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ON THE CONVERGENCE OF DOUBLE FOURIER SERIES OF FUNCTIONS OF BOUNDED PARTIAL GENERALIZED VARIATION

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The convergence of double Fourier series of functions of bounded partial Λ-variation is investigated. Sufficient and necessary conditions on the sequence $\Lambda = \{\lambda_n\}$ are found for the convergence of Fourier series of functions of bounded partial Λ-variation.

Key words and phrases: Double Fourier series, partial Λ-variation.

1. Classes of functions of bounded generalized variation

In 1881 Jordan [1] introduced the class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was generalized by many authors (quadratic variation, Φ-variation, Λ-variation ets., see [2]-[5]). In two dimensional case the class BV of functions of bounded variation was introduced by Hardy [6].

Let f be a real function of two variables of period 2π with respect to each variable. Given intervals $I = (a, b)$, $J = (c, d)$ and points x, y from $T := [0, 2\pi]$ we denote

$$f(I, y) := f(b, y) - f(a, y), \qquad f(x, J) = f(x, d) - f(x, c)$$

and

$$f(I, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).$$

Let $E = \{I_i\}$ be a collection of non-overlapping intervals from T ordered in arbitrary way and let Ω be the set of all such collections E. Denote by Ω_n the set of all collections $E \in \Omega$ with $|\Omega| = n$.

For a sequence of positive numbers $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ we denote

$$\Lambda V_1(f) = \sup_y \sup_{E\in\Omega} \sum_i \frac{|f(I_i, y)|}{\lambda_i} \quad (E = \{I_i\}),$$

$$\Lambda V_2(f) = \sup_x \sup_{F\in\Omega} \sum_j \frac{|f(x, J_j)|}{\lambda_j} \quad (F = \{J_j\}),$$

$$\Lambda V_{1,2}(f) = \sup_{F,\, E\in\Omega} \sum_i \sum_j \frac{|f(I_i, J_j)|}{\lambda_i \lambda_j}.$$

Definition 1. *We say that the function f has Bounded Λ-variation on $T = [0, 2\pi]^2$ and write $f \in \Lambda BV$, if*

$$\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) + \Lambda V_{1,2}(f) < \infty.$$

We say that the function f has Bounded Partial Λ-variation and write $f \in P\Lambda BV$ if

$$P\Lambda V(f) := \Lambda V_1(f) + \Lambda V_2(f) < \infty.$$

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty,\ n = 1, 2, \ldots$) the classes ΛBV and $P\Lambda BV$ coincide with the Hardy class BV and PBV respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{I_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,

$$1 < \lambda_1 \le \lambda_2 \le \ldots, \qquad \lim_{n\to\infty} \lambda_n = \infty. \tag{1.1}$$

In the case $\lambda_n = n,\ n = 1, 2\ldots$ we say *Harmonic Variation* instead of Λ-variation and write H instead of Λ (HBV, $PHBV$, $HV(f)$, ets).

The notion of Λ-variation was introduced by D. Waterman [4] in the one-dimensional case and A. Sahakian [10] in the two-dimensional case.

Definition 2. *Let Φ be a strictly increasing continuous function on $[0, \infty)$ with $\Phi(0) = 0$. We say that the function f has bounded partial Φ-variation on T^2 and write $f \in PBV_\Phi$, if*

$$V_\Phi^{(1)}(f) := \sup_y \sup_{\{I_i\}\in\Omega_n} \sum_{i=1}^{n} \Phi(|f(I_i, y)|) < \infty, \quad n = 1, 2, ...,$$

$$V_\Phi^{(2)}(f) := \sup_x \sup_{\{J_j\}\in\Omega_m} \sum_{j=1}^{m} \Phi(|f(x, J_j)|) < \infty, \quad m = 1, 2, ...$$

In the case when $\Phi(u) = u^p$, $p \ge 1$, the notion of bounded partial p-variation (class PBV_p) was introduced in [8].

Theorem 1. *Let* $\Lambda = \{\lambda_n \gamma_n\}$ *and* $\gamma_n \geq \gamma_{n+1} > 0, \; n = 1, 2, \ldots$.

1) If

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} < \infty, \tag{1.2}$$

then $P\Lambda BV \subset HBV$.

