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# Integrability and separation of variables in Calogero-Coulomb-Stark and two-center Calogero-Coulomb systems

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We propose the integrable $N$-dimensional Calogero-Coulomb-Stark and two-center Calogero-Coulomb systems and construct their constants of motion via the Dunkl operators. Their Schrödinger equations decouple in parabolic and elliptic coordinates into the set of three differential equations like for the Coulomb-Stark and two-center Coulomb problems. The Calogero term preserves the energy levels, but changes their degrees of degeneracy.

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## I. INTRODUCTION

One of the most important features of the Coulomb problem is its maximal superintegrability caused by the conservation of the Runge-Lenz vector. As a consequence, its Hamiltonian admits separation of variables in several coordinate systems and, hence, a few types of integrable perturbations with separation of variables. The textbook examples of such perturbations are (see, for example, Ref. [1]):

(1) The Coulomb problem in a constant electric field (the Coulomb-Stark problem) with the Stark potential $\delta V(\boldsymbol{r}) = \boldsymbol{F}\cdot\boldsymbol{r}$, which admits a separation of variables in parabolic coordinates.
(2) The two-center Coulomb problem, which admits a separation of variables in elliptic coordinates.

These systems remain integrable despite the partially broken rotational symmetry along the highlighted direction. The latter coincides with the direction of the electric field $\boldsymbol{F}$ for the Coulomb-Stark problem, and with the vector $\boldsymbol{a}$ connecting two Coulomb charges for the two-center Coulomb problems. As a result, the orthogonal angular momentum components and the modified longitudinal component of the Runge-Lenz vector are preserved.

However, the aforementioned systems are not solvable exactly. In the Coulomb-Stark problem, one can get analytically only the perturbative spectrum, while in the two-center Coulomb system, the energy spectrum can be constructed only numerically, except for some special cases [2]. Nevertheless, the separation of variables is crucial in the study of these systems.

Recently, together with O. Lechtenfeld, we have observed that the $N$-dimensional Coulomb problem deformed by the rational Calogero potential [3] (and by its generalization associated with an arbitrary root system [4]) remains superintegrable and has the same energy spectrum as the original Coulomb model [5]. The Calogero-Coulomb Hamiltonian is given by the expression

$$\mathcal{H}_\gamma = \sum_{i=1}^{N}\frac{p_i^2}{2} + \sum_{i<j}^{N}\frac{g(g-1)}{(x_i-x_j)^2} - \frac{\gamma}{\sqrt{\sum_i x_i^2}}. \tag{1.1}$$

In Ref. [6], we have revealed an explicit expression for the analog of the Runge-Lenz vector in this system. We also observed that, being formulated in terms of the Dunkl operators [7], the conserved quantities and their algebra look pretty similar to those in the conventional Coulomb problem. This led us to claim that most of the properties of the conventional Coulomb problem and its integrable perturbations have their Calogero-Coulomb counterparts.

Thus, one may ask the following:

Are the high-dimensional Coulomb-Stark and two-center Coulomb problems with the Calogero potential (we will refer to them as Calogero-Coulomb-Stark and two-center Calogero-Coulomb problems) integrable systems? If they are, do they admit separation of variables, at least partially?

In this paper, we give positive answer to both questions:

(i) The proper choice of the direction of the vectors $\boldsymbol{F}$ and $\boldsymbol{a}$ makes the related systems integrable. The corresponding Hamiltonians are given by the expressions

$$\mathcal{H}_{\gamma,F} = \sum_{i=1}^{N}\frac{p_i^2}{2} + \sum_{i<j}^{N}\frac{g(g-1)}{(x_i-x_j)^2} - \frac{\gamma}{\sqrt{\sum_i x_i^2}} + \frac{F}{\sqrt{N}}\sum_{i=1}^{N} x_i \tag{1.2}$$

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$$\mathcal{H}_{\gamma_1,\gamma_2} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma_1}{\sqrt{\sum_i (x_i - a/\sqrt{N})^2}} - \frac{\gamma_2}{\sqrt{\sum_i (x_i + a/\sqrt{N})^2}}. \quad (1.3)$$

The constants of motion are the Dunkl-operator deformations of the corresponding integrals in the underlying Coulomb systems with the subsequent symmetrization over the coordinates.

