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ALGEBRA

Hyperidentities of Weakly Idempotent Lattices

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Abstract—The paper is devoted to the characterization of hyperidentities of the variety of weakly idempotent lattices that is nilpotent closure of the variety of lattices. The existence of a finite base for such hyperidentities is established.

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1. INTRODUCTION

Various extensions of classical concept of the lattice have been considered in literature. For instance, in papers [1], [2] the notion of a weakly associative lattice was introduced, while in papers [3] - [5] algebras with a system of identities were considered, which we call weakly idempotent lattices.

Definition 1.1. *An algebra with one binary operation* $(L; \wedge)$ *is called a weakly idempotent semilattice, if it satisfies the following identities:*

$$a \wedge b = b \wedge a \text{ (commutativity)}, \tag{1.1}$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c) \text{ (associativity)}, \tag{1.2}$$

$$a \wedge (b \wedge b) = a \wedge b \text{ (weak idempotency)}. \tag{1.3}$$

Adding the idempotency identity: $a \wedge a = a$, we obtain a semilattice. The set of all idempotent elements of each weakly idempotent semilattice forms a semilattice.

Definition 1.2. (see [3] - [5]) *An algebra* $(L; \wedge, \vee)$ *with two binary operations is called weakly idempotent lattice, if its reducts* $(L; \wedge)$ *and* $(L; \vee)$ *are weakly idempotent semilattices, and the following identities are satisfied :*

$$a \wedge (b \vee a) = a \wedge a, a \vee (b \wedge a) = a \vee a \text{ (weak absorption)}, \tag{1.4}$$

$$a \wedge a = a \vee a \text{ (equalization)}. \tag{1.5}$$

The set of all idempotents of a weakly idempotent lattice forms a lattice. There exist algebras that form weakly idempotent lattices but not lattices. For example the algebra $(Z \setminus \{0\}; \wedge, \vee)$, where $x \wedge y = (|x|, |y|)$ and $x \vee y = [|x|, |y|]$, for which $(|x|, |y|)$ and $[|x|, |y|]$ are the greatest common divisor and the least common multiple of elements $|x|$ and $|y|$, respectively forms a weakly idempotent lattice but not a lattice, because for a negative x we have $x \wedge x \neq x$. A weakly idempotent lattice $(L; \wedge, \vee)$ is said to be distributive if it satisfies both distributivity identities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. To each weakly idempotent lattice corresponds a quasi-order θ, defined by: $x\theta y \leftrightarrow x \wedge y = x \wedge x$.

Note that the operations of a weakly idempotent lattice preserve its quasi-order. Recall that a hyperidentity is a second-order formula of the form $\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2)$, where $X_1, \ldots, X_m$ are

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functional variables, while $x_1, \dots, x_n$ are object variables in words (terms) w_1, w_2. The hyperidentities usually are presented without quantifier prefix, that is, as the equality: $w_1 = w_2$. A hyperidentity $w_1 = w_2$ is said to be satisfied in the algebra $(Q; F)$, if this equality holds whenever every functional variable and every object variable are replaced by an arbitrary operation of the corresponding arity from F and by an arbitrary element from Q respectively (see [6] − [8]).

It is clear that a weakly idempotent lattice $L = (L; \wedge, \vee)$ is distributive if and only if in L the hyperidentity $X(Y(x, y), z) = Y(X(x, z), X(y, z))$ is satisfied.

Characterizations of hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean and De Morgan algebras given in papers [7]− [12]. For a basis of hyperidentities in termal (polynomial) algebras we refer to [13] − [15], while applications of hyperidentities in discrete mathematics can be found in [16].

A hyperidentity is said to be satisfied in the variety V, if it is satisfied in each algebra of the variety V. In this case, the hyperidentity is called hyperidentity of the variety V. A hyperidentity (or identity) $w_1 = w_2$ is said to be homogeneous (or regular in the sense of A. I. Mal'cev), if the words w_1 and w_2 contain the same object variables. Each hyperidentity of the variety of weakly idempotent lattices is homogeneous. In this paper we characterize hyperidentities of the variety of weakly idempotent lattices.