2) If, in addition, for some $\delta > 0$

$$\gamma_n = O(\gamma_{n^{[1+\delta]}}) \quad as \quad n \to \infty \tag{1.3}$$

and

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n} = \infty, \tag{1.4}$$

then $P\Lambda BV \not\subset HBV$.

Proof. 1) Let $f \in P\Lambda BV$ and

$$\sum_{i,j=1}^{\infty} \frac{|f(I_i, J_j)|}{ij} = \sum_{i \leq j} \frac{|f(I_i, J_j)|}{ij} + \sum_{i > j} \frac{|f(I_i, J_j)|}{ij} := I_1 + I_2.$$

Then according to (1.2),

$$\begin{aligned} I_1 &= \sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=i}^{\infty} \frac{|f(I_i, J_j)|}{j} \\ &\leq 2 \sum_{i=1}^{\infty} \frac{1}{i} \sup_x \sum_{j=i}^{\infty} \frac{|f(x, J_j)|}{\lambda_j} \frac{\lambda_j}{j} \\ &\leq 2 \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \sup_x \sum_{j=i}^{\infty} \frac{|f(x, J_j)|}{\lambda_j} \\ &\leq 2 \Lambda V_2(f) \sum_{i=1}^{\infty} \frac{\lambda_i}{i^2} \leq C \cdot \Lambda V_2(f) < \infty. \end{aligned}$$

Similarly, $I_2 \leq C \cdot \Lambda V_1(f) < \infty$.

2) For the proof of the second statement of Theorem 1 we use the following well-known lemma.

Lemma 1. *Let* u_i *and* v_i, $i = 1, 2, \ldots, j$ *be two increasing (decreasing) sequences of positive numbers. Then for any rearrangement* $\{\sigma(i)\}$ *of the set* $\{1, 2, \ldots, j\}$

$$\sum_{i=1}^{j} u_i v_{j-i+1} \leq \sum_{i=1}^{j} u_i v_{\sigma(i)} \leq \sum_{i=1}^{j} u_i v_i.$$

Let (1.3) and (1.4) be fulfilled. We define

$$f(x,y) := \begin{cases} t_j, & x = \frac{1}{i},\ y = \frac{1}{j},\ j < i \le j + m_j,\ i,j = 1,2,\dots \\ 0, & \text{otherwise,} \end{cases}$$

where

$$t_j := \left(\sum_{i=1}^{m_j} \frac{1}{\lambda_j} \right)^{-1}, \qquad m_j = \left[j^{1+\delta} \right], \quad j = 1,2,\dots \tag{1.5}$$

Let $x = 1/i$ and let $j(i)$ be the smallest integer satisfying

$$j(i) + m_{j(i)} \ge i. \tag{1.6}$$

Since t_j is decreasing and λ_j is increasing, using Lemma 1 we can write

$$\begin{aligned} &\sup_{F\in\Omega} \sum_{j=1}^{\infty} \frac{|f(1/i, J_j)|}{\lambda_j} \\ &= \sum_{j=j(i)}^{i-1} \frac{t_j}{\lambda_{j-j(i)}} \le t_{j(i)} \sum_{j=1}^{i-j(i)} \frac{1}{\lambda_j} \le t_{j(i)} \sum_{j=1}^{m_{j(i)}} \frac{1}{\lambda_j} = 1. \end{aligned}$$

Hence

$$\Lambda V_2(f) \le 1. \tag{1.7}$$

For $y = 1/j$ we have

$$\sup_{E\in\Omega} \sum_{i=1}^{\infty} \frac{|f(I_i, 1/j)|}{\lambda_i} = t_j \sum_{i=1}^{m_j} \frac{1}{\lambda_i} = 1.$$

Consequently,

$$\Lambda V_1(f) \le 1. \tag{1.8}$$

Combining (1.7) and (1.8) we conclude that $f \in P\Lambda BV$.

