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SYSTEMS THEORY
AND GENERAL CONTROL THEORY

Control and Optimization in a Collision Avoidance Problem in Oscillating Systems

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Abstract—The development of control algorithms, including optimal control ones, in the collision avoidance problem for a system of two pendulums with a controllable common base is considered. Two problems are solved. The first one searches for the law of variation of the bounded control force that makes the system move from its initial state of rest to the given final state of rest during a finite time and ensures the pendulums do not collide in the process of oscillatory motions. The second problem searches for the performance-optimal law of variation of acceleration of the base and the bounded force that generates the acceleration. The algorithms for constructing the sought controls that use Kalman controllability conditions and Pontryagin's maximum principle method are presented. The dynamics of the system involved is simulated for the constructed control laws. The numerical results of both problems are compared to find that implementation of the developed performance-optimal control algorithm can help significantly decrease the releasing time of the pendulums while preventing a possible collision.

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INTRODUCTION

The development of control algorithms, including the optimal control ones, that ensure dynamic plants move towards their specified targets, while helping avoid collision is studied within the modern theory of controllable systems. What makes the subject topical is the importance of the issues, since they originate from many applied problems. For instance, such problems include moving pilotless vehicles from their initial positions to the required positions to avoid collisions or, in robotics, preventing the collision of manipulators or mobile robots for their group control when their active areas overlap. There are a significant number of works that deal with collision avoidance problems in various statements. We can conditionally split these works into two groups. The first group includes problems considered within the theory of differential games (approach–evasion problems). In these problems, one of the plants tends to approach the other at an inadmissible distance resulting in a collision, while the other, in contrast, tries to evade it and thus avoid a collision. The second group consists of problems treated as multidimensional problems of control theory, where both plants tend to their specified terminal states and avoid collision [1–6]. These problems are contiguous with problems of collision avoidance with an obstacle that one can interpret as control problems with phase restrictions taken into account [7–10]. It is logically connected with the survival problem [11], i.e., the problem of constructing the control that keeps the system's trajectory in the set given beforehand [12–14].

In this work, we consider two problems from the second group: the collision avoidance problems for the system of two physical pendulums with suspensions on a common moving base controlled by a bounded force. This system has multiple applications in the applied control problems of moving objects. The control problem for such system has been already considered in [15, 16], albeit, without the collision avoidance condition, which, here, means that the distance between the centers of masses of the pendulums is not greater than the given value. In the first problem, we develop the algorithm for constructing the law of the bounded control force so that the law ensures the system moves from its initial state of rest to the given state of rest during a finite time period without a collision of the pendulums. In the second problem, we develop the algorithm for constructing the law of variation of acceleration of the base's motion and the bounded control force that generates it such that the system moves from its state of rest by the given distance, with the oscillations damped, during the minimum possible time without a collision of the pendulums. We performed numerical simulation of the system's dynamics to prove the practical efficiency of the developed control algorithms. Some results of this work are presented in [17–19].

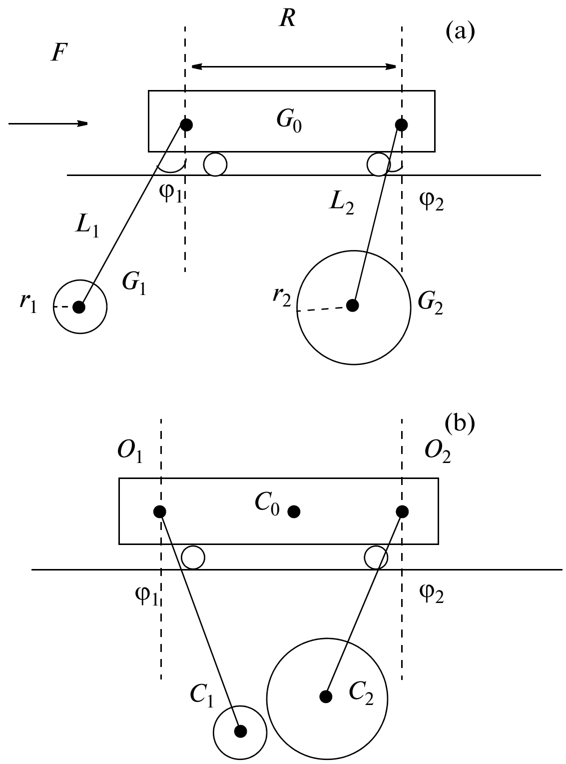

Fig. 1.

1. CALCULATION MODEL AND STATEMENT OF THE CONTROL PROBLEM

We consider the plane motion of a mechanical system consisting of three solid bodies G_0, G_1, G_2 (Fig. 1a). The body G_0 with mass m_0 can move steadily without friction along axis Ox, subjected to force F applied to body G_0. The bodies G_1, G_2 are physical pendulums with suspension points O_1 and O_2, respectively, on the moving base G_0. The pendulums consist of weightless rods and ball bobs of masses m_1 and m_2 rigidly connected to them, respectively. We use I_1 and I_2 to designate the moments of inertia of the pendulums with respect to the axes of the suspensions, L_1 and L_2 to designate the distances from the axes of suspensions O_1 and O_2 to the centers of inertia C_1 and C_2 of the bobs, $R = |O_1O_2|$ to designate the distance between suspensions, and r_i and r to designate the radii of the ball bobs and their sum, respectively, $r = r_1 + r_2 < R$, $L_1 + r_1 = L_2 + r_2$. Suppose the suspension points O_1 and O_2 are equidistant from the center of gravity C_0 of the body G_0: $|O_1C_0| = |C_0O_2| = R/2$. We do not take into account the resistance of the medium.

Bodies G_1, G_2 with masses m_1 and m_2 are subjected to the gravity force and the reaction force that make them oscillate in the plane xy. Force $F(t)$ can be directed parallel to the horizontal axis Ox, in the direction of increasing ($F > 0$) or decreasing ($F < 0$) of x. In order to describe the system's motion, we choose the abscissa x of the center of gravity of base G_0 and the angles φ_1 and φ_2 of deviation of the pendulums from the vertical axes as independent coordinates (we take the direction shown in Fig. 1a as the positive one).

The equations of oscillatory motion of the mechanical model are described by Lagrange's equations

$$
\begin{aligned}
&(m_0 + m_1 + m_2)\ddot{x} - m_1L_1\ddot{\varphi}_1\cos\varphi_1 - m_2L_2\ddot{\varphi}_2\cos\varphi_2 + m_1L_1\dot{\varphi}_1^2\sin\varphi_1 + m_2L_2\dot{\varphi}_2^2\sin\varphi_2 = F,\\
&I_1\ddot{\varphi}_1 - m_1L_1\ddot{x}\cos\varphi_1 = -m_1L_1g\sin\varphi_1,\\
&I_2\ddot{\varphi}_2 - m_2L_2\ddot{x}\cos\varphi_2 = -m_2L_2g\sin\varphi_2.
\end{aligned}
\tag{1.1}
$$

Suppose the restriction on the control force $F(t)$ is

$$
|F(t)| \le F^0, \quad t \in [0, T], \tag{1.2}
$$

where F^0 is the positive constant.

In the course of controllable oscillatory motion of the pendulums, the distance between the centers of inertia C_1 and C_2 of the ball bobs as the function of time $d = d(t)$ can be calculated in the moving coordinates C_0xy with the origin at the center of gravity C_0 of body G_0 by the formula

$$d(t) = \sqrt{[x_2(t) - x_1(t)]^2 + [y_2(t) - y_1(t)]^2}, \tag{1.3}$$

$$x_1(t) = -R/2 - L_1 \sin\varphi_1(t), \quad y_1(t) = L_1 \cos\varphi_1(t), \tag{1.4}$$

$$x_2(t) = R/2 - L_2 \sin\varphi_2(t), \quad y_2(t) = L_2 \cos\varphi_2(t), \tag{1.5}$$

where x_1, y_1, and x_2, y_2 are the projections of the centers of inertia of the ball bobs onto the axes C_0x and C_0y.