2. PLONKA NONIDEMPOTENT FUNCTIONS

An algebra $\mathfrak{U} = (U; \Sigma)$ is said to be a sum of its mutually disjoint subalgebras $(U_i; \Sigma)$, where $i \in I$, if the following conditions are satisfied (cf. [17]-[19]):
i) $U_i \cap U_j = \emptyset$ for all $i, j \in I, i \neq j$; ii) $U = \bigcup_{i \in I} U_i$;
iii) On the set of indices I there is a relation "$\leq$" such that $(I; \leq)$ is an upper semilattice possessing the following properties:
iv) if $i \leq j$, then there exists a homomorphism $\varphi_{i,j} : (U_i; \Sigma) \mapsto (U_j; \Sigma)$, where $\varphi_{i,i}(x) = F_t(x, \dots, x)$ for any operation $F_t \in \Sigma$, $x \in U_i$ and $\varphi_{i,j} \cdot \varphi_{j,k} = \varphi_{i,k}$, $i \leq j \leq k$;
v) for all $A \in \Sigma$ and for all $x_1, \dots, x_n \in Q$ we have $A(x_1, \dots, x_n) = A(\varphi_{i_1,i_0}(x_1), \dots, \varphi_{i_n,i_0}(x_n))$, where the arity $|A| = n, x_1 \in U_{i_1}, \dots, x_n \in U_{i_n}, i_1, \dots, i_n \in I, i_0 = sup\{i_1, \dots, i_n\}$.

Note that a homogeneous identity, which holds in all mutually disjoint subsystems, remains valid also in their sum. This follows from item v) of the definition. Therefore, each homogeneous hyperidentity that is valid in all mutually disjoint subalgebras, remains valid also in their sum.

Definition 2.1. *Let $\mathfrak{U} = (U; \Sigma)$ be an arbitrary algebra. A binary function $f : U \times U \to U$ is called Plonka nonidempotent function of $\mathfrak{U}$, if it satisfies the following identities (cf. [17]-[19]):*

1.$f(f(x, y), z) = f(x, f(y, z))$; 2.$f(x, f(y, z)) = f(x, f(z, y))$;

3.$f(x, x) = F_t(x, \dots, x)$ for any operation $F_t \in \Sigma$;

4.$f(F_t(x_1, \dots, x_{n(t)}), y) = F_t(f(x_1, y), \dots f(x_{n(t)}, y))$ for any operation $F_t \in \Sigma$;

5.$f(y, F_t(x_1, \dots, x_{n(t)})) = f(y, F_t(f(y, x_1), \dots f(y, x_{n(t)})))$ for any operation $F_t \in \Sigma$;

6.$f(F_t(x_1, \dots, x_{n(t)}), x_i) = F_t(x_1, \dots, x_{n(t)})$ (for all $1 \leq i \leq n(t)$) for any operation $F_t \in \Sigma$;

7.$f(F_t(x_1, \dots, x_{n(t)}), F_t(x_1, \dots, x_{n(t)})) = F_t(x_1, \dots, x_{n(t)})$ for any operation $F_t \in \Sigma$.

8. $f(x, f(x, y)) = f(x, y)$.

Theorem 2.1. *To each Plonka nonidempotent function of the algebra $\mathfrak{U} = (U; \Sigma)$ corresponds a representation of $\mathfrak{U}$ in the form of sum of its mutually disjoint subalgebras.*

