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PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

Constants of Motion in Deformed Oscilaltor and Coulomb Systems[1]

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Abstract—In this note we propose a unified description for the constants of motion for superintegrable deformations of the oscillator and Coulomb systems on N-dimensional Euclidean space, sphere and hyperboloid.

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The oscillator and Coulomb systems, as well as their generalizations to sphere and hyperboloid [1] are the most known and most important examples of bound superintegrable systems (the N-dimensional systems with $2N-1$ functionally independent constants of motion). The rational Calogero model is the most known nontrivial example of unbound superintegrable system. The superintegrability was established for the classical [3] and quantum [4] rational Calogero models, as well as for its generalization, associated with an arbitrary finite Coxeter group [5]. Few years ago in Ref. [6] it was proposed to construct the integrable deformations of the N-dimensional Euclidean oscillator and Coulomb systems by replacing their angular part by an $(N-1)$-dimensional integrable system:

$$\mathcal{H} = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2} + V(r), \tag{1}$$

where $\{p_r, r\} = 1$,

$$V_{osc}(r) = \frac{\omega^2 r^2}{2}, \quad V_{Coul}(r) = -\frac{\gamma}{r}. \tag{2}$$

Similar deformations of oscillator and Coulomb systems on N-dimensional sphere and (two-sheet) hyperboloid have also been proposed (here and through the text we assume the unit radius of the sphere and hyperboloid),

$$\begin{aligned} \mathbb{S}^N : \mathcal{H} &= \frac{p_\chi^2}{2} + \frac{\mathcal{I}}{\sin^2\chi} + V(\chi), \\ \mathbb{H}^N : \mathcal{H} &= \frac{p_\chi^2}{2} + \frac{\mathcal{I}}{\sinh^2\chi} + V(\chi) \end{aligned} \tag{3}$$

with $\{p_\chi, \chi\} = 1$ and $\mathcal{I}$ being a $(N-1)$-dimensional integrable system formulated in terms of the action-angle variables. The exact forms for the potential are:

$$\begin{aligned} \mathbb{S}^N : V_{osc}(\chi) &= \frac{\omega^2 \tan^2\chi}{2}, \\ V_{Coul}(\chi) &= -\gamma\cot\chi, \end{aligned} \tag{4}$$

$$\begin{aligned} \mathbb{H}^N : V_{osc}(\chi) &= \frac{\omega^2 \tanh^2\chi}{2}, \\ V_{Coul}(\chi) &= -\gamma\coth\chi. \end{aligned} \tag{5}$$

It appeared that the proposed deformations remain superintegrable if the "angular Hamiltonian" $\mathcal{I}$ has the following dependence on the action variables [7]

$$\mathcal{I} = \frac{1}{2}\left(\sum_{a=1}^{N-1} k_a I_a + c_0\right)^2, \quad k_a = \frac{n_a}{m_a}, \quad n_a, m_a \in \mathbb{N}. \tag{6}$$

Surprisingly, the "angular Calogero model", associated with arbitrary Coxeter root systems, analyzed from the various viewpoints in Refs. [9], has precisely such form. This leads to conclusion that the oscillator and Coulomb systems supplemented by the Calogero potential, preserve the superintegrability. For the A_N Calogero-Coulomb model and for the spherical generalizations of A_N Calogero-oscillator and Calogero-Coulomb problems, the explicit expressions of the hidden symmetry generators were presented in [8].

Note that the superintegrability has an explicit formulation in terms of the action-angle variables. Namely, consider an integrable N-dimensional system with the following Hamiltonian in action-angle variables:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}(k_1 I_1 + k_2 I_2, I_3, \ldots, I_N), \\ \{I_i, \Phi_j\} &= \delta_{ij}, \quad \Phi_i \in [0, 2\pi), \end{aligned} \tag{7}$$

[1] The article is published in the original.

where $k_{1,2}$ are integers. The system has a hidden symmetry, given by the additional constant of motion $K_{hidden} = \mathrm{Re}A(I_i)e^{\imath(m\Phi_1 - n\Phi_2)}$, where $A(I_i)$ is an arbitrary complex function. Respectively, for the Hamiltonian

$$\mathcal{H} = \mathcal{H}(k_1 I_1 + k_2 I_2 + \ldots k_N I_N), \tag{8}$$

where $k_1, \ldots, k_N$ are integer numbers, all the functions

$$K_{ij} = \mathrm{Re}A_{ij}(I)e^{\imath(k_j\Phi_i - k_i\Phi_j)} \tag{9}$$

are constants of motion distinct from the Liouville integrals. The Liouville integrals together with the additional $N-1$ integrals constitute a full set of $2N-1$ functionally independent constants of motion, ensuring the maximal superintegrability. This reasoning can be extended obviously to the case when k_i are rational numbers. In Ref. [6] it was revealed that the Hamiltonians of the deformed oscillator- and Coulomb-like systems have the following functional dependence on the action variables,

$$\mathcal{H}_{osc} = \mathcal{H}_{osc}(2I_r + \sqrt{2\mathcal{I}}),\quad \mathcal{H}_{Coul} = \mathcal{H}_{Coul}(I_r + \sqrt{2\mathcal{I}}). \tag{10}$$