Suppose condition $d(0) = R > r_1 + r_2$ is met at the initial instant for distance (1.3). We restrict ourselves to considering small oscillations when the angles φ_i are small

$$|\varphi_i(t)| \le \varphi^*, \quad i = 1,2, \quad t \in [0,T], \tag{1.6}$$

and $\sin\varphi_i \approx \varphi_i$, $\cos\varphi_i \approx 1$. In (1.6), φ^* is the given variable.

In this case, Eqs. (1.1) take the form

$$(m_0 + m_1 + m_2)\ddot{x} - m_1L_1\ddot{\varphi}_1 - m_2L_2\ddot{\varphi}_2 = F(t), \quad t \in [0,T], \tag{1.7}$$

$$I_1\ddot{\varphi}_1 + m_1gL_1\varphi_1 = m_1L_1\ddot{x}, \quad I_2\ddot{\varphi}_2 + m_2gL_2\varphi_2 = m_2L_2\ddot{x}.$$

Using linearized formulas (1.4) and (1.5) $x_1 = -R/2 - L_1\varphi$, $x_2 = R/2 - L_2\varphi_2$, $y_1 = L_1$, and $y_2 = L_2$, we can write distance $d = d(t)$ (1.3) as

$$d(t) = \sqrt{(R - L_2\varphi_2(t) + L_1\varphi_1(t))^2 + (L_2 - L_1)^2}, \quad t \in [0,T]. \tag{1.8}$$

We say that for control $F(t)$ (1.2), the pendulums G_1 and G_2 collide if there exists instant $t^* \in [0,T]$ such that the distance between the centers of masses of the ball bobs equals the given number for the first time

$$d(t^*) = r, \quad r = r_1 + r_2; \tag{1.9}$$

they do not collide otherwise; i.e., if during the entire oscillating process, the distance between the centers of masses of the ball bobs is greater than the given number

$$d(t) > r, \quad t \in [0,T]. \tag{1.10}$$

P r o b l e m 1. Find control force $F(t)$ that moves system (1.7) from its initial state of rest

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \varphi_i(0) = 0, \quad \dot{\varphi}_i(0) = 0, \quad i = 1,2, \tag{1.11}$$

by the given distance a while damping its oscillations

$$x(T) = a, \quad \dot{x}(T) = 0, \quad \varphi_i(T) = 0, \quad \dot{\varphi}_i(T) = 0, \quad i = 1,2, \tag{1.12}$$

and ensures the no-collision condition (1.10) and the restrictions on the angles (1.6) are met. Here, T is the termination time of the process that is still unknown.

2. ALGORITHM FOR CONSTRUCTING THE CONTROL

We describe some characteristic positions of the system of pendulums that do collide. For the given L_1, L_2, r, φ^*, we consider the pair of angles $(\varphi_1^R, \varphi_2^R)$ such that

$$\varphi_1^R < 0, \quad \varphi_2^R > 0, \quad \varphi_2^R = -\varphi_1^R = \varphi^R, \quad 0 \le \varphi^R \le \varphi^*. \tag{2.1}$$

Pair $(\varphi_1^R, \varphi_2^R)$ (2.1) is matched with the value of parameter R that can be found from (1.9) given (1.8)

$$r < R \le R^*, \tag{2.2}$$

$$R = \varphi^R(L_1 + L_2) + 2\sqrt{r_1r_2}, \quad \text{if} \quad 0 \le \varphi^R < \varphi^*, \tag{2.3}$$

$$R^* = \varphi^*(L_1 + L_2) + 2\sqrt{r_1r_2}, \quad \text{if} \quad \varphi^R = \varphi^*. \tag{2.4}$$

Pair (R, φ^R) (2.3) is matched with the positions of the pendulums for which there is a collision (Fig. 1b) and pair (R^*, φ^*) (2.4) is matched with the limiting position of the system of pendulums for which there is collision.

If $R > R^*$, no control (1.2) can result in a collision since $d(t) > r$ for any $t \in [0, T]$. Indeed,

$$d^2(t) = (R - L_2\varphi_2(t) + L_1\varphi_1(t))^2 + (L_2 - L_1)^2 > (R^* - L_2\varphi_2(t) + L_1\varphi_1(t))^2 + (L_2 - L_1)^2$$

$$> (R^* - \varphi^*(L_1 + L_2))^2 + (L_2 - L_1)^2 \geq r^2.$$

Suppose, $r < R \leq R^*$. Then, in order to avoid a possible collision (1.9) in the course of the oscillatory motion, it is sufficient to meet the inequalities

$$|\varphi_i(t)| \leq \overline{\varphi}^R, \quad i = 1, 2, \quad t \in [0, T] \tag{2.5}$$

for any $\overline{\varphi}^R$ such that

$$0 < \overline{\varphi}^R < \varphi^R \tag{2.6}$$

when restriction (1.2) is met. This imposes the condition on time T of the process. Majorizing and simplifying the left-hand sides of inequalities (2.5) and (1.2), we can obtain the sufficient conditions on the finite time T.

2.1. Constructing the Law of Variation of the Control Acceleration without Taking into Account Restrictions on the Angles

We introduce new variables and designations

$$\omega_i^2 = m_i g L I_i^{-1}, \quad \varphi_t' = m_i L I_i^{-1}\varphi_i, \quad p_i = m_i^2 L^2 I_i^{-1}, \quad m = m_0 + m_1 + m_2. \tag{2.7}$$

In designations (2.7) (the primes are omitted in what follows), system (1.7) can be written as

$$m\ddot{x} - p_1\ddot{\varphi}_1 - p_2\ddot{\varphi}_2 = F, \tag{2.8}$$

$$\ddot{\varphi}_1 + \omega_1^2\varphi_1 = \ddot{x}, \tag{2.9}$$

$$\ddot{\varphi}_2 + \omega_2^2\varphi_2 = \ddot{x}. \tag{2.10}$$

Note that by (2.7), if $I_1 > I_2$ for the ball bobs, the oscillation frequencies of the pendulums satisfy the inequality $0 < \omega_1 < \omega_2$ and vice versa. In what follows, suppose the first case holds.

First, we consider Eqs. (2.9) and (2.10) only. After we introduce the variables

$$z_1 = x, \quad z_2 = \dot{x}, \quad u = \ddot{x}, \quad z_3 = \omega_1\varphi_1, \quad z_4 = \dot{\varphi}_1, \quad z_5 = \omega_2\varphi_2, \quad z_6 = \dot{\varphi}_2, \tag{2.11}$$

$$z^R = \begin{cases} \overline{\varphi}^R, & r < R \leq R^*, \\ \varphi^*, & R^* < R. \end{cases}$$

The equations of oscillatory motions (2.9) and (2.10), boundary conditions (1.11) and (1.12), and restrictions (2.5) take the form

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = u, \quad \dot{z}_3 = \omega_1 z_4, \quad \dot{z}_4 = -\omega_1 z_3 + u, \quad \dot{z}_5 = \omega_2 z_6, \quad \dot{z}_6 = -\omega_2 z_5 + u, \tag{2.12}$$

$$z_i(0) = 0, \quad i = \overline{1,6}, \tag{2.13}$$

$$z_1(T) = a, \quad z_i(T) = 0, \quad i = \overline{2,6}, \tag{2.14}$$

$$|z_3(t)| \leq \omega_1 z^R, \quad |z_5(t)| \leq \omega_2 z^R, \quad t \in [0, T]. \tag{2.15}$$

We introduce the designation

$$\Omega = \min(\omega_1, \omega_2 - \omega_1) > 0, \quad 0 < \omega_1 < \omega_2. \tag{2.16}$$

Our immediate aim is to find the control acceleration $u(t)$ that moves system (2.12) from state (2.13) to the final state (2.14) without taking into account phase restrictions (2.15).