Proof. On the set U define the relation $\alpha \subseteq U \times U$ by $a \alpha b \leftrightarrow f(a, b) = f(a, a), \quad f(b, a) = f(b, b)$, where f is the Plonka nonidempotent function of given algebra $\mathfrak{U}$. Observe that α is an equivalence relation on the set U, and denote by $U_i, i \in I$ the corresponding equivalence classes. Thus, we obtain a partition of the set U into mutually disjoint subsets $U_i \subseteq U, i \in I$. We show that U_i are subalgebras. Indeed, if $a_1, \dots, a_{n(t)} \in U_i$, $i \in I$, then for any $F_t \in \Sigma$, ($|F_t| = t$) we have:

$$f(F_t(a_1, \dots, a_{n(t)}), a_1) \stackrel{6}{=} F_t(a_1, \dots a_{n(t)}) \stackrel{7}{=} f(F_t(a_1, \dots a_{n(t)}), F_t(a_1, \dots, a_{n(t)}));$$

$$f(a_1, F_t(a_1, \dots, a_{n(t)})) \stackrel{5}{=} f(a_1, F_t(f(a_1, a_1), \dots f(a_1, a_{n(t)})))$$

$$\stackrel{2}{=} f(a_1, F_t(F_t(a_1, \dots, a_1), \dots, F_t(a_1, \dots, a_1))) \stackrel{2,7}{=} f(a_1, F_t(a_1, \dots, a_1)) = F_t(a_1, \dots, a_1) \stackrel{2}{=} f(a_1, a_1),$$

that is, $F_t(a_1, \dots, a_{n(t)}), a_1 \in U_i$.

Next, on the set of indices I define the order "$\leq$" as follows: $i_1 \leq i_2$ if and only if there exist $a \in U_{i_1}$, $b \in U_{i_2}$ such that $f(b,a) = f(b,b)$. Note that this definition converts the set I into an upper semilattice. Finally, we define the mapping $\varphi_{i_1,i_2} : U_{i_1} \mapsto U_{i_2}$ for $i_1 \leq i_2$ as $\varphi_{i_1,i_2}(a) = f(a,b)$, where $b \in U_{i_2}, a \in U_{i_1}$, and the result follows. Theorem 2.1 is proved.

3. ON SUBDIRECTLY IRREDUCIBLE WEAKLY IDEMPOTENT QUASILATTICES

Definition 3.1. *A binary algebra $\mathfrak{U} = (U; \Sigma)$ is called a weakly idempotent quasilattice, if it satisfies the following hyperidentities:*

$$X(x,x) = Y(x,x), \tag{3.1}$$
$$X(x,y) = X(y,x), \tag{3.2}$$
$$X(x, X(y,z)) = X(X(x,y),z), \tag{3.3}$$
$$X(x, X(y,y)) = X(x,y), \tag{3.4}$$
$$X(Y(X(x,y),z), Y(x,z)) = Y(X(x,y),z). \tag{3.5}$$

Note that each weakly idempotent lattice and each weakly idempotent semilattice satisfies the hyperidentities (3.1) - (3.5). Now we proceed to prove a number of hyperidentities that are valid in all weakly idempotent quasilattices. We start with the the following hyperidentity:

$$X(x, X(Y(x,y)), Y(y,z)) = X(Y(z,y),x). \tag{3.6}$$

Observe first that the following hyperidentities immediately follow from hyperidentities (3.5) and (3.2):

$$X(Y(X(Y(z,y),x), X(y,x)), Y(x, X(y,x)) = Y(X(Y(z,y),x), X(y,x)), \tag{3.7}$$
$$X(Y(X(x,y),x), Y(y,x)) = Y(X(x,y),x). \tag{3.8}$$

Indeed, to prove the hyperidentity (3.7), note that

$X(Y(X(Y(z,y),x), X(y,x)), Y(x.X(y,x))) \overset{(3.2)}{=} X(Y(X(y,x), X(x, Y(z,y))), Y(X(y,x),x))$
$\overset{(3.5)}{=} Y(X(y,x), X(x, Y(z,y))) \overset{(3.2)}{=} Y(X(Y(z,y),x), X(y,x))$.

The validity of hyperidentity (3.8) follows from hyperidentity (3.5) with $z = x$. Therefore, we get

$Y(X(Y(z,y),x), X(y,x)) \overset{(3.7)}{=} X(Y(X(Y(z,y),x), X(y,x)), Y(x, X(y,x))$
$\overset{(3.8)}{=} X(Y(X(Y(z,y),x)X(y,x)), X(Y(X(x,y),x), Y(y,x))$
$\overset{(3.3)}{=} X(X(Y(X(Y(z,y),x), X(y,x)), Y(X(x,y),x)), Y(y,x)) \overset{(3.7)}{=} (Y(X(Y(z,y),x), X(y,x)), Y(y,x))$.

