\int_R F_k(x)\,dx = 0, \quad R \in \mathcal{R}^{\text{dyadic}},\ \operatorname{len}(R) \ge 2^{-l_k}, \tag{66}$$

and for any $x \in [0,1)^2$ there exists a dyadic rectangle $R_k(x) \subset [0,1)^2$ such that

$$2^{-l_k} > \operatorname{len}(R_k(x)) \ge \operatorname{wd}(R_k(x)) \ge 2^{-l_k-\alpha(L_k)}, \tag{67}$$

$$\frac{1}{|R_k(x)|}\Big|\int_{R_k(x)} F_k(t)\,dt\Big| \ge L_k. \tag{68}$$

Denote

$$F(x) = \sum_{k=1}^{\infty} \frac{F_k(x)}{2^k}. \tag{69}$$

From (63) and (54) it follows that $\|F_k\|_1 \le 2$, and so $\|F\|_1 \le 2$. Let $x \in [0,1)^2$. From (53) and (67) we get $\operatorname{len}(R_k(x)) \ge 2^{-l_{k+1}} \ge 2^{-l_j}$ if $j > k$. Thus, using (66), we obtain

$$\int_{R_k(x)} F_j(t)\,dt = 0, \quad j > k. \tag{70}$$

On the other hand, (65) and (55) imply

$$\left|\frac{1}{|R_k(x)|}\int_{R_k(x)} \sum_{j=1}^{k-1} \frac{F_j(t)}{2^j}\,dt\right| \le \beta(L_{k-1}) < L_k/2, \quad k \ge 2. \tag{71}$$

From (68), (70) and (71) we get

$$\left|\frac{1}{|R_k(x)|}\int_{R_k(x)} F(t)\,dt\right| \ge \frac{1}{|R_k(x)|}\left|\int_{R_k(x)} F_k(t)\,dt\right| - \frac{L_k}{2} > \frac{L_k}{2},$$

which yields

$$\limsup_{\operatorname{len}(R)\to 0,\, x\in R\in\mathcal{R}^{\mathrm{dyadic}}} \left|\frac{1}{|R|}\int_R F(t)\,dt\right| = \infty, \quad x \in [0,1)^2. \tag{72}$$

Now take an arbitrary rectangle $R \in \mathcal{R}^{\mathrm{dyadic}}_{\Delta}$. We have

$$\operatorname{len}(R) = 2^{-\nu_k} \ge \operatorname{wd}(R) = 2^{-\nu_t}. \tag{73}$$

From (66) we get

$$\int_R F_j(t)\,dt = 0 \quad \text{if } l_j \ge \nu_k. \tag{74}$$

On the other hand, if $l_j < \nu_k$, then from (54) it follows that $l_j + \alpha(L_j) < \nu_k$, and therefore by (64) we get

$$\operatorname{wd}(\operatorname{supp}(F_j)) \ge 2^{-l_j-\alpha(L_j)} \ge 2^{-\nu_k}. \tag{75}$$

Thus, using simple properties of dyadic rectangles, we conclude that

$$l_j < \nu_k,\ R \not\subset \operatorname{supp} F_j \ \Rightarrow\ R \cap \operatorname{supp} F_j = \emptyset. \tag{76}$$

Consider the sets

$$G_1 = \{x \in [0,1)^2 : \delta_{\mathcal{R}}(x, F_k) = 0,\ k = 1,2,\ldots\},$$
$$G_2 = \bigcup_{k=1}^{\infty} \bigcap_{j:\, l_j \ge \nu_k}^{\infty} ([0,1)^2 \setminus \operatorname{supp} F_j), \qquad G = G_1 \cap G_2.$$

Since $F_k(x)$ is bounded, the equality $\delta_{\mathcal{R}}(x, F_k) = 0$ holds almost everywhere, and so $|G_1| = 1$. From (63) it follows that $|G_2| = 1$ and therefore $|G| = 1$.

Take $x \in G$. We have

$$x \notin \operatorname{supp} F_j, \quad j > k_0, \tag{77}$$

for some k_0. Consider the rectangle $R \in \mathcal{R}_\Delta^{\text{dyadic}}$ such that $x \in R$. Suppose we have (73) and $k > k_0$. Then from (76) and (77) we get

$$R \cap \operatorname{supp} F_j = \emptyset \quad \text{if } j > k_0 \text{ and } l_j < \nu_k. \tag{78}$$

From (74) and (78) we conclude that

$$\frac{1}{|R|} \int_R F(t)\,dt = \sum_{j=1}^{k_0} \frac{1}{2^j \cdot |R|} \int_R F_j(t)\,dt.$$

Thus we obtain

$$\lim_{\operatorname{len}(R)\to 0,\, x\in R\in\mathcal{R}_\Delta^{\text{dyadic}}} \frac{1}{|R|} \int_R F(t)\,dt = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}. \tag{79}$$

On the other hand, (77) implies

$$F(x) = \sum_{j=1}^{k_0} \frac{F_j(x)}{2^j}. \tag{80}$$

From (72), (79) and (80) we conclude that $F \in \mathcal{F}(\mathcal{R}_\Delta^{\text{dyadic}}) \setminus \mathcal{F}(\mathcal{R}^{\text{dyadic}})$, which completes the proof of the theorem. ■

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