Now we prove that $f \notin HBV$. From (1.3) and (1.5) it follows that

$$\sum_{i=1}^{m_j} \frac{1}{\lambda_i} = \sum_{i=1}^{m_j} \frac{1}{i\gamma_i} \le C \frac{\log m_j}{\gamma_{m_j}} \le C \frac{\log j}{\gamma_j}.$$

Hence

$$t_j \cdot \log j \ge c\gamma_j, \qquad j = 2,3,\dots \tag{1.9}$$

and from the definition of f, (1.5) and (1.4) we obtain

$$\sup_{E,F\in\Omega}\sum_{i,j}\frac{|f(I_i,J_j)|}{ij}$$
$$\geq\sum_{j=1}^{\infty}\frac{t_j}{j}\sum_{i=j+1}^{j+m_j}\frac{1}{i}\geq c\sum_{j=1}^{\infty}\frac{t_j}{j}\log(j+m_j)\geq c\sum_{j=1}^{\infty}\frac{\gamma_j}{j}=\infty.$$

Theorem 1 is proved. □

Substituting $\lambda_n\equiv 1$ and $\lambda_n=n$ in Theorem 1, we obtain the following corollaries:

Corollary 1. *$PBV\subset HBV$ and $PHBV\not\subset HBV$.*

Corollary 2. *Let Φ and Ψ are conjugate functions in the sense of Yung ($ab\leq\Phi(a)+\Psi(b)$) and let for some $\{\lambda_n\}$ satisfying (1.1)*

$$\sum_{n=1}^{\infty}\Psi\left(\frac{1}{\lambda_n}\right)<\infty. \tag{1.10}$$

Then $PBV_\Phi\subset HBV$. In particular, $PBV_p\subset HBV$ for any $p\geq 1$.

Indeed, from the inequality $\frac{a}{\lambda}\leq\Phi(a)+\Psi(\frac{1}{\lambda})$ it follows that $PBV_\Phi\subset P\Lambda BV$ under assumption (1.10), and $P\Lambda BV\subset HBV$ if (1.1) holds.

Definition 3 ([9]). *The partial modulus of variation of a function f are the functions $v_1(n,f)$ and $v_2(m,f)$ defined by*

$$v_1(n,f):=\sup_y\sup_{\{I_i\}\in\Omega_n}\sum_{i=1}^{n}|f(I_i,y)|,\quad n=1,2,\ldots,$$

$$v_2(m,f):=\sup_x\sup_{\{J_k\}\in\Omega_m}\sum_{i=1}^{m}|f(x,J_k)|,\quad m=1,2,\ldots.$$

For functions of one variable the concept of the modulus variation was introduced by Chanturia [5].

Theorem 2. *If $f\in B$ is bounded on T^2 and*

$$\sum_{n=1}^{\infty}\frac{\sqrt{v_j(n,f)}}{n^{3/2}}<\infty,\quad j=1,2,$$

then $f\in HBV$.

Proof. Using Abel transformation we write

$$\begin{aligned}\sum_{k=1}^{m}\frac{|f(x,J_k)|}{k} &= \sum_{k=1}^{m-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\sum_{l=1}^{k}|f(x,J_l)|+\frac{1}{m}\sum_{k=1}^{m}|f(x,J_k)\\ &\leq \sum_{k=1}^{m-1}\frac{1}{k^2}\left(\sum_{l=1}^{k}|f(x,J_l)|\right)^{1/2}\left(\sum_{l=1}^{k}|f(x,J_l)|\right)^{1/2}+c\\ &\leq c\sum_{k=1}^{m-1}\frac{\sqrt{k}}{k^2}\left(\sum_{l=1}^{k}|f(x,J_l)|\right)^{1/2}+c\\ &\leq c\sum_{k=1}^{\infty}\frac{\sqrt{v_2(k,f)}}{k^{3/2}}+c\leq c<\infty.\end{aligned}$$

Consequently,

$$HV_2(f)<\infty. \tag{1.11}$$

Analogously, we can prove that

$$HV_1(f)<\infty. \tag{1.12}$$

Using Hardy transformation we obtain

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{|f(I_i,J_j)|}{ij}\\ &=\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\left(\frac{1}{i}-\frac{1}{i+1}\right)\left(\frac{1}{j}-\frac{1}{j+1}\right)\sum_{l=1}^{i}\sum_{s=1}^{j}|f(I_l,J_s)|\\ &+\frac{1}{n}\sum_{j=1}^{m-1}\left(\frac{1}{j}-\frac{1}{j+1}\right)\sum_{l=1}^{n}\sum_{s=1}^{j}|f(I_l,J_s)|\\ &+\frac{1}{m}\sum_{i=1}^{n-1}\left(\frac{1}{j}-\frac{1}{j+1}\right)\sum_{l=1}^{i}\sum_{s=1}^{m}|f(I_l,J_s)|\\ &+\frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}|f(I_i,J_j)|\\ &=I+II+III+IV.\end{aligned} \tag{1.13}$$