Since for $\Omega > 0$ system (2.12) is completely controllable [15], then, following the Kalman approach [16], we search for the control acceleration $u(t)$ in the form of the linear combination of the proper motions of homogeneous system (2.12).

The fundamental matrix of solutions of homogeneous system (2.12) and its inverse matrix have the form

$$\Phi(t) = \begin{pmatrix} \Phi_1(t) & (0) \\ (0) & \Phi_2(t) \end{pmatrix}, \quad \Phi_1(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \Phi_2(t) = \begin{pmatrix} \cos\omega_1 t & \sin\omega_1 t & 0 & 0 \\ -\sin\omega_1 t & \cos\omega_1 t & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & \sin\omega_2 t \\ 0 & 0 & -\sin\omega_2 t & \cos\omega_2 t \end{pmatrix},$$

$$\Phi^{-1}(t) = \begin{pmatrix} \Phi_1^{-1}(t) & (0) \\ (0) & \Phi_2^{-1}(t) \end{pmatrix}, \quad \Phi_1^{-1}(t) = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad \Phi_2^{-1}(t) = \begin{pmatrix} \cos\omega_1 t & -\sin\omega_1 t & 0 & 0 \\ \sin\omega_1 t & \cos\omega_1 t & 0 & 0 \\ 0 & 0 & \cos\omega_2 t & -\sin\omega_2 t \\ 0 & 0 & \sin\omega_2 t & \cos\omega_2 t \end{pmatrix},$$

where (0) is the zero matrix of the respective dimension.

We introduce the matrix

$$R(t) = \int_0^t Q(\tau) Q^{\mathrm{T}}(\tau) d\tau, \tag{2.17}$$

$$Q(t) = \Phi^{-1}(t) B, \quad B = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^{\mathrm{T}}, \tag{2.18}$$

$$Q^{\mathrm{T}}(t) = \begin{pmatrix} -t & 1 & -\sin\omega_1 t & \cos\omega_1 t & -\sin\omega_2 t & \cos\omega_2 t \end{pmatrix},$$

where B is the vector of the coefficients for the control u in the system of Eqs. (2.12).

The elements R_{ij} of matrix $R(T)$ (2.17) have the following values

$$R_{11} = T^3/3, \quad R_{22} = T, \quad R_{33} = T/2 - (\sin 2T\omega_1)/(4\omega_1), \quad R_{44} = T/2 + (\sin 2T\omega_1)/(4\omega_1), \tag{2.19}$$

$$R_{55} = T/2 - (\sin 2T\omega_2)/(4\omega_2), \quad R_{66} = T/2 + (\sin 2T\omega_2)/(4\omega_2), \quad R_{12} = R_{21} = -T^2/2,$$

$$R_{13} = R_{31} = (\sin T\omega_1 - T\omega_1 \cos T\omega_1)/\omega_1^2, \quad R_{14} = R_{41} = -(T\omega_1 \sin T\omega_1 + \cos T\omega_1 - 1)/\omega_1^2,$$

$$R_{15} = R_{51} = (\sin T\omega_2 - T\omega_2 \cos T\omega_2)/\omega_2^2, \quad R_{16} = R_{61} = -(T\omega_2 \sin T\omega_2 + \cos T\omega_2 - 1)/\omega_2^2,$$

$$R_{23} = R_{32} = (\cos T\omega_1 - 1)/\omega_1, \quad R_{24} = R_{42} = (\sin T\omega_1)/\omega_1, \quad R_{25} = R_{52} = (\cos T\omega_2 - 1)/\omega_2,$$

$$R_{26} = R_{62} = (\sin T\omega_2)/\omega_2, \quad R_{34} = R_{43} = -(\sin T\omega_1)^2/(2\omega_1),$$

$$R_{35} = R_{53} = (\omega_2 \cos T\omega_2 \sin T\omega_1 - \omega_1 \cos T\omega_1 \sin T\omega_2)/(\omega_2^2 - \omega_1^2),$$

$$R_{36} = R_{63} = (\omega_1 \cos T\omega_1 \cos T\omega_2 + \omega_2 \sin T\omega_1 \sin T\omega_2 - \omega_1)/(\omega_1^2 - \omega_2^2),$$

$$R_{45} = R_{54} = (\omega_2 - \omega_2 \cos T\omega_1 \cos T\omega_2 - \omega_1 \sin T\omega_1 \sin T\omega_2 - \omega_1)/(\omega_1^2 - \omega_2^2),$$

$$R_{46} = R_{64} = (\omega_1 \cos T\omega_2 \sin T\omega_1 - \omega_2 \cos T\omega_1 \sin T\omega_2)/(\omega_1^2 - \omega_2^2),$$

$$R_{56} = R_{65} = -(\sin T\omega_2)^2/(2\omega_2).$$

Since system (2.12) is completely controllable and matrix $R(T)$ is positive definite for some $t = T > 0$; i.e., it is nonsingular [20].

We can write the expression for the control function $u(t)$ that moves system (2.12) from coordinate origin (2.13) of the phase space to the final state (2.14) as

$$u(t) = (\Phi^{-1}(t)B)^{\mathrm{T}} R^{-1}(T) z^*, \tag{2.20}$$

where

$$z^* = \Phi^{-1}(T)z^1 = z^1, \quad z^1 = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}},$$

and the solution of system (2.12) that corresponds to control (2.20) as

$$z(t) = W(t)R^{-1}(T)z^*, \quad W(t) = \Phi(t)R(t). \tag{2.21}$$

2.2. Meeting Restrictions on Angles

First, we show that, choosing a sufficiently large termination time T of the process, we can ensure restrictions (2.15) are met in the course of the motion under control (2.20). We use the theorem similar to Theorem 5.1 from [16].

T h e o r e m. Suppose for some $T > 0$ the matrix $R(T)$ is nonsingular and suppose for any six-dimensional vector v the following inequalities are met:

$$\|W_1(t)K(T)v\| \le \mu_1(T)\|v\|, \tag{2.22}$$

$$\|W_2(t)K(T)v\| \le \mu_2(T)\|v\| \quad \text{for} \quad t \in [0, T], \tag{2.23}$$

$$\|R(T)K(T)v\| \ge \lambda_i(T)\|v\|, \quad i = 1, 2, \tag{2.24}$$

where $W_1(t)$ and $W_2(t)$ are, respectively, the third and fifth rows of matrix (2.21), $K(T) = E_6$ is the unit matrix of dimension 6×6; $\mu_i(T)$, $\lambda_i(T)$, and $i = 1, 2$ are the positive scalars; and $\|\cdot\|$ is the Euclidean vector nor. Then, if the conditions

$$z^R\lambda_1(T)\mu_1^{-1}(T) \ge \|z^*\|\omega_1^{-1}, \quad z^R\lambda_2(T)\mu_2^{-1}(T) \ge \|z^*\|\omega_2^{-1} \tag{2.25}$$

are met, control (2.20) moves system (2.12) from state (2.13) to state (2.14) at instant T, while restrictions (2.15) are met.

