5.Возможные приложения предложенного метода.

Как уже указывалось, предложенный метод доказательства полноты на основе модифицированной Леммы Кальмара может быть легко обобщен на любые другие варианты k - значных логик при любом k≥3 с видоизменением и дополнением ряда фомул, приведенных в Утверждении пункта 3. в качестве выводимых, и даже на системы нечетких логик с учетом вышеприведенного замечания. Важно также, что этот метод позволяет построить формальные системы выводов Гильбертовского типа для всех указнных логик. Действительно, достаточно взять в качестве схем аксиом формулы из уже указанного списка выводимых формул, дополнив их рядом других (в зависимости от рассматриваемой логики) и правило Modus ponens. Аналогичная система для двузначной классической логики построена в [4]. Полнота и непротиворечивость этой системы просматривается непосредственно из самой аксиоматики.

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СИСТЕМЫ ФРЕГЕ НЕ МОНОТОННЫ

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FREGE SYSTEMS ARE NO MONOTONOUS

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АННОТАЦИЯ

В настоящей статье мы исследуем соотношение между сложностными характеристиками выводов в системах Фреге для минимальных тавтологий и результатов подстановок в них. Мы показываем, что существует много последовательностей пар минимальных тавтологий φ_n и формул ψ_n, являющихся результатом подстановок в φ_n таких, что для каждого n: 1) длины φ_n и ψ_n по порядку равны n, 2) количество шагов выводов ψ_n в системах Фреге ограничены константой, а длины тех же выводов ограничены линейной функцией от n, в то время как 3) количество шагов выводов φ_n в системах Фреге по порядку не менее n, а длины тех же выводов ограничены снизу функцией порядка n^2. Таким образом доказано, что результат подстановки в минимальную тавтологию может быть выведен в системах Фреге гораздо проще, чем сама минимальная тавтология, слеовательно системы Фреге не монотонны ни по шагам, ни по длинам выводов.

ABSTRACT

In this paper we investigate the relations between the Frege proof complexities of minimal tautologies and of results of substitutions in them. We show that there are many sequences of pairs of minimal tautologies φ_n and formulae ψ_n, which are the results of some substitution in φ_n such, that for every n 1) the sizes of φ_n and ψn are equal n by order, 2) the lines of Frege proofs for ψ_n are bounded by some constant and the sizes of Frege proofs for ψn are bounded by linear function in n, just as 3) the lines of Frege proofs for φ_n are at least n by order and the sizes of Frege proofs for φ_n are at least n^2 by order. So the result of substitution can be proved in Frege systems more easier than corresponding minimal tautology, therefore the Frege systems are no monotonous neither by lines nor by size.

Ключевые слова: *минимальная тавтология; системы Фреге; сложностные величины выводов; монотонные системы.*

Keywords: *minimal tautology; Frege systems; proof complexity measures; monotonous system.*

Introduction

The minimal tautologies, i.e. tautologies, which are not a substitution of a shorter tautology, play main role in proof complexity area. Really all propositional formulaes, proof complexities of which are investigated in many well known papers, are minimal tautologies. There is traditional assumption that minimal tautology must be no harder than any substitution in it. This idea was revised at first by Anikeev in

[1]. He has introduced the notion of monotonous proof system and has given two types of no complete propositional proof systems : monotonous system, in which the proof lines of all minimal tautologies are no more, than the proof lines for results of a substitutions in them, and no monotonous system, the proof lines of substituted formulas in which can be less than the proof lines of corresponding minimal tautologies. The analogous question about Frege systems was still open. In this paper we introduce for the propositional proof systems the notions of monotonous by lines and monotonous by sizes of proofs, and prove that Frege systems are no monotonous neither by lines nor by size.

2.Preliminaries

We will use the current concepts of a propositional formula, a classical tautology, Frege proof systems for classical propositional logic, proof and proof complexity [3]. Let us recall some of them.

A Frege system $\mathcal{F}$ uses a denumerable set of propositional variables, a finite, complete set of propositional connectives; $\mathcal{F}$ has a finite set of inference rules defined by a figure of the form $\frac{A_1A_2\ldots A_m}{B}$ (the rules of inference with zero hypotheses are the axioms schemes); $\mathcal{F}$ must be sound and complete, i.e. for each rule of inference $\frac{A_1A_2\ldots A_m}{B}$ every truth-value assignment, satisfying $A_1A_2\ldots A_m$, also satisfies B, and $\mathcal{F}$ must prove every tautology.

The particular choice of a language for presented propositional formulas is immaterial in this consideration. However, because of some technical reasons we assume that the language contains the propositional variables p, q and $p_i(i\geq 1)$, logical connectives $\neg$, $\supset$ and parentheses (,). Note that some parentheses can be omitted in generally accepted cases. We assume also that $\mathcal{F}$ has well known inference rule modus ponens.

By $|\varphi|$ we denote the size of a formula φ, defined as the number of all variable entries in it. It is obvious that the full size of a formula, which is understood to be the number of all symbols, or the number of all entries of logical signs, is bounded by some linear function in $|\varphi|$.

Definition 1. A tautolgy is called minimal if it is not a substitution of a shorter tautology.

We denote by $S(\varphi)$ the set of all formulas, every of which is result of some substitution in a minimal tautology φ.