Note that the integrability of the Coulomb-Stark and two-center Coulomb systems (even without the Calogero interaction, even at the classical level) does not follow from the existence of the Runge-Lenz vector or the properties of the symmetry algebra of the usual Coulomb problem. The preservation of the constructed constants of motion in the systems (1.2) and (1.3) will be verified independently, based on the properties of the Dunkl operators.

(ii) After the transition to the Jacobi (center-of-mass) coordinates, the above listed systems admit a complete separation of variables in parabolic or elliptic coordinates for $N = 2$, 3 and a partial separation for $N > 3$. In fact, the Schrödinger equation decouples into three parts, only one of which depends on the Calogero inverse-square term. The latter can be treated as a deformation of the Schrödinger equation for the $SO(N-1)$ angular momentum, usually referred to as an angular Calogero Hamiltonian [8–11].

The paper is organized as follows:

In Sec. II, we describe the Calogero-Coulomb problem and its symmetry algebra, which contains the deformations of the angular momentum and Runge-Lenz vector by means of the Dunkl operator [5,6]. Then we perform the orthogonal transformation to the Jacobi coordinates and show that the transition to the spherical coordinates factorizes the original Schrödinger equation into three parts. The first two are just the radial and angular Schrödinger equations of the standard Coulomb problem. The third part produces the Schrödinger equation of the angular Calogero model with the excluded center of mass, which has been solved in Ref. [9].

In Sec. III, we introduce the Calogero-Coulomb-Stark problem and present explicit expressions for its constants of motion. They include the components of the Dunkl angular momentum, which are orthogonal to the direction of the electric field $\boldsymbol{F}$ and the modified component of the Runge-Lenz vector along the field direction. We go on to show that after the transition to the Jacobi coordinates, the system possesses a partial separation of variables in parabolic coordinates as in spherical coordinates. Namely, the Schrödinger equation decouples into three parts, where the first two coincide with the respective Schrödinger equations of the standard Coulomb problem in parabolic coordinates, while the third one produces the Schrödinger equation of the angular Calogero model with the excluded center of mass [9].

In Sec. IV, we introduce the two-center Calogero-Coulomb problem and perform the similar study for it. Again, the components of the Dunkl angular momentum, orthogonal to the symmetry axis of the system given now by the vector $\boldsymbol{a}$, are preserved. The last integral is a deformation of similar construction applied for the underlying two-center Coulomb system [12]. The latter is based on the peculiar mixture of the longitudinal components of the two Runge-Lenz vectors, corresponding to each charge. After transition to the Jacobi coordinates, the system acquires a partial separation of variables in elliptic coordinates.

## II. CALOGERO-COULOMB PROBLEM

### A. Coulomb problem

The $N$-dimensional Coulomb problem is defined by the Hamiltonian

$$\mathcal{H}^0_\gamma = \frac{\boldsymbol{p}^2}{2} - \frac{\gamma}{r} \quad \text{with} \quad r = \sqrt{\boldsymbol{x}^2}. \quad (2.1)$$

It possesses the maximum number $(2N-1)$ of functionally independent constants of motion. First, due to the rotational invariance, the system possesses conserving angular momentum tensor

$$L_{ij} = x_i p_j - x_j p_i. \quad (2.2)$$

Its components satisfy the standard commutation rules for the $SO(N)$ generators:

$$[L_{ij}, L_{kl}] = \imath\delta_{lj} L_{ik} + \imath\delta_{ki} L_{jl} - \imath\delta_{kj} L_{il} - \imath\delta_{li} L_{jk}. \quad (2.3)$$

The angular momentum provides the system with $N-1$ commuting quadratic (in momentum) constants of motion

$$\boldsymbol{L}_k^2 = \sum_{i<j\le k} L_{ij}^2 \quad \text{with} \quad 2 \le k \le N, \qquad \boldsymbol{L}_N^2 = \boldsymbol{L}^2. \quad (2.4)$$

They correspond to the quadratic Casimir elements, constricted from the generators of the sequence of the natural embeddings:

$$SO(2) \subset \ldots \subset SO(k) \subset \ldots \subset SO(N).$$

Together with the Hamiltonian (2.1), these constants of motion form the complete set of Liouville integrals.