Note that by condition (2.16),

$$\omega_i \ge \Omega, \quad \omega_1 + \omega_2 \ge 3\Omega. \tag{2.26}$$

Taking (2.26) into account, we have the following estimates for the elements of matrix (2.19)

$$|R_{13}| \le 1/\Omega^2 + T/\Omega, \quad |R_{14}| \le 2/\Omega^2 + T/\Omega, \quad |R_{15}| \le 1/\Omega^2 + T/\Omega, \quad |R_{16}| \le 2/\Omega^2 + T/\Omega, \tag{2.27}$$

$$|R_{23}| \le 2/\Omega, \quad |R_{24}| \le 1/\Omega, \quad |R_{25}| \le 2/\Omega, \quad |R_{26}| \le 1/\Omega, \quad |R_{34}| \le 1/(2\Omega), \quad |R_{35}| \le 1/(3\Omega),$$

$$|R_{36}| \le 5/(3\Omega), \quad |R_{45}| \le 5/(3\Omega), \quad |R_{46}| \le 5/(3\Omega), \quad |R_{56}| \le 1/(2\Omega).$$

We find $\lambda_i(T)$, $\mu_i(T)$, and $i = 1, 2$. We estimate the left-hand side of inequality (2.22); to do that, we apply the Cauchy-Bunyakovsky-Schwarz inequality and use estimates (2.27)

$$\|W_i(t)K(T)v\| \le \|W_i(t)\|\|v\| \le [(0.5 + 2/\Omega^2)T^2 + (6/\Omega^3 + 1/(2\Omega))T + 5/\Omega^4 + 821/(72\Omega^2)]^{1/2}\|v\|.$$

Hence, we can put

$$\mu_1(T) = \mu_2(T) = [(0.5 + 2/\Omega^2)T^2 + (6/\Omega^3 + 1/(2\Omega))T + 5/\Omega^4 + 821/(72\Omega^2)]^{1/2} \tag{2.28}$$

in (2.22). We estimate the left-hand side of inequality (2.23). For any vector v, we have

$$\|R(T)v\| = \|0.4T^3v + [R(T) - 0.4T^3E_6]v\| \ge 0.4T^3\|v\| - \|Mv\|, \quad M = R(T) - 0.4T^3E_6. \tag{2.29}$$

Here, we introduced the symmetric matrix M of dimension 6×6. By the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$\|Mv\|^2 \le \|v\|^2 \sum_{i,j=1}^{6} M_{i,j}^2.$$

Then, taking into account the symmetry of matrix M and the estimates for its elements obtained using relations (2.27) for the elements of matrix $R(T)$, we have

$$\|Mv\| \le g(T), \quad g(T) = [2T^6/75 + T^4/2 + 8T^2/\Omega^2 + 24T/\Omega^3 + 293/(9\Omega^2) + 20/\Omega^4]^{1/2}. \tag{2.30}$$

Using inequalities (2.29) and (2.30), we have

$$\|R(T)v\| \ge (0.4T^3 - g(T))\|v\|. \tag{2.31}$$

Hence, condition (2.34) is met if

$$0.4T^3 - g(T) > 0. \tag{2.32}$$

Studying it with the expression for the function $g(T)$ (2.30) taken into account, we can see that there exists number $T_0 > 0$ such that the latter inequality holds for all $T \in (T_0, +\infty)$.

Then, comparing (2.24) and (2.31), we have

$$\lambda_i(T) = 0.4T^3 - g(T) > 0, \quad i = 1, 2. \tag{2.33}$$

Substituting expressions (2.28) and (2.33) into inequalities (2.25), we have that restrictions (2.15) are met if we choose the finite time T from the following set

$$M = \bigcap_{i=0}^{2} M_i, \tag{2.34}$$

where

$$M_0 = \{T : T \in (T_0, +\infty), T_0 > 0\}, \quad M_i = \{T > 0 : z^R \lambda_i(T)\mu_i^{-1}(T) > \|z^*\|\omega_i^{-1}\}, \quad i = 1, 2. \tag{2.35}$$

2.3. Meeting the Restriction on the Control Function

Now, we discuss the choice of the variable $T \in M$ that ensures restriction (1.2) is met. From (2.7)–(2.9), we have

$$F(t) = (m - p_1 - p_2)u(t) + \omega_1^2 p_1 \varphi_1(t) + \omega_2^2 p_2 \varphi_2(t), \quad t \in [0, T]. \tag{2.36}$$

We estimate

$$\begin{aligned} |F| &= \left|(m - p_1 - p_2)u(t) + \omega_1^2 p_1 \varphi_1(t) + \omega_2^2 p_2 \varphi_2(t)\right| \\ &\le |(m - p_1 - p_2)u(t)| + \omega_1^2 p_1 |\varphi_1(t)| + \omega_2^2 p_2 |\varphi_2(t)| \le |(m - p_1 - p_2)u(t)| + (\omega_1^2 p_1 + \omega_2^2 p_2)\varphi^R \le F^0. \end{aligned} \tag{2.37}$$

We have from inequality (2.37)

$$|u(t)| \le u^0, \tag{2.38}$$

where

$$u^0 = (F^0 - (\omega_1^2 p_1 + \omega_2^2 p_2)\varphi^R)/|m - p_1 - p_2|.$$

Thus, for restriction (1.2) to meet for control acceleration (2.20), inequality (2.38) is sufficient to be met for all $t \in [0, T]$, i.e.,

$$|u(t)| = \left|Q^{\mathrm{T}}(t)R^{-1}(T)z^*\right| \le u^0 \quad Q(t) = \Phi^{-1}(t)B, \quad t \in [0, T]. \tag{2.39}$$

We can show that all hypotheses of Theorem 5.1 from [16] hold; the theorem states that if the following inequalities

$$\left\|Q^{\mathrm{T}}(t)K(T)v\right\| \le \mu_u(T)\|v\|, \quad t \in [0, T], \tag{2.40}$$

$$\|R(T)K(T)v\| \ge \lambda_u(T)\|v\|, \tag{2.41}$$

$$u^0 \lambda_u(T)\mu_u^{-1}(T) \ge \|z^*\| \tag{2.42}$$

are met for controllable system (2.12) and for any six-dimensional vector v, where $K(T) = E_6$, but $\lambda_u(T) > 0$ and $\mu_u(T) > 0$ are positive scalars, then restriction (2.39) is met for constructed control acceleration (2.20).

Indeed, since the left-hand sides of inequalities (2.24) and (2.41) coincide, we can take

$$\lambda_u(T) = \lambda(T) \tag{2.43}$$

as $\lambda_u(T)$.

In order to find $\mu_u(T)$, we estimate the left-hand side of inequality (2.40).

Applying the Cauchy-Bunyakovsky-Schwarz inequality and using expression (2.18) for the components of the vector $Q(t)$, we have

$$\left\|Q^{\mathrm{T}}(t)K(T)v\right\| \le \left\|Q^{\mathrm{T}}(t)\right\|\|v\| \le \sqrt{T^2+3}\|v\|.$$

Hence, we can take

$$\mu_u(T) = \sqrt{T^2+3} \tag{2.44}$$

in (2.40).

Since inequalities (2.40) and (2.41) are met for the found $\mu_u(T)$ and $\lambda_u(T)$, restriction (2.39) and, hence, restriction (2.38) hold if we choose time T from the set

$$M_3 = \{T > 0 : u^0\lambda_u(T)\mu_u^{-1}(T) \ge \|z^*\|\}. \tag{2.45}$$

Thus, for the constructed law of variation of acceleration (2.20) and respective control force (2.36) problem (1.2), (1.7), (1.11), and (1.12) can be solved if

$$T \in \bigcap_{i=0}^{3} M_i, \tag{2.46}$$

where M_i is found by (2.35) and (2.45).