Since

$$\sum_{l=1}^{i}\sum_{s=1}^{j}|f(I_l,J_s)|\leq 2i\sup_x\sum_{s=1}^{j}|f(x,J_s)|\leq 2iv_2(j,f)$$

and

$$\sum_{l=1}^{i}\sum_{s=1}^{j}|f(I_l,J_s)|\leq 2j\sup_{y}\sum_{l=1}^{i}|f(I_l,y)|\leq 2jv_1(i,f),$$

we can write

$$\begin{aligned} I &\leq \sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\frac{1}{i^2j^2}\left(\sum_{l=1}^{i}\sum_{s=1}^{j}|f(I_l,J_s)|\right)^{1/2}\left(\sum_{l=1}^{i}\sum_{s=1}^{j}|f(I_l,J_s)|\right)^{1/2} \\ (1.14)\quad &\leq 2\sum_{i=1}^{n-1}\sum_{j=1}^{m-1}\frac{\sqrt{ijv_2(j,f)v_1(i,f)}}{i^2j^2} \\ &\leq 2\sum_{i=1}^{\infty}\frac{\sqrt{v_1(i,f)}}{i^{3/2}}\sum_{j=1}^{\infty}\frac{\sqrt{v_2(j,f)}}{j^{3/2}}<\infty, \end{aligned}$$

$$\begin{aligned} II &\leq \frac{1}{n}\sum_{j=1}^{m-1}\frac{1}{j^2}\left(\sum_{l=1}^{n}\sum_{s=1}^{j}|f(I_l,J_s)|\right)^{1/2}\left(\sum_{l=1}^{n}\sum_{s=1}^{j}|f(I_l,J_s)|\right)^{1/2} \\ (1.15)\quad &\leq \frac{1}{n}\sum_{j=1}^{m-1}\frac{\sqrt{njv_2(j,f)}}{j^2} \\ &\leq \frac{\sqrt{v_1(n,f)}}{\sqrt{n}}\sum_{j=1}^{\infty}\frac{\sqrt{v_2(j,f)}}{j^{3/2}}\leq c<\infty,\quad n=1,2,\ldots, \end{aligned}$$

Analogously, we can prove that

$$III\leq c<\infty, \tag{1.16}$$

$$IV\leq 2\sqrt{\frac{v_1(n,f)}{n}\frac{v_2(m,f)}{m}}\leq c<\infty,\ n,m=1,2,\ldots. \tag{1.17}$$

Combining (1.11) - (1.17), we conclude that $f\in HBV$. The proof of Theorem 2 is complete. □

2. Convergence of two-dimensional trigonometric Fourier series

Let $f\in L^1(T^2)$, $T^2:=[0,2\pi]^2$. The Fourier series of f with respect to the trigonometric system is the series

$$S[f]:=\sum_{m,n=-\infty}^{+\infty}\widehat{f}(m,n)e^{imx}e^{iny},$$

where

$$\widehat{f}(m,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}f(x,y)e^{-imx}e^{-iny}dxdy$$

are the Fourier coefficients of the function f. The rectangular partial sums are defined as follows:

$$S_{M,N}\left[f,(x,y)\right]:=\sum_{m=-M}^{M}\sum_{n=-N}^{N}\widehat{f}(m,n)\,e^{imx}e^{iny},$$

In this paper we consider convergence of **rectangular partial sums only** (convergence in the sense of Pringsheim) of double Fourier series.

We denote by $C(T^2)$ the space of continuous and 2π-periodic with respect to each variable functions with the norm

$$\|f\|_C:=\sup_{(x,y)\in T^2}|f(x,y)|.$$

For a function f defined on T^2 we denote by $f(x\pm 0,y\pm 0)$ the open coordinate quadrant limits (if exist) at the point (x,y) and denote

$$\begin{aligned}(2.1)\qquad &\sum f(x\pm 0,y\pm 0)\\ &=\left\{f(x+0,y+0)+f(x+0,y-0)+f(x-0,y+0)+f(x-0,y-0)\right\}.\end{aligned}$$

The well known Dirichlet-Jordan theorem (see [7]) states that the Fourier series of a function $f(x),\ x\in T$ of bounded variation converges at every point x to the value $\left[f(x+0)+f(x-0)\right]/2$. If in addition f is continuous on T, then the Fourier series converges uniformly on T.