Apart from the angular momentum, the $N$-dimensional Coulomb motion preserves the Runge-Lenz vector defined by expression

$$A_i = \frac{1}{2}\sum_j \{L_{ij}, p_j\} - \frac{\gamma x_i}{r}, \tag{2.5}$$

where the curly braces mean the anticommutator: $\{a, b\} = ab + ba$.

The angular momentum tensor and the Runge-Lenz vector define the complete set of constants of motion for the Coulomb system.

### B. Calogero-Coulomb Hamiltonian

The Calogero-Coulomb problem introduced in Ref. [13] contains the additional inverse-square interaction term *a la* Calogero, as already mentioned in the Introduction. It is a mixture of the $N$-particle rational Calogero model [see Refs. [4,14] for review] and of the $N$-dimensional Coulomb system. In vector notations the Hamiltonian (1.1) can be rewritten as follows:

$$\mathcal{H}_\gamma = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma}{r}. \tag{2.6}$$

It inherits most of the properties of the original Coulomb system (2.1) and possesses hidden symmetries given by an analog of Runge-Lenz vector [6]. It is convenient to describe this system by means of the Dunkl operators which makes transparent its analogy with the Coulomb problem. Namely, instead of the $\mathcal{H}_\gamma$, let us consider the more general Hamiltonian,

$$\mathcal{H}^{\rm gen}_\gamma = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma}{r} = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j} \frac{g(g - s_{ij})}{(x_i - x_j)^2} - \frac{\gamma}{r}, \tag{2.7}$$

where the momentum modified by the Dunkl exchange operators [7] is used,

$$\boldsymbol{\pi} = -\imath\boldsymbol{\nabla}, \qquad \nabla_i = \partial_i - \sum_{j\neq i} \frac{g}{x_i - x_j} s_{ij}. \tag{2.8a}$$

The operator $s_{ij}$ in Eqs. (2.7) and (2.8a) interchanges the $i$th coordinate with the $j$th coordinate,

$$s_{ij}\Psi(\ldots, x_i, \ldots, x_j, \ldots) = \Psi(\ldots, x_j, \ldots, x_i, \ldots). \tag{2.8b}$$

On the symmetric wave functions, the generalized Hamiltonian $\mathcal{H}^{\rm gen}_\gamma$ reduces to the Calogero-Coulomb Hamiltonian (2.6).

The Dunkl operators commute mutually like ordinary partial derivatives. However, their commutations with coordinates are nontrivial deformations of the Heisenberg algebra relations [7],

$$[\pi_i, \pi_j] = 0, \qquad [\pi_i, x_j] = -\imath S_{ij}. \tag{2.9a}$$

Here the operators $S_{ij}$ are related with the permutations in the following way:

$$S_{ij} = \begin{cases} -g s_{ij} & \text{for } i \neq j, \\ 1 + g\sum_{k\neq i} s_{ik} & \text{for } i = j. \end{cases} \tag{2.9b}$$

Using the above relations, it can be verified that the operator-valued tensors $[S_{ij}, x_k]$ and $[S_{ij}, \pi_k]$ are symmetric in all three indexes [8],

$$[S_{ik}, x_j] = [S_{jk}, x_i], \qquad [S_{ik}, \pi_j] = [S_{jk}, \pi_i]. \tag{2.10}$$

It must also be mentioned that the symmetries of the two-dimensional Calogero-Coulomb problem associated with the dihedral group $D_2$ have been recently studied in Ref. [15] in terms of the Dunkl operators.