2.4. Control Calculation and Numerical Simulation

In order to calculate the bounded control acceleration (2.20) and its generating force (2.8), we do the following.

(1) For the given parameters φ^*, m_i, L_i, r_i, $i = 1, 2$ of system (1.7), we use formulas (2.2)–(2.4) to find the values R and R^* and formulas (2.7) and for $I_i = m_i\left(L_i^2 + 2r_i^2/5\right)$ to find the parameters p_i, ω_i, and I_i, $i = 1, 2$.

(2) Next, we find sets M_0, M_i, $i = 1, 2$ (2.35), M_3 (2.45), and then set M (2.34).

(3) For the given a (1.12), we choose the finite time T from condition (2.46) and construct the law of variation of the control acceleration by formula (2.20).

(4) The control force on the interval $0 \le t \le T$ that corresponds to the constructed law of variation of acceleration is given by expression (2.36), where φ_1 and φ_2 are found by integrating system (2.8), (2.9), and (1.11) for acceleration (2.20).

We used this algorithm to perform the numerical simulation for system (1.7) with the following dimensional parameters

$$L_1 = 6.1 \text{ m}, \quad L_2 = 4.8 \text{ m}, \quad R = 3 \text{ m}, \quad m_0 = 50 \text{ kg}, \quad m_1 = 10 \text{ kg}, \quad m_2 = 15 \text{ kg}, \quad F^0 = 160 \text{ N}, \tag{2.47}$$

$$a = 4 \text{ m}, \quad g = 9.8 \text{ м s}^{-2}, \quad r_1 = 0.4 \text{ m}, \quad r_2 = 1.7 \text{ m}, \quad r = 2.1 \text{ m}, \quad \varphi^* = 0.17 \text{ rad}, \quad \varphi^R = 0.15 \text{ rad}.$$

Since by (2.2) $R < R^* = 3.5$ m, we have $z^R = \overline{\varphi}^R$ from (2.10).

Thus, the parameters we need for the calculation are

$$\overline{\varphi}^R = 0.12 \text{ rad}, \quad T = 35 \text{ s}, \quad M_0 = (11.88 \text{ s}, +\infty), \quad M_1 = (32.67 \text{ s}, +\infty),$$

$$M_2 = (31.96 \text{ s}, +\infty), \quad M_3 = (13.94 \text{ s}, +\infty).$$

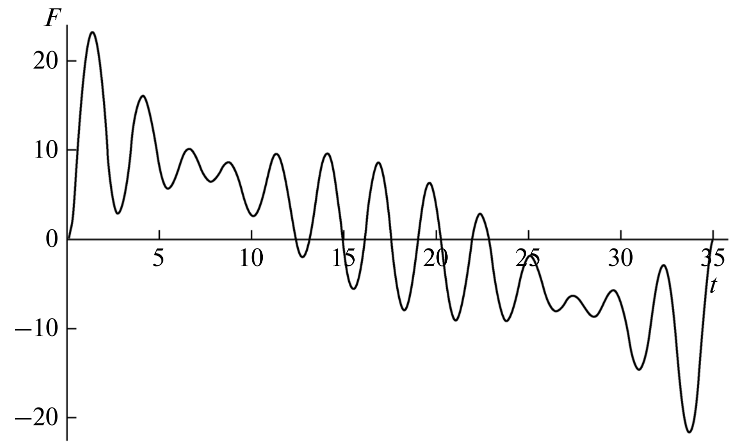

Fig. 2.

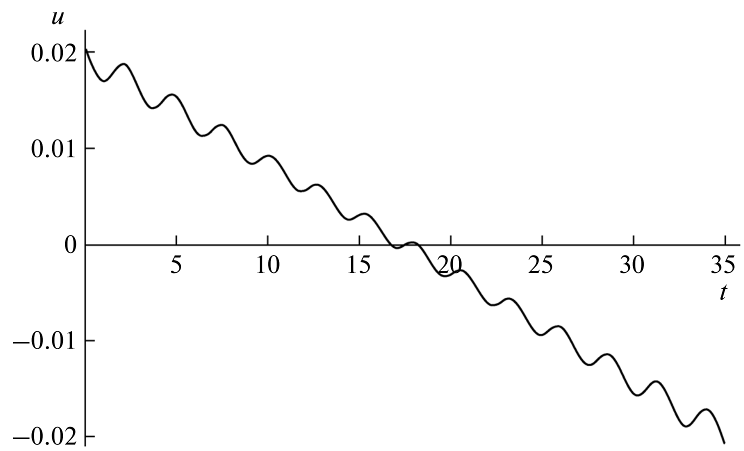

Fig. 3.

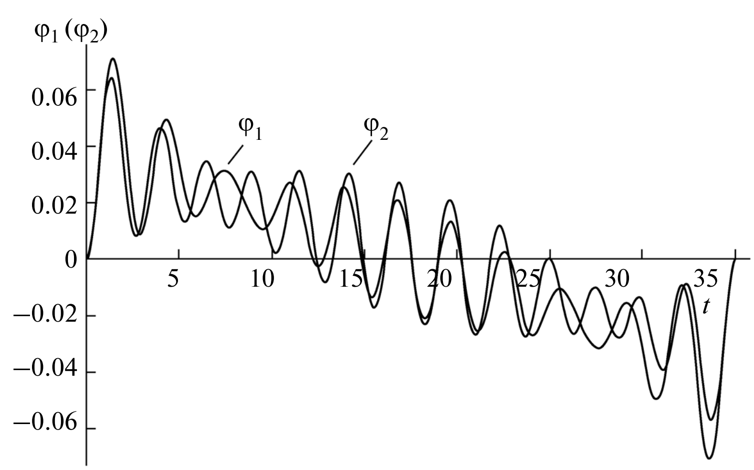

Fig. 4.

The control force, acceleration, and angles of the pendulums depending on time (Figs. 2–4, respectively) show that the proposed control algorithm is admissible in practice.

3. PERFORMANCE-OPTIMAL CONTROL PROBLEM

We consider system (1.7) under restrictions (1.2) and (1.6). Using designations (2.7) (with the primes omitted in what follows) and $u = \ddot{x}$, we write system (1.7) in the form

$$\ddot{x} = u, \tag{3.1}$$

$$\ddot{\varphi}_1 + \omega_1^2\varphi_1 = u, \quad \ddot{\varphi}_2 + \omega_2^2\varphi_2 = u, \tag{3.2}$$

$$mu - p_1\ddot{\varphi}_1 - p_2\ddot{\varphi}_2 = F. \tag{3.3}$$

We impose the restriction on the acceleration $u = \ddot{x}(t)$

$$|u(t)| \le u_0, \quad t \in [0, T], \tag{3.4}$$

where $u_0 > 0$ is the constant unknown for now.

We consider the following problem.

P r o b l e m 2. Find the law of variation of acceleration $u(t)$ (3.4) and its generating force $F(t)$ (3.3) that satisfies restriction (1.2) such that system (3.1) and (3.2) moves from its initial state of rest

$$x(0) = 0, \quad \dot{x}(0) = 0, \quad \varphi_i(0) = 0, \quad \dot{\varphi}_i(0) = 0, \quad i = 1, 2 \tag{3.5}$$

by the given distance a with its oscillations damped

$$x(T) = a, \quad \dot{x}(T) = 0, \quad \varphi_i(T) = 0, \quad \dot{\varphi}_i(T) = 0, \quad i = 1, 2 \tag{3.6}$$

during the minimal possible time T if the no-collision condition (1.10) and restrictions on angles (1.6) are met.