Thus, we obtain the following hyperidentity:

$$Y(X(Y(z,y),x), X(y,x)) = X(Y(X(Y(z,y),x), X(y,x)), Y(y,x)). \tag{3.9}$$

Now we prove the hyperidentity (3.6). We have

$X(x, X(Y(x,y), Y(y,z)) \overset{(3.2),(3.3)}{=} X(X(x, Y(y,z)), Y(x,y))$
$\overset{(3.5)}{=} X(Y(X(x, Y(y,z)), X(x,y)), Y(x,y)) \overset{(3.9),(3.2)}{=} Y(X(Y(z,y),x), X(y,x)) \overset{(3.5)}{=} X(Y(z,y),x)$.

Thus, the hyperidentity (3.6) is proved. Replacing y by $Y(x,y)$ in the hyperidentity (3.6), we obtain

$$X(x, X(Y(x, Y(x,y)), Y(Y(x,y),z))) = X(Y(z, Y(x,y)),x).$$

In view of (3.2)-(3.4) we obtain $Y(x, Y(x,y)) = Y(Y(x,x),y) = Y(y, Y(x,x)) = Y(y,x) = Y(x,y)$. Therefore

$$X(x, X(Y(x,y), Y(Y(x,y),z))) = X(x, Y(z, Y(x,y))); \tag{3.10}$$
$$Y(X(Y(z,y),x), X(y,x)) = X(Y(X(Y(z,y),x), X(y,x)), Y(y,x)). \tag{3.11}$$

The hyperidentity

$$X(Y(y,z), X(x, Y(x,y))) = X(Y(z,y),x) \tag{3.12}$$

follows from the hyperidentities (3.2), (3.3) and (3.6). Indeed, we have
$X(Y(y,z),X(x,Y(x,y))) \overset{(3.1),(3.3)}{=} X(X(x,Y(y,z)),Y(x,y)) \overset{(3.3),(3.6)}{=} X(x,Y(y,z))$.
Substituting $y=z$ into (3.12), we obtain

$$X(y,X(x,Y(x,y)))=X(y,x). \tag{3.13}$$

Now we prove the hyperidentity

$$Y(Y(x,X(y,Y(y,z))),z)=Y(x,Y(y,z)). \tag{3.14}$$

To this end, we have to prove the following two hyperidentities:

$$X(Y(x,y),z)=X(X(Y(x,y),z),Y(Y(x,y),z), \tag{3.15}$$

$$X(y,Y(y,z))=Y(y,X(y,z)). \tag{3.16}$$

We first prove the hyperidentity (3.15). We have

$$X(Y(x,y),z) \overset{(3.12)}{=} X(Y(x,y),X(z,Y(z,Y(x,y)))) \overset{(3.3),(3.2)}{=} X(X(Y(x,y),z),Y(z,Y(x,y))).$$

Observe that in view of (3.15), we can use the hyperidentities (3.4) and (3.1) for $x=y$, to obtain

$$X(y,z)=X(X(y,z),Y(y,z)). \tag{3.17}$$

Indeed, we have
$X(X(y,z),Y(y,z)) \overset{(3.4)}{=} X(X(X(y,y),z),Y(Y(y,y),z) \overset{(3.1)}{=} X(X(Y(y,y),z),Y(Y(y,y),z)$
$\overset{(3.15)}{=} X(Y(y,y),z) \overset{(3.1)}{=} X(X(y,y),z) \overset{(3.4)}{=} X(y,z)$.