### C. Constants of motion for $\mathcal{H}^{\rm gen}_\gamma$

Let us define the deformed angular momentum operator via the Dunkl momentum [8,16,17]:

$$L_{ij} = x_i\pi_j - x_j\pi_i. \tag{2.11}$$

Its components satisfy the following commutation relations [8]:

$$[L_{ij}, L_{kl}] = \imath L_{ik}S_{lj} + \imath L_{jl}S_{ki} - \imath L_{il}S_{kj} - \imath L_{jk}S_{li}. \tag{2.12}$$

The Dunkl angular momentum inherits many properties of the ordinary angular momentum (2.2). In particular, the equations

$$\sum_k \{S_{ik}, \pi_k\} = 2\pi_i, \qquad \sum_k \{S_{ik}, x_k\} = 2x_i, \tag{2.13}$$

which follow from the relations (2.9a) and (2.9b), imply that the operators $L_{ij}$ commute with the Dunkl momentum square and with the radius vector length [8]:

$$[\boldsymbol{\pi}^2, L_{ij}] = [r, L_{ij}] = 0. \tag{2.14}$$

Hence, the Dunkl angular momentum is a constant of motion of the generalized Calogero-Coulomb Hamiltonian [5]:

$$[\mathcal{H}^{\rm gen}_\gamma, L_{ij}] = 0. \tag{2.15}$$

The deformation of the Runge-Lenz vector (2.5) is given by the expression [6]

$$A_i = \frac{1}{2}\sum_j \{L_{ij}, \pi_j\} + \frac{\imath}{2}[\pi_i, S] - \frac{\gamma x_i}{r}. \tag{2.16}$$

It contains the permutation-group invariant element which vanishes in the absence of the Calogero term ($g = 0$):

$$S = \sum_{i<j} S_{ij}: \quad [S, S_{ij}] = 0. \tag{2.17}$$

The deformed Runge-Lenz vector is also a constant of motion:

$$[\mathcal{H}_\gamma^{\rm gen}, A_i] = 0. \tag{2.18}$$

The proof is given in Ref. [6], and we repeat it here for completeness.

According to (2.14), the commutator of the Hamiltonian with the first term from the rhs of Eq. (2.16) is proportional to

$$\left[\sum_j \{L_{ij}, \pi_j\}, \frac{\gamma}{r}\right] = -\frac{\imath\gamma}{r^3}\sum_j \{L_{ij}, x_j\} = \left\{\frac{\imath\gamma}{r}, \pi_i\right\} - \imath\gamma \sum_j \left\{\frac{x_i x_j}{r^3}, \pi_j\right\}, \tag{2.19}$$

and its commutator with the last term is proportional to

$$\left[\boldsymbol{\pi}^2, \frac{x_i}{r}\right] = \imath \sum_j \left\{\frac{x_i x_j}{r^3} - \frac{S_{ij}}{r}, \pi_j\right\}. \tag{2.20}$$

Combining the relations (2.19) and (2.20) and using the definition (2.17), we obtain

$$\left[\frac{1}{2}\sum_j \{L_{ij}, \pi_j\} - \frac{\gamma x_i}{r}, \mathcal{H}_\gamma^{\rm gen}\right] = \imath \sum_j \left\{\frac{\gamma S_{ij}}{2r}, \pi_i - \pi_j\right\} = \frac{\imath}{2}[[S, \pi_i], \mathcal{H}_\gamma^{\rm gen}]. \tag{2.21}$$

In the last equation we used the identity

$$\sum_j (\pi_j - \pi_i) S_{ij} = [S, \pi_i].$$

Finally, Eq. (2.21) completes the proof of the conservation condition (2.18).

The deformed Runge-Lenz vector can be written directly through the coordinate and Dunkl momentum by avoiding the invariant $S$:

$$\boldsymbol{A} = \boldsymbol{x}\left(\boldsymbol{\pi}^2 - \frac{\gamma}{r}\right) - \left(r p_r + \frac{N-1}{2\imath}\right)\boldsymbol{\pi}. \tag{2.22}$$

Here $p_r$ is the ordinary radial momentum canonically conjugate to the radial coordinate: $[p_r, r] = -\imath$. It turns into its $g = 0$ analog under the replacement $\boldsymbol{\pi} \to \boldsymbol{p}$.