4. ALGORITHM FOR CONSTRUCTING PERFORMANCE-OPTIMAL CONTROL

In Section 2, we described some characteristic positions of a system of pendulums that collide. We noted that pair (R, φ^R) (2.3) is matched with the positions of the pendulums for which there is a collision (Fig. 1b) and pair (R^*, φ^*) (2.4) is matched with the limiting position of the system of pendulums for which there is a collision.

Thus, in Problem 2, as in Problem 1, when $r < R \le R^*$, we can avoid possible collision (1.9) if, in the process of oscillatory motion, we ensure inequalities (2.5) are met for any $\overline{\varphi}^R$ that corresponds to restriction (2.6).

If $R > R^*$, no control (1.2) can result in a collision since $d(t) > r$ for any $t \in [0, T]$.

Thus, the problem involved is reduced to finding the optimal law of variation of acceleration (3.4) in the performance problem for system (3.1), (3.2), (3.5), and (3.6) and the law of variation of the respective bounded control force (1.2) and (3.3) only for the restrictions on the angles of form (1.6) when $R > R^*$ and restrictions on the angles of form (2.5) when $r < R \le R^*$.

4.1. Optimal Control Acceleration in Performance Problem without Taking into Account Restrictions on Angles

We construct the optimal law of variation of acceleration in the linear performance problem for system (3.1), (3.2), and (3.4)–(3.6).

After we introduce the variables

$$z_1 = x, \quad z_2 = \dot{x}, \quad u = \ddot{x}, \quad z_3 = \varphi_1, \quad z_4 = \dot{\varphi}_1, \quad z_5 = \varphi_2, \quad z_6 = \dot{\varphi}_2, \tag{4.1}$$

Eqs. (3.1) and (3.2), boundary conditions (3.5) and (3.6), and restriction (3.4) take the form

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = u, \quad \dot{z}_3 = z_4, \quad \dot{z}_4 = -\omega_1^2 z_3 + u, \quad \dot{z}_5 = z_6, \quad \dot{z}_6 = -\omega_2^2 z_5 + u, \tag{4.2}$$

$$z_i(0) = 0, \quad i = \overline{1, 6}, \tag{4.3}$$

$$z_1(T) = a, \quad z_i(T) = 0, \quad i = \overline{2, 6}, \tag{4.4}$$

$$|u| \le u_0. \quad (4.5)$$

The solution to the performance-optimal control problem for system (4.2)–(4.5) exists, is unique, and can be found from the maximum principle if the following two conditions are met: the condition for the common position and the condition for the point $u = 0$ to belong to the interior of the domain of restrictions on the control [21]. One can easily see that the second condition is met. The condition for the common position is reduced to checking that the following determinant Δ formed of vector column $b, Ab, \ldots A^5 b$, where A is the matrix of linear system (4.2) and b is the vector of coefficients of control u

$$A = \begin{pmatrix} A_1 & (0) \\ (0)^{\mathrm{T}} & A_2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}^{\mathrm{T}},$$

is not zero.

In our case, $\Delta = -\omega_1^2 \omega_2^4 (\omega_1^2 - \omega_2^2)^2$. Since the frequencies ω_1, ω_2 are different, $\Delta \ne 0$. Hence, the maximum principle is the necessary and sufficient optimality condition [21].

The Hamilton function for the system for the performance problem is

$$H = q_1 z_2 + q_2 u + q_3 z_4 + q_4(u - \omega_1^2 z_3) + q_5 z_6 + q_6(u - \omega_2^2 z_5). \quad (4.6)$$

Here, q_i are conjugate variables that satisfy the system of equations

$$\dot{q}_1 = 0, \quad \dot{q}_2 = -q_1, \quad \dot{q}_3 = \omega_1^2 q_4, \quad \dot{q}_4 = -q_3, \quad \dot{q}_5 = \omega_2^2 q_6, \quad \dot{q}_6 = -q_5. \quad (4.7)$$

We find the optimal control $u(t)$ from the condition of the maximum of the Hamilton function

$$u(t) = u_0 \operatorname{sign}(q_2 + q_4 + q_6). \quad (4.8)$$

Thus, the problem of finding the optimal modes is reduced to solving the boundary-value problem for the system of differential Eqs. (4.2)–(4.4), into which we need to substitute the control from (4.8). The condition $H(T) \ge 0$ is also to be met, while the conjugate variables are not all to be identically zero.

Condition (4.8) yields that the optimal control $u(t)$ is a relay function that takes the values $\pm u_0$. The number of switching points and their positions are unknown and the expected result of the solving process.

Now, we construct the relay controls with five switching points for system (4.2)–(4.5) in the performance problem. We show that the control of the form

$$u = u_0 \quad \text{for} \quad t \in (0, t_1), \quad t \in (t_2, t_3), \quad t \in (t_4, t_5), \quad (4.9)$$
$$u = -u_0 \quad \text{for} \quad t \in (t_1, t_2), \quad t \in (t_3, t_4), \quad t \in (t_5, T)$$

helps move system (4.2) from its initial state (4.3) to its final state (4.4). Here, t_1, t_2, t_3, t_4, t_5 are the switching instants and T is the time when the process terminates.

We substitute control (4.9) into system (4.2) and integrate them for the initial conditions (4.3). Meeting boundary conditions (4.4) for $t = T$, we have the system of transcendental equations for $t_1, t_2, t_3, t_4, t_5, T$

$$t_1 - t_2 + t_3 - t_4 + t_5 = T/2, \quad t_1^2 - t_2^2 + t_3^2 - t_4^2 + t_5^2 = T^2/2 - a/u_0, \quad (4.10)$$
$$2(\cos\omega_i t_1 - \cos\omega_i t_2 + \cos\omega_i t_3 - \cos\omega_i t_4 + \cos\omega_i t_5) - 1 - \cos\omega_i T = 0,$$
$$2(\sin\omega_i t_1 - \sin\omega_i t_2 + \sin\omega_i t_3 - \sin\omega_i t_4 + \sin\omega_i t_5) - \operatorname{sins}\omega_i T = 0, \quad i = 1, 2.$$

Since we can permute the beginning and the end of the trajectories by inverting the motions, we search for the parameters t_1, t_2, t_3, t_4, t_5 so that control (4.9) is symmetric with respect to the instant $t = T/2$

$$t_1 = T/2 - \xi_1, \quad t_2 = T/2 - \xi_2, \quad t_3 = T/2, \quad t_4 = T/2 + \xi_2, \quad t_5 = T/2 + \xi_1. \quad (4.11)$$

Here, ξ_1 and ξ_2 are unknown constants. We substitute (4.11) into system (4.10). The first equation of (4.10) is satisfied identically, while the other equations, after certain transformations, can be reduced to

$$2(\xi_1^2 - \xi_2^2) = T^2/4 - a/u_0, \quad (4.12)$$

$$\cos(\omega_i T/2)[2\cos\omega_i\xi_1 - 2\cos\omega_i\xi_2 + 1 - \cos(\omega_i T/2)] = 0, \quad i = 1,2,$$

$$\sin(\omega_i T/2)[2\cos\omega_i\xi_1 - 2\cos\omega_i\xi_2 + 1 - \cos(\omega_i T/2)] = 0, \quad i = 1,2.$$

The latter two equations hold if the system

$$2\cos\omega_i\xi_1 - 2\cos\omega_i\xi_2 + 1 - \cos(\omega_i T/2) = 0, \; i = 1,2, \tag{4.13}$$

together with Eq. (4.12) has a solution with respect to ξ_1, ξ_2, and T.