Next, we prove the hyperidentity (3.16). We have
$X(y,Y(y,z)) \overset{(3.5),(3.4)}{=} Y(X(y,Y(y,z)),y) \overset{(3.2)}{=} Y(y,X(y,Y(y,z)))$
$\overset{(3.5)}{=} Y(y,Y(X(y,Y(y,z)),X(y,z))) \overset{(3.13),(3.2)}{=} Y(y,Y(X(y,Y(y,z)),X(X(y,Y(y,z)),z)))$
$\overset{(3.10),(3.3)}{=} Y(y,X(X(y,Y(y,z)),z)) \overset{(3.3)}{=} Y(y,X(X(y,z),Y(y,z))) \overset{(3.17)}{=} Y(y,X(y,z))$.

Now we establish the hyperidentity (3.14). We have
$Y(Y(x,X(y,Y(y,z))),z) \overset{(3.16)}{=} Y(Y(x,Y(y,X(y,z))),z)$
$\overset{(3.3)}{=} Y(x,Y(Y(y,z),X(y,z))) \overset{(3.17)}{=} Y(x,Y(y,z))$.
According to (3.14) we have

$$Y(Y(x,X(z,Y(y,z))),y)=Y(x,Y(y,z)). \tag{3.18}$$

In what follows, we also have to prove the following hyperidentity:

$$X(x,Y(x,X(y,Y(y,z))))=X(x,Y(x,X(z,Y(y,z)))). \tag{3.19}$$

We have

$$\begin{aligned} X(x,Y(x,X(y,Y(y,z))) &\overset{(3.5)}{=} X(x,X(Y(x,X(y,Y(y,z)))),Y(x,Y(y,z))) \\ &\overset{(3.14)}{=} X(x,X(Y(x,Y(y,Y(y,z))),Y(Y(x,X(z,Y(y,z)),y)))) \\ &\overset{(3.10)}{=} X(x,Y(Y(x,X(z,Y(y,z))),y)) \overset{(3.14)}{=} Y(x,Y(y,z)). \end{aligned}$$

$$\begin{aligned} X(x,Y(x,X(z,Y(y,z))) &\overset{(3.5)}{=} X(x,X(Y(x,X(z,Y(y,z)))),Y(x,Y(y,z))) \\ &\overset{(3.18)}{=} X(x,X(Y(x,Y(z,Y(y,z))),Y(Y(x,X(z,Y(y,z)),y)))) \\ &\overset{(3.10)}{=} X(x,Y(Y(x,X(z,Y(y,z))),y)) \overset{(3.14)}{=} Y(x,Y(y,z)). \end{aligned}$$

Therefore

$$X(x,Y(x,X(y,Y(y,z)))=Y(x,Y(y,z)), \tag{3.20}$$

$$X(x,Y(x,X(z,Y(y,z)))=Y(x,Y(y,z)). \tag{3.21}$$

From hyperidentities (3.20) and (3.21) follows the hyperidentity (3.19).

Lemma 3.1. *Each weakly idempotent quasilattice $(Q; A, B)$ with two binary operations is either a weakly idempotent lattice or a sum of its mutually disjoint subalgebras that are weakly idempotent lattices.*

Proof. Define the mapping $f : Q \times Q \to Q$ by $f(x, y) = A(x, B(x, y)) = B(x, A(x, y))$, and show that f is a Plonka nonidempotent function. The correctness of mapping f immediately follows from hyperidentity (3.16). Now we verify the conditions of Definition 2.1. We have

1. $f(f(x, y), z) = f(A(x, B(x, y)), z) = A(A(x, B(x, y)), B(A(x, B(x, y)), z))$
$\overset{(3.5)}{=} A(A(x, B(x, y)), A(B(A(x, B(x, y)), z)), B(B(x, y), z))$
$\overset{(3.18)}{=} A(A(x, B(x, y)), A(B(A(x, B(x, y)), z), B(B(z, A(x, B(x, y))), z)))$
$\overset{(3.10)}{=} A(A(x, B(x, y)), B(B(z, A(x, B(x, y))), y)) \overset{(3.18)}{=} A(A(x, B(x, y)), B(B(x, y), z))$
$\overset{(3.3)}{=} A(x, A(B(x, y), B(B(x, y), z))) \overset{(3.10)}{=} A(x, B(B(x, y), z))$.
$f(x, f(y, z)) = f(x, A(y, B(y, z))) = A(x, B(x, A(y, Y(y, z)))) \overset{(3.20)}{=} A(x, B(x, B(y, z))))$.