Hardy [6] generalized the Dirichlet-Jordan theorem to the double Fourier series and proved that if a function $f(x,y)$ has bounded variation in the sense of Hardy ($f\in BV$), then $S[f]$ converges at any point (x,y) to the value $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$. If, in addition, f is continuous on T^2, then $S[f]$ converges uniformly on T^2.

Theorem S (Sahakian [10]). *The Fourier series of a function $f(x,y)\in HBV$ converges to $\frac{1}{4}\sum f(x\pm 0,y\pm 0)$ at any point (x,y), where the quadrant limits (2.1) exist. The convergence is uniform on any compact K, where the function f is continuous.*

Theorem S was proved in [10] under the assumption that the function f is continuous on some open set containing K while Sargsyan noticed in [11], that the continuity of f on the compact K is sufficient.

Analogs of Theorem S for higher dimensions can be found in [12] and [13]. Convergence of spherical and other partial sums of double Fourier series of functions of bounded Λ-variation was investigated in details by Dyachenko (see [14], [15] and references therein).

The first named author [9] has proved that if f is continuous and has bounded partial p-variation ($f \in PBV_p$) for some $p \in [1, +\infty)$, then $S[f]$ converges uniformly on T^2. Moreover, the following is true:

Theorem G (Goginava [9]). *Let $f \in C(T^2)$ and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \ j = 1, 2.$$

Then $S[f]$ converges uniformly on T^2.

Theorems 1, 2, Corollary 2 and Theorem S imply

Theorem 3. *Let $f \in P\Lambda BV$ with*

$$\sum_{j=1}^{\infty} \frac{\lambda_j}{j^2} < \infty, \quad \frac{\lambda_j}{j} \downarrow 0.$$

Then $S[f]$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point (x, y) where the quadrant limits (2.1) exist. The convergence is uniform on any compact K where the function f is continuous.

Theorem 4. *Let $f \in B$ and*

$$\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2.$$

Then $S[f]$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point (x, y) where the quadrant limits (2.1) exist. The convergence is uniform on any compact K where the function f is continuous.

Corollary 3. *Let $f \in B$ and let $v_1(k, f) = O(k^\alpha), v_2(k, f) = O(k^\beta)$, where $0 < \alpha, \beta < 1$. Then $S[f]$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point (x, y) where the quadrant limits (2.1) exist. The convergence is uniform on any compact K where the function f is continuous.*

Theorem 5. *Let $f \in PBV_p$, where $p \geq 1$. Then $S[f]$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point (x, y) where the quadrant limits (2.1) exist. The convergence is uniform on any compact K where the function f is continuous.*

From Theorem 3 it follows that for any $\delta > 0$ the Fourier series of the function $f \in P\left\{\frac{n}{\log^{1+\delta} n}\right\}BV$ converges to $\frac{1}{4}\sum f(x \pm 0, y \pm 0)$ at any point (x, y) where the quadrant limits (2.1) exist. Moreover, one can not take here

$\delta = 0$ (see Corollary 4). It is interesting to compare this result with the one obtained by M. Dyachenko and D. Waterman in [16].

For the sequence $\Lambda = \{\lambda_n\}_{n=1}^{\infty}$ denote

$$\Lambda^* V_1(f) = \sup_{\{y_i\}} \sup_{E \in \Omega} \sum_n \frac{|f(I_i, y_i)|}{\lambda_i},$$

$$\Lambda^* V_2(f) = \sup_{\{x_j\}} \sup_{F \in \Omega} \sum_m \frac{|f(x_j, J_j|}{\lambda_j}.$$

Definition 4 ([16]). *We say that the function f belongs to the class $\Lambda^* BV$, if*

$$\Lambda^* V(f) := \Lambda^* V_1(f) + \Lambda^* V_2(f) < \infty.$$

Observe that this definition differs from but is equivalent to that given in [16]. It is easy to see that $P\Lambda V(f) \leq \Lambda^* V(f)$ and hence $\Lambda^* BV \subset P\Lambda BV$.