Using (4.12), we can write Eqs. (4.13) as

$$2\cos\omega_i\xi_1 - 2\cos\omega_i\xi_2 - \cos[\omega_i(2\xi_1^2 - 2\xi_2^2 + a/u_0)^{1/2}] + 1 = 0, \quad i = 1,2. \tag{4.14}$$

For an arbitrarily given a, $u_0 > 0$ we find ξ_1 and ξ_2 from (4.14) and then T from (4.12). Then, we use (4.11) to find the sought switching instants.

Control (4.9) that moves system (4.2) from its initial state (4.3) to its final state (4.4) for any a, $u_0 > 0$ is constructed. In order to prove it is optimal, we specify the respective nonzero vector of conjugate variables. We consider the conjugate variables

$$q_1(t) = 1, \quad q_2(t) = T/2 - t, \tag{4.15}$$

$$q_3(t) = c_1\omega_1\cos\omega_1(T/2 - t), \quad q_4(t) = c_1\sin\omega_1(T/2 - t),$$

$$q_5(t) = c_2\omega_2\cos\omega_2(T/2 - t), \quad q_6(t) = c_2\sin\omega_2(T/2 - t),$$

where

$$c_1 = (\xi_1\sin\omega_2\xi_2 - \xi_2\sin\omega_2\xi_1)/(\sin\omega_1\xi_2\sin\omega_2\xi_1 - \sin\omega_1\xi_1\sin\omega_2\xi_2),$$

$$c_2 = (\xi_2\sin\omega_1\xi_1 - \xi_1\sin\omega_1\xi_2)/(\sin\omega_1\xi_2\sin\omega_2\xi_1 - \sin\omega_1\xi_1\sin\omega_2\xi_2).$$

Functions (4.15) satisfy system (4.7) and their respective control (4.8) is

$$u(t) = u_0\operatorname{sign}\{T/2 - t + c_1\sin\omega_1(T/2 - t) + c_2\sin\omega_2(T/2 - t)\}. \tag{4.16}$$

One can easily check that control (4.16) is switched at instants (4.11) and coincides with (4.8). Condition $H(T) \geq 0$ is also met.

Since the constructed control (4.16) given by (4.11), (4.12), and (4.14) satisfies the maximum principle for the linear performance problem (4.2)–(4.5), it is optimal for all $a > 0$ and $u_0 > 0$. This unique control is the solution to performance problem (4.2)–(4.5).

4.2. Meeting Restrictions on the Angles and the Control Force

Now, we discuss the choice of the variable $u^0 > 0$ that ensures restrictions (1.2) and (1.6) or (2.5) are met.

We eliminate $\ddot{\varphi}_1$ and $\ddot{\varphi}_2$ from (3.3) and represent the control force as

$$F(t) = (m - p_1 - p_2)u(t) + \omega_1^2 p_1\varphi_1(t) + \omega_2^2 p_2\varphi_2(t). \tag{4.17}$$

For the optimal acceleration law (4.9), we estimate the value of the control force (4.17) on the time interval $0 \leq t \leq T$

$$|F(t)| \leq |m - p_1 - p_2| u_0 + \omega_1^2 p_1 |\varphi_1(t)| + \omega_2^2 p_2 |\varphi_2(t)| \leq F^0. \tag{4.18}$$

We find the explicit estimates for the modules $|\varphi_1(t)|, |\varphi_2(t)|$ on the interval $0 \leq t \leq T$. Since the functions $\varphi_i(t)$ and $i = 1,2$ are symmetric with respect to $t = T/2$, it is sufficient to consider only the interval $0 \leq t \leq T/2$. Integrating system (4.2) successively on the intervals $[0,t_1), [t_1,t_2)$, and $[t_2,t_3)$ for control acceleration (4.9), we find

$$|\varphi_i(t)| = u_0 g^{-1}|1 - \cos\omega_i t| \leq 2u_0 g^{-1}, \quad t \in [0,t_1), \tag{4.19}$$

$$|\varphi_i(t)| = u_0 g^{-1}|2\cos\omega_i(t - t_1) - 1 - \cos\omega_i t| \leq 4u_0 g^{-1}, \quad t \in [t_1,t_2),$$

$$|\varphi_i(t)| \leq 6u_0 g^{-1}, \quad t \in [t_2,t_3).$$

It follows from (4.19) that

$$|\varphi_i(t)| \le 6u_0 g^{-1}, \quad i = 1,2, \quad t \in [0,T]. \tag{4.20}$$

Using (4.20), we have from (4.18)

$$|F(t)| \le [|m - p_1 - p_2| + 12(p_1 + p_2)]u_0 \le F^0, \quad t \in [0,T]. \tag{4.21}$$

Taking into account (4.20) and (4.21), we can conclude that to meet restrictions (1.2) and (1.6) or (2.5) in the performance problem involved, we need to choose the maximal value u^0 that ensures the inequalities hold

$$[|m - p_1 - p_2| + 12(p_1 + p_2)]u_0 \le F^0 \tag{4.22}$$

and

$$6u_0 g^{-1} \le \varphi^*, \quad i = 1,2, \quad \text{if} \quad R^* \le R \tag{4.23}$$

or

$$6u_0 g^{-1} \le \bar{\varphi}^R, \quad i = 1,2, \quad \text{if} \quad r < R \le R^*, \tag{4.24}$$

where $\bar{\varphi}^R$ is given from interval (2.6).

The sought value u^0 is found as follows:

$$u_0 = \min\{\varphi^* g/6,\ F^0 \big/ (12(p_1 + p_2) + |m - p_1 - p_2|)\} \tag{4.25}$$

for $R^* \le R$ and

$$u_0 = \min\{\bar{\varphi}^R g/6,\ F^0 \big/ (12(p_1 + p_2) + |m - p_1 - p_2|)\} \tag{4.26}$$

for $r < R \le R^*$.

4.3. Control Calculation and Numerical Simulation

In order to calculate the optimal law of variation (4.9) of acceleration and the bounded control force generating it (4.17), we do the following.

(1) For the given parameters φ^*, m_i, L_i, r_i, and $i = 1,2$ of system (1.7), we use formulas (2.2)–(2.4) to find the values R and R^* and formulas (2.7) and for $I_i = m_i(L_i^2 + 2r_i^2/5)$ to find the parameters p_i, ω_i, and I_i, $i = 1,2$.

(2) We calculate u^0 by (4.25) or (4.26), depending on whether relation $R^* \le R$ or $r < R \le R^*$ holds. In the second case, we can arbitrarily give the value of the parameter $\bar{\varphi}^R$ that satisfies restriction (2.6).

(3) For the given a (4.6) and found u^0 (4.25) and (4.26), we find the solutions of system (4.12) and (4.13) with respect to ξ_1, ξ_2, and T, and then find the switching instants of the optimal control law (4.9).

(4) The control force on the interval $0 \le t \le T$ that corresponds to the constructed law of acceleration is given by (4.17), where φ_1 and φ_2 are found by integrating system (4.2) and (4.5) for the optimal law of acceleration (4.9).

We used this algorithm to numerically simulate system (1.7) with dimensional parameters (2.47). These values are matched with case (4.26) since $R < R^* = 3.5$ m by (2.2).

For the chosen value $\bar{\varphi}^R$, the parameters required for calculating the control are

$$\varphi^R = 0.12 \text{ rad}, \quad \omega_1 = 1.27 \text{ s}^{-1}, \quad \omega_2 = 1.39 \text{ s}^{-1}, \quad p_1 = 4.99 \text{ kg},$$

$$p_2 = 9.52 \text{ kg}, \quad u_0 = 0.198 \text{ m s}^{-2}, \quad \xi_1 = 2.29 \text{ s}, \quad \xi_2 = 2.169 \text{ s}, \quad T = 9.228 \text{ s}.$$

Figures 5–7 give the optimal law of acceleration variation u, the bounded control force F, and the angles of the pendulums φ_1 and φ_2 depending on time and calculated by the described algorithm.