2. $f(x, x) = A(x, B(x, x)) \overset{(3.1)}{=} A(x, A(x, x)) \overset{(3.4)}{=} A(x, x) = B(x, x)$.
3. $f(x, f(y, z)) = f(x, A(y, B(y, z))) = A(x, B(x, A(y, B(y, z))))$;
$f(x, f(z, y)) = f(x, A(z, B(z, y))) = A(x, B(x, A(z, B(z, y))))$.
According to hyperidentity (3.18), we obtain $f(x, f(y, z)) = f(x, f(z, y))$.

In the proof of conditions 4-8 of Definition 2.1, without loos of generality, we can assume that $F_t = A$.

According to Theorem 2.1, the algebra $(Q; A, B)$ is a sum of its mutually disjoint subalgebras U_i, $i \in I$. Hence, it remains to prove that U_i are weakly idempotent lattices. For subalgebras U_i we have to check only the identities of weak absorption (1.4): $x \wedge (x \vee y) = x \wedge x, x \vee (x \wedge y) = x \vee x$. Indeed, $x, y \in U_i$ if and only if $f(x, y) = f(x, x)$, $f(y, x) = f(y, y)$. Computing the right and left hand sides of equality $f(x, y) = f(x, x)$ (for $A = \wedge, B = \vee$), we obtain $f(x, y) = x \wedge (x \vee y)$ and $f(x, x) = x \wedge (x \vee x) = x \wedge (x \wedge x) = x \wedge x$, implying that $x \wedge (x \vee y) = x \wedge x$. Similarly we can obtain the second identity.

Theorem 3.1. *For a subdirectly irreducible weakly idempotent quasilattice $\mathfrak{U} = (U; \Sigma)$ we have $|\Sigma| \le 2$.*

Proof. Let $\mathfrak{U} = (U; \Sigma)$ be a weakly idempotent quasilattice. We show that if $|\Sigma| \ge 3$, then $\mathfrak{U}$ is not subdirectly irreducible. Since $|\Sigma| \ge 3$, then there exist pairwise distinct binary operations $A_1, A_2, A_3 \in \Sigma$. Define $f_{ij}(x, y) = A_i(x, A_j(x, y))$. Define the relations $\widetilde{\theta}_{i,j}$ on the set U by $x\widetilde{\theta}_{i,j}y \leftrightarrow f_{i,j}(x, y) = x, f_{i,j}(y, x) = y$. Then $\theta_{i,j} = \widetilde{\theta}_{i,j} \cup \{x = x\}$ is an equivalence relation on the set U. $\theta_{i,j}$ are congruences on algebra $\mathfrak{U}$.

Now we show that $\theta_{1,2} \cap \theta_{1,3} \cap \theta_{2,3} = \omega$. If $x(\theta_{1,2} \cap \theta_{1,3} \cap \theta_{2,3})y$, then $x\theta_{1,2}y$, $x\theta_{1,3}y$, $x\theta_{2,3}y$, and hence $f_{1,2}(x, y) = x$, $f_{1,2}(y, x) = y, f_{1,3}(x, y) = x$, $f_{1,3}(y, x) = y, f_{2,3}(x, y) = x$, $f_{2,3}(y, x) = y$, or $x = y$. In the first case, from hyperidentity (3.7), replacing z by $Z(x, y)$, we obtain
$A_1(A_2(x, y), A_3(x, y)) = A_1(A_1(A_2(x, A_3(x, y)), A_2(x, y)), A_3(x, y))$
$= A_1(A_1(x, A_2(x, y)), A_3(x, y)) = A_1(x, A_3(x, y)) = x$;
$A_1(A_2(y, x), A_3(y, x)) = A_1(A_1(A_2(y, A_3(y, x)), A_2(y, x)), A_3(y, x)) = A_1(y, A_3(y, x)) = y$.
Therefore, in both cases we have $x = y$. It remains to show that all the three congruences $\theta_{1,2}, \theta_{1,3}$ and $\theta_{2,3}$ are nontrivial. For instance, let us prove the nontriviality of congruence $\theta_{1,2}$. Since $A_1 \ne A_2$, there exist elements $x, y \in U$ such that $A_1(x, y) \ne A_2(x, y)$, and hence $A_1(x, y)\theta_{1,2}A_2(x, y)$. Indeed, according to hyperidentity (3.17), for $y = X(x, y), z = Y(x, y)$ we have