Theorem DW ([16]). *If $f \in \left\{\frac{n}{\log n}\right\}^* BV$, then at any point (x, y) the quadrant limits (2.1) exist and the double Fourier series of f converges to $\frac{1}{4} \sum f(x \pm 0, y \pm 0)$.*

Moreover, the sequence $\left\{\frac{n}{\log n}\right\}$ can not be replaced with any sequence $\left\{\frac{n\alpha_n}{\log n}\right\}$, where $\alpha_n \to \infty$.

It is easy to show (see[16]) that $\left\{\frac{n}{\log n}\right\}^* BV \subset HBV$, hence the convergence part of Theorem DW follows from Theorem S. It is essential that the condition $f \in \left\{\frac{n}{\log n}\right\}^* BV$ guaranties the existence of quadrant limits.

The next theorem in particular shows that the condition $HV_{1,2}(f) < \infty$ in Theorem S is necessary, i.e the boundedness of partial harmonic variation is not sufficient for the convergence of Fourier series of continuous function.

Theorem 6. *Let $\Lambda = \{\lambda_n \gamma_n\}$, where γ_n is an decreasing sequence satisfying (1.3) and (1.4). Then there exists a continuous function $f \in P\Lambda BV$ such that*

$$\sup_N |S_{N,N}[f, (0,0)]| = \infty, \tag{2.2}$$

where $S_{N,N}[f, (x, y)]$ is the square partial sum of the Fourier series of f.

We need the following simple lemma that easily follows from Lemma 1.

Lemma 2. *Let the function $g(t)$ be defined on T and*

$$0 = t_1 < t_2 < \ldots < t_{2m} = 2\pi.$$

Suppose g is increasing on $[t_i, t_{i+1}]$ if i is odd and is decreasing, if i is even. If

$$|g(t_{i+1}) - g(t_i)| > |g(t_{i+2}) - g(t_{i+1})|, \quad i = 1, 2, \dots, 2m-2,$$

then

$$\Lambda BV(g) = \sum_{i=1}^{2m-1} \frac{|g(t_{i+1}) - g(t_i)|}{\lambda_i},$$

for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1).

Proof of Theorem 6. It is not difficult to see that for any sequence $\Lambda = \{\lambda_n\}$ satisfying (1.1) the class $C(T^2) \cap P\Lambda BV$ is a Banach space with the norm

$$\|f\|_* := \|f\|_C + P\Lambda V(f).$$

Let $\Lambda = \{\lambda_n\}$ be as in Theorem 6, and denote

$$A_{i,j} = \left[\frac{\pi i}{N+1/2}, \frac{\pi(i+1)}{N+1/2}\right) \times \left[\frac{\pi j}{N+1/2}, \frac{\pi(j+1)}{N+1/2}\right).$$

Define t_j and m_j as in (1.5) and consider the function

$$f_N(x,y) = \sum_{(i,j)\in W} t_j \, \chi_{A_{i,j}}(x,y) \sin\left(N + \frac{1}{2}\right) x \cdot \sin\left(N + \frac{1}{2}\right) y, \tag{2.3}$$

where $\chi_A(x,y)$ is the characteristic function of the set $A \subset T^2$ and

$$W := \{(i,j) : j < i < j + m_j, \quad 1 \le j < N_\delta\}, \qquad N_\delta = \left(\frac{N}{2}\right)^{\frac{1}{1+\delta}}.$$

Each summand in the sum (2.3) is continuous on the rectangle $A_{i,j}$ and vanishes on its boundary, hence $f_N \in C(T^2)$.

Next, in view of Lemma 2, using the same arguments as in the proof of (1.7) and (1.8), we get

$$\Lambda V_1(f_N) \le 1, \qquad \Lambda V_2(f_N) \le 1.$$

Hence $f_N \in P\Lambda BV$ and

$$\|f_N\|_* \le 3, \quad N = 1, 2, \dots. \tag{2.4}$$

Observe that $N_\delta < N$ and $j + m_j < N$, if $j < N_\delta$, hence $A_{i,j} \subset T^2$, if $(i,j) \in W$. Taking into account (1.5) and (1.9) we get