2.1. Proof complexities

In the theory of proof complexity two main characteristics of the proof are: t- complexity, defined as the number of proof steps (lines), l-complexity, defined as total number of variable entries in proof (size) (formal definitions are for example in [4]).

Let ϕ be a proof system and φ be a tautology. We denote by $t_\varphi^\phi(l_\varphi^\phi)$ the minimal possible value of t-complexity (l-complexity) for all ϕ-proofs of tautology φ.

Definition 2. The proof system ϕ is called t-monotonous (l-monotonous) if for every minimal tautology φ and for all formulas $\Psi\in S(\varphi)$ $t_\varphi^\phi \leq t_\psi^\phi$ $(l_\varphi^\phi \leq l_\psi^\phi)$.

2.2. Essential subformulas of tautologies

For proving the main results we use also the notion of essential subformulas, introduced in [2].

Let F be some formula and $Sf(F)$ be the set of all non-elementary subformulas of formula F.

For every formula F, for every $\phi\in Sf(F)$ and for every variable p by F_φ^p is denoted the result of the replacement of the subformulas φ everywhere in F by the variable p. If $\phi\notin Sf(F)$, then F_φ^p is F.

We denote by $Var(F)$ the set of variables in F.

Definition 3. Let p be some variable that $p\notin Var(F)$ and $\phi\in Sf(F)$ for some tautology F. We say that φ is an essential subformula in F iff F_φ^p is non-tautology.

The set of essential subformulas in tautology F we denote by Essf(F), the number of essential subformulas - by Nessf(F) and the sum of sizes of all essential subformulas by Sessf(F).

If F is minimal tautology, then $Essf(F)=Sf(F)$.

In [2] the following statement is proved.

Proposition. Let F be a minimal tautology and $\phi\in Essf(F)$, then in every $\mathcal{F}$-proof of F subformula φ must be essential either at least in some axiom, used in proof or in formulae $A_1\supset(A_2\supset(\ldots\supset A_m)\ldots)\supset B$ for some used in proof inference rule $\frac{A_1A_2\ldots A_m}{B}$.

Remark 1. For every Frege system the number of mentioned essential subformulas is bounded with some constant.

Definition 4. The sequence of minimal tautolgies φ_n is called standard if 1) the size of φn is n by order, 2) $Nessf(\varphi_n)$ also is n by order and 3) $Seesf(\varphi_n)$ is n^2 by order.

There are many standard sequences, for example the sequence of formulas

$\varphi_n=p_1\supset(p_2\supset(p_3\supset\ldots\supset(p_n\supset p_1)..))$.

Remark 2.Frege proofs of every formula φn from standard sequence require by order n lines and n^2 sizes.

3. Main result

Here we give the main theorem, but at first we must give the following easy proved auxiliary statements.

Lemma 1. a) The minimal t-complexity and l-complexity of Frege proofs for formula $p\supset p$ are bounded by constant.

b) For each formulae A and B the minimal t-complexity of Frege proofs for formula $A\supset(\neg A\supset B)$ is bounded by constant, just as its minimal l-complexity is bounded by $c\bullet\max(|A|,|B|)$ for some constant c.

Lemma 2. If F is minimal tautology and p is some variable that $p\notin Var(F)$, then formula $p\supset F$ is minimal tautology also.

Lemma 3. If φ is some minimal tautology, p is some variable of φ and ψ is the result of substitution of some formula

A instead of p in φ, then $t_{\psi}^{F} \leq t_{\varphi}^{F}$ and $l_{\psi}^{F} \leq |A|\, l_{\varphi}^{F})$ for every Frege system $\mathcal{F}$.

Theorem. Every Frege system $\mathcal{F}$ is neither t-monotonous nor l-monotonous.

Proof. Let us consider the tautologies $\psi_n = p \supset \varphi_n$, where every φ_n is formula from some standard sequence and variable is not belong to $Var(\varphi_n)$. According to statement of Lemma 2. every formula ψ_n is minimal tautology. Let the minimal possible values of t-complexity and l-complexity of ψ_n in some fixed Frege system $\mathcal{F}$ be t(n) and l(n) accordingly.

Let us consider the tautologies:

$\alpha_n = \neg (p \supset p) \supset \varphi_n$ and

$\beta_n = (p \supset p) \supset \varphi_n$,

every of which belong to $S(\psi_n)$, therefore by Lemma 3. the minimal possible values of t-complexity and l-complexity of both α_n and β_n in $\mathcal{F}$ are t(n) and 2l(n). Now we can prove the formula $p \supset p$ and using proof of β_n can prove by modus ponens formula φ_n with no more than $t(n)+c_1$ lines and $2l(n)+c_2$ size for some constants c_1 and c_2. Using statements of Lemma 1. and remark 2., we obtain that t(n) and l(n) must be by order more than n and n^2 accordingly. According to statements of Lemma 1. we can prove every of the formulae $(p \supset p) \supset \alpha_n$ with no more than c_3 lines and no more than $c_4(n+1)$ sizes for some constants c_3 and c_4.. Again using the proof of formulae $p \supset p$, we can prove α_n with constant bounded lines and linear in n bounded sizes. □

4.Conclusion

We show that in Frege systems some of minimal tautologies (ψ_n) can proved more harder than some formulas from $S(\psi_n)$. Such formulae are for example α_n, which are results of some substitution in ψ_n.

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