Thus, having in mind that the Hamiltonians of superintegrable systems should be of the form (8), we deduce that the oscillator and Coulomb systems remain superintegrable being deformed by the angular Hamiltonian (6). This construction can be viewed as a higher-dimensional generalizations of the so-called Tremblay–Turbiner–Winternitz [10] and Post–Winternitz [11] systems and of their spherical/hyperbolic analogs [6]. These systems are superintegrable deformations of the two-dimensional oscillator and Coulomb problems with the Poschl–Teller type angular Hamiltonian

$$\mathcal{I}_{PT} = \frac{p_\varphi^2}{2} + \frac{k^2\alpha^2}{\sin^2 k\varphi} + \frac{k^2\beta^2}{\sin^2 k\varphi},\quad k \in \mathbb{N}. \tag{11}$$

Although these systems coincide with the two-dimensional rational Calogero-oscillator and Calogero-Coulomb systems associated with dihedral group [12], they were invented as an independent superintegrable models and attracted much attention (probably, because of their simplicity). In particular, M. Ranada suggested to describe them within the so-called κ-dependent formalism, which describes, in a uniform way, the systems on Euclidean plane, sphere and hyperboloid. He also suggested a very nice formulation of their symmetries calling it a "complex factorization" [13]. In this note we present the extension of Ranada's approach to the generic superintegrable deformations of oscillator and Coulomb systems.

Let us introduce the complex variables

$$u_a = \sqrt{I_a}e^{\imath\Phi_a},\quad z = p_r + \frac{\imath\sqrt{2\mathcal{I}}}{T},\quad \text{where } T = \begin{cases} r & \text{for } \mathbb{R}^N, \\ \tan\chi & \text{for } \mathbb{S}^N, \\ \tanh\chi & \text{for } \mathbb{H}^N. \end{cases} \tag{12}$$

These variables form the following nonvanishing Poisson brackets,

$$\{u_a, \bar{u}_b\} = \imath\delta_{ab},\quad \{z, \bar{z}\} = -\frac{\imath}{2\sqrt{2\mathcal{I}(u)}}(z - \bar{z})^2 + \kappa 2\imath\sqrt{2\mathcal{I}}, \tag{13}$$

$$\{z, u_a\} = -u_a k_a \frac{\imath(\bar{z} - z)}{2\sqrt{2\mathcal{I}}},\quad \{z, \bar{u}_b\} = \bar{u}_b k_b \frac{\imath(\bar{z} - z)}{2\sqrt{2\mathcal{I}}}, \tag{14}$$

where $\kappa = 0$ for $\mathbb{R}^N$, $\kappa = 1$ for $\mathbb{S}^N$, and $\kappa = -1$ for $\mathbb{H}^N$. Then the angular Hamiltonian reads

$$\mathcal{I} = \frac{1}{2}\left(\sum_{a=1}^{N-1} k_a u_a \bar{u}_a\right)^2. \tag{15}$$

In the absence of potential $V(r)$ the Hamiltonian takes the form

$$\mathcal{H}_0 = \frac{z\bar{z}}{2} + \kappa\mathcal{I}. \tag{16}$$

It possesses the $N-1$ functionally independent complex constants of motion:

$$\mathcal{M}_a = z^{n_a}(u_a)^{m_a},\quad \{\mathcal{M}_a, \mathcal{H}_0\} = 0. \tag{17}$$

Hence, this system is superintegrable with $2N-1$ functionally independent real constants of motion. When $\kappa = 0$, it possesses dynamical conformal symmetry $so(1,2)$

$$\{\mathcal{H}_0, \mathcal{D}\} = 2\mathcal{H}_0,\quad \{\mathcal{H}_0, \mathcal{K}\} = \mathcal{D},\quad \{\mathcal{K}, \mathcal{D}\} = -2\mathcal{K}, \tag{18}$$

$$\mathcal{H}_0 = \frac{z\bar{z}}{2},\quad \mathcal{D} = \sqrt{2\mathcal{I}(u)}\frac{z + \bar{z}}{\imath(\bar{z} - z)},\quad \mathcal{K} = \frac{4\mathcal{I}(u)}{(\imath(\bar{z} - z))^2}, \tag{19}$$

The superintegrable deformations of N-dimensional oscillator and Coulomb systems are:

$$\mathcal{H}_{osc} = \frac{z\bar{z}}{2} + \omega^2\frac{4\mathcal{I}(u)}{(\imath(\bar{z} - z))^2} + \kappa\mathcal{I},\quad \mathcal{H}_{Coul} = \frac{z\bar{z}}{2} - \frac{\imath\gamma(\bar{z} - z)}{2\sqrt{2\mathcal{I}}} + \kappa\mathcal{I}. \tag{20}$$

Their constants of motion reads

$$\begin{aligned} \mathcal{M}_{osc} &= \left(z^2 + \frac{2\omega^2\sqrt{2\mathcal{I}}}{(\imath(\bar{z} - z))^2} \right)^{n_a} u_a^{2m_a}, \\ \mathcal{M}_{Coul} &= \left(z - \frac{\imath\gamma}{\sqrt{2}(\sqrt{2\mathcal{I}})^{3/2}} \right)^{n_a} u_a^{m_a}. \end{aligned} \tag{21}$$

In summary, we have presented a unified formulation of superintegrable deformations of the N-dimensional oscillator and Coulomb problems on Euclidean space, sphere and hyperboloid. In the given formulation the space type (flat, spherical or hyperbolical) appears only in the Poisson bracket relations leading to the difference in symmetry algebra. We suppose to present the detail study of this formulation in forthcoming paper.

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