Figures 8a and 8b give the trajectories of the simulated oscillatory motions of two pendulums on the phase planes $(\varphi_1, \dot{\varphi}_1)$ and $((\varphi_2, \dot{\varphi}_2)$, respectively.

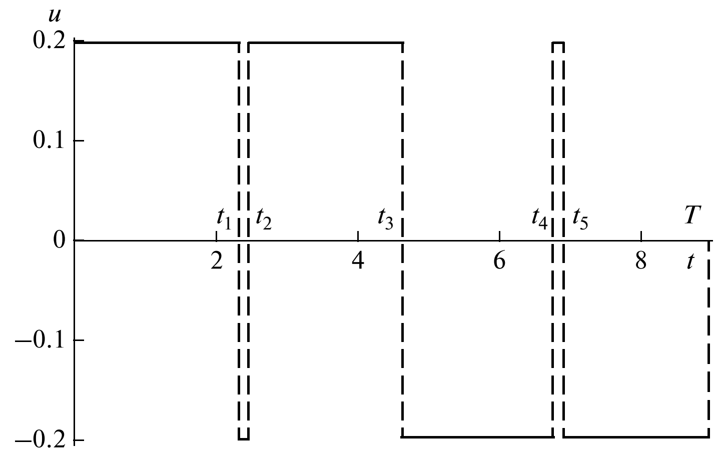

Fig. 5.

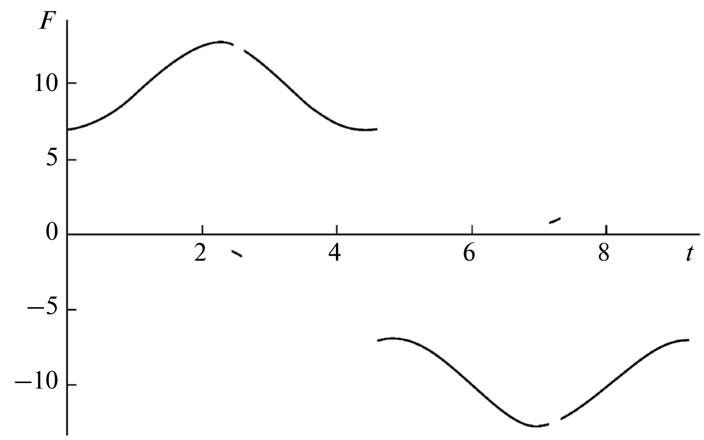

Fig. 6.

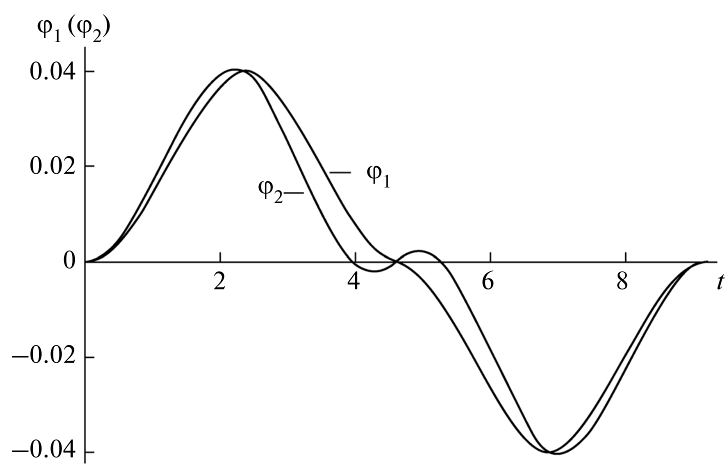

Fig. 7.

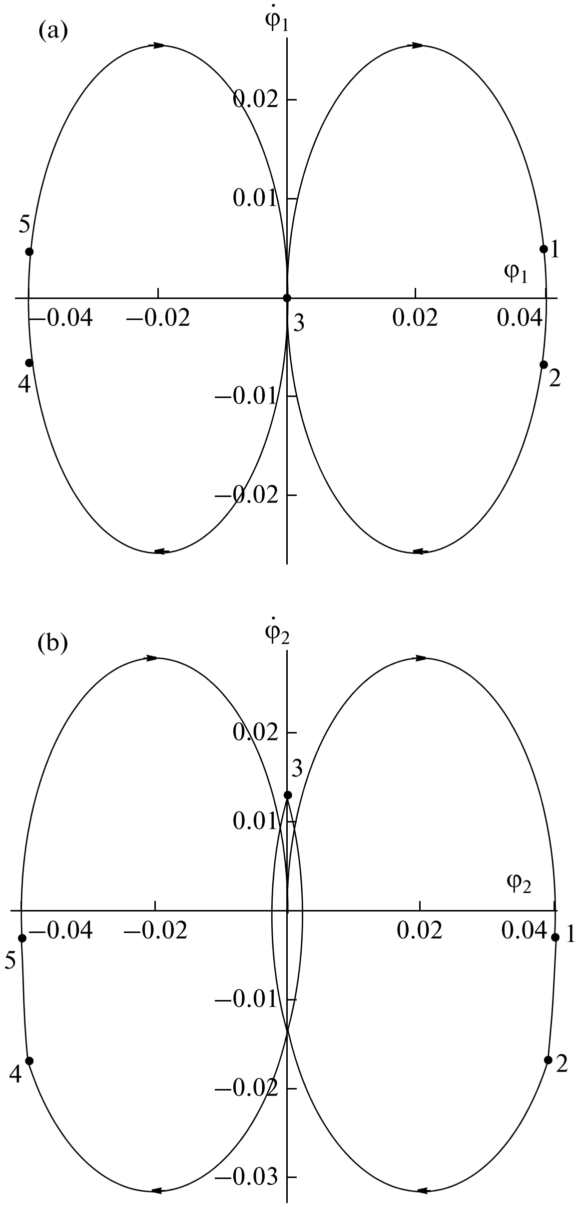

Fig. 8.

The bold points depict the state of the respective pendulum when the control acceleration is switched. Numbers 1–5 designate the successive numbers of switching instants (4.11) of acceleration (4.9), while the arrows designate the direction of motion of the depicting point. The minimal time it takes the pendulums to come from the initial state of rest to the final state of rest is $T = 9.228$ s. Comparing this result to time $T = 35$ s obtained in Section 2.3 for Problem 1, we can see that the developed algorithm helps significantly reduce the settling time of the pendulums, while avoiding a possible collision.

CONCLUSIONS

We developed a constructive approach to solving problem of avoiding a collision of two pendulums with a common moving base during their oscillatory motions via a bounded control force applied to the base. We constructed the control force law and obtained the sufficient condition such that this law ensures the system moves from its initial state of rest to the given state of rest during a finite time period so that the pendulums do not collide while moving, with the restriction on the control force being met. We performed the numerical simulation of the system's dynamics for the constructed control law; the results showed that the proposed calculation technique can be used in practice. We developed the algorithm for constructing the bounded control force and its respective law of variation of acceleration of the base's motion that ensure the system of two pendulums moves from its initial state of rest to the given final state of rest during the minimal time for dumping oscillations so that the pendulums do not collide while moving. We proved that the relay law of variation of the control acceleration with five switching points is optimal in the linear problem of the settling of the pendulum and solved the problem of choosing the maximal value of acceleration variation so that the restriction on the control force and restrictions on the angles of the pendulums that ensure there is no collision are met. The numerical simulation of the system's dynamics helped prove the practical efficiency, in terms of performance, of the developed control algorithm.

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