$$X(X(x, y), Y(x, y)) = X(X(X(x, y), Y(x, y)), Y(X(x, y), Y(x, y)).$$

This and the hyperidentity (3.13) imply $X(x, y) = X(X(x, y), Y(X(x, y), Y(x, y)))$. From the last equality for $X = A_1, Y = A_2$ we have

$$f_{1,2}(A_1(x, y), A_2(x, y)) = A_1(A_1(x, y), A_2(A_1(x, y), A_2(x, y))) = A_1(x, y).$$

Using (3.16) we obtain $f_{1,2}(A_2(x, y), A_1(x, y)) = A_1(A_2(x, y), A_2(A_2(x, y), A_1(x, y)))$
$= A_2(A_2(x, y), A_1(A_2(x, y), A_1(x, y))) = A_2(x, y)$.

This shows that the underlying algebra is not subdirectly irreducible, which is a contradiction. Thus, the cardinality of the set of operations of subdirectly irreducible weakly idempotent quasilattice is at most two. Theorem 3.1 is proved.

4. THE MAIN RESULT

Theorem 4.1. *Any hyperidentity of the variety of weakly idempotent lattices is a consequence of the hyperidentities (3.1) - (3.5).*

Proof. By Theorem 3.1, the cardinality $|\Sigma|$ of a subdirectly irreducible weakly idempotent quasilattice $(U;\Sigma)$ is at most two. Hence, by Birkhoff theorem on subdirect products, each weakly idempotent quasilattice is isomorphic to a subdirect product of weakly idempotent quasilattices with one or two binary operations. Since a weakly idempotent quasilattice with one binary operation is p a weakly idempotent semilattice, then any homogeneous hyperidentity is satisfied in a weakly idempotent quasilattice with one binary operation. On the other hand, from Lemma 3.1 we infer that any weakly idempotent quasilattice with two binary operations is either a weakly idempotent lattice or is a sum of its mutually disjoint subalgebras, which are weakly idempotent lattices. Therefore, each homogeneous hyperidentity which is satisfied on the variety of weakly idempotent lattices, also will be satisfied on each weakly idempotent quasilattice with two binary operations. This completes the proof of Theorem 4.1.

Corollary 4.1. ([7], [8], [10]) *Each hyperidentity of the variety of lattices is a consequence of hyperidentities (3.2), (3.3), (3.5) and the idempotency hyperidentity:* $X(x,x)=x$.

Now we show that the hyperidentity (3.5) is a consequence of the hyperidentities (3.1) - (3.4) and the distributivity hyperidentity

$$X(Y(x,y),z)=Y(X(x,z),X(y,z)). \tag{4.1}$$

Indeed, we have

$$X(Y(X(x,y),z),Y(y,z)) \overset{(4.1)}{=} X(X(Y(x,z),Y(y,z)),Y(y,z)) \overset{(3.3)}{=}$$
$$X(Y(x,z),X(Y(y,z),Y(y,z)) \overset{(1.3)}{=} X(Y(x,z),Y(y,z)) \overset{(4.1)}{=} Y(X(x,y),z).$$

Corollary 4.2. *Each hyperidentity of the variety of the distributive weakly idempotent lattices is a consequence of the hyperidentities (3.1) - (3.4) and the distributivity hyperidentity (4.1).*

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