$$\begin{aligned}
\pi \cdot S_{N,N}[f_N, (0,0)] &= \int_{T^2} f_N(x,y) D_N(x) D_N(y) dxdy \\
&= \sum_{(i,j)\in W} t_j \int_{A_{i,j}} \frac{\sin^2\left(N+\frac{1}{2}\right)x \cdot \sin^2\left(N+\frac{1}{2}\right)y}{4 \sin\frac{x}{2} \sin\frac{y}{2}} dxdy \qquad (2.5) \\
&\ge c \sum_{j=1}^{N_\delta} \frac{t_j}{j} \sum_{i=j+1}^{j+m_j} \frac{1}{i} \ge c \sum_{j=1}^{N_\delta} \frac{t_j}{j} \log(j + m_j) \ge c \sum_{j=1}^{N_\delta} \frac{\gamma_j}{j} \to \infty
\end{aligned}$$

as $N \to \infty$, where c is an absolute constant.

Applying the Banach-Steinhaus Theorem, from (2.4) and (2.5) we obtain that there exists a continuous function $f \in P\Lambda BV$ satisfying (2.2) $\square$

Taking $\gamma_n = \frac{1}{\log n}$ in Theorem 6 we get

Corollary 4. *There exists a continuous function $f \in P\left\{\frac{n}{\log n}\right\}BV$ the Fourier series of which diverges at $(0,0)$.*

References

[1] C. Jordan, Sur la series de Fourier, *C.R. Acad. Sci. Paris* **92** (1881), 228–230.

[2] N. Wiener, The quadratic variation of a function and its Fourier coefficients, *Massachusetts J. Math.* **3** (1924), 72–94.

[3] J. Marcinkiewicz, On a class of functions and their Fourier series. *Compt. Rend. Soc. Sci. Warsowie* **26** (1934), 71–77.

[4] D. Waterman, On convergence of Fourier series of functions of generalized bounded variation, *Studia Math.* **44**, 1(1972), 107–117.

[5] Z. A. Chanturia, The modulus of variation of a function and its application in the theory of Fourier series, *Soviet. Math. Dokl.* **15** (1974), 67–71.

[6] G. H. Hardy, On double Fourier series and especially which represent the double zeta function with real and incommensurable parameters, *Quart. J. Math. Oxford Ser.* **37** (1906), 53–79.

[7] A. Zygmund, *Trigonometric Series*, Cambridge University Press, Cambridge, 1959.

[8] U. Goginava, Uniform convergence of Cesaro means of negative order of double Walsh-Fourier series, *J. Approx. Theory.* **124** (2003), 96–108.

[9] U. Goginava, On the uniform convergence of multiple trigonometric Fourier series, *East J. Approx.* **3**, 5(1999), 253–266.

[10] A. A. Sahakian, On the convergence of double Fourier series of functions of bounded harmonic variation, *Izv. Akad. Nauk Armyan. SSR Ser. Mat.* **21** 6(1986), 517–529 (in Russian); English translation: *Soviet J. Contemp. Math. Anal.* **21**, 6(1986), 1–13.

[11] O. G. Sargsyan, On the convergence and the Gibbs phenomenon of multiple Fourier series for functions of bounded harmonic variation, Izv. Akad. Nauk Armenii Mat. **28**, 3(1993), 3–20 (in Russian); English translation: *J. Contemp. Math. Anal.* **28** 3(1993).

[12] A. I. Sablin, Λ-variation and Fourier series, *Izv. Vyssh. Uchebn. Zaved. Mat.* **10** (1987), 66–68 (in Russian); English translation: *Soviet Math. (Iz. VUZ)* **31** (1987).

[13] A. N. Bakhvalov, Continuity in Λ-variation of functions of several variables and the convergence of multiple Fourier series, *Mat. Sb.* **193**, 12(2002), 3–20 (in Russian); English translation: *Sb. Math.* **193**, 11-12(2002), 1731–1748.

[14] M. I. Dyachenko, Waterman classes and spherical partial sums of double Fourier series, *Anal. Math.* **21** (1995), 3–21

[15] M. I. Dyachenko, Two-dimensional Waterman classes and u-convergence of Fourier series, *Mat. Sb.* **190**, 7(1999), 23–40 (in Russian).; English translation: *Sb. Math.* **190**, 7-8(1999), 955–972.

[16] M. I. Dyachenko and D. Waterman, Convergence of double Fourier series and W-classes, *Trans. Amer. Math. Soc.* **357** (2005), 397–407.

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