The relation (2.22) can be checked by inserting Eq. (2.11) into Eq. (2.16) and applying the identity

$$\boldsymbol{\pi}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{\pi} = 2(\boldsymbol{x}\cdot\boldsymbol{p}) - \imath N = 2r p_r - \imath N. \tag{2.23}$$

This identity is a direct consequence of the deformed canonical commutation relations (2.9a).

The commutation relations among the Dunkl angular momentum components (2.12) are supplemented by those which incorporate the deformed Runge-Lenz momentum and Hamiltonian [6]:

$$[A_i, L_{kl}] = \imath A_k S_{li} - \imath A_l S_{ki}, \qquad [A_i, A_j] = -2\imath \mathcal{H}_\gamma^{\rm gen} L_{ij}. \tag{2.24}$$

In the absence of the inverse-square interaction, the exchange operator $S_{ij}$ is reduced to the Kronecker delta, and we get the familiar algebra, formed by the symmetry generators of the Coulomb problem. On a negative constant-energy surface ${\mathcal{H}_\gamma}^0 = E < 0$, it reduces to the orthogonal algebra $so(N+1)$, and on the positive one, ${\mathcal{H}_\gamma}^0 = E > 0$, to the $so(N,1)$.

### D. Constants of motion for $\mathcal{H}_\gamma$

Let us remember that the Calogero-Colomb problem can be obtained by the restriction of the generalized Hamiltonian (2.7) to symmetric wave functions (2.8b). Therefore, its constants of motion can be constructed by taking the symmetric polynomials depending on the components of the Dunkl angular momentum and Runge-Lenz vector [5,6]:

$$\mathcal{L}_{2k} = \sum_{i<j} L_{ij}^{2k}, \tag{2.25}$$

$$\mathcal{A}_k = \sum_i A_i^k. \tag{2.26}$$

The expressions above demonstrate that the Calogero-Coulomb problem is a superintegrable system, like the pure Calogero [18] and Coulomb models. Among these integrals, only the two are quadratic on momenta. The first one is the square of the Dunkl angular momentum,

$$\mathcal{L}_2 = 2\mathcal{I} + S(S - N + 2), \tag{2.27}$$

where $\mathcal{I}$ is the angular part of the Calogero (Calogero-Coulomb) Hamiltonian [8]:

$$\mathcal{H}_\gamma = \frac{p_r^2}{2} - \imath\frac{N-1}{2r}p_r - \frac{\gamma}{r} + \frac{\mathcal{I}}{r^2}. \tag{2.28}$$

It coincides with the Casimir element of the Dunkl angular momentum algebra and is also a conserved quantity:

$$[L_{ij}, \mathcal{I}] = 0, \qquad [\mathcal{I}, \mathcal{H}_\gamma] = 0. \tag{2.29}$$

So it can be used instead of $\mathcal{L}_2$ as a constant of motion.

The second quadratic integral is given by the sum of the components of the deformed Runge-Lenz vector defined in Ref. [5]. For convenience, we take a rescaled expression,

$$\frac{1}{\sqrt{N}}\mathcal{A}_1 = \frac{1}{\sqrt{N}}\sum_j \{L_{ij}, \pi_j\} - \frac{\gamma x_0}{r} = x_0\left(2\mathcal{H}_\gamma^{\text{gen}} + \frac{\gamma}{r}\right) - \left(rp_r + \frac{N-1}{2\imath}\right)p_0. \quad (2.30)$$

It depends on the normalized center-of-mass coordinate and momentum

$$x_0 = \frac{1}{\sqrt{N}}\sum_{i=1}^N x_i, \qquad p_0 = \frac{1}{\sqrt{N}}\sum_{i=1}^N p_i. \quad (2.31)$$

The existence of two second-order integrals suggests that the Calogero-Coulomb problem admits a separation of variables for dimension $N \le 3$.

### E. Jacobi coordinates

The Jacobi coordinates [10,19,20] guarantee the separations of the center-of-mass (2.31) from the relative motion, keeping the kinetic term's shape. They are defined in the following way:

$$y_0 = x_0 = \frac{1}{\sqrt{N}}(x_1 + \cdots + x_N),$$
$$y_k = \frac{1}{\sqrt{k(k+1)}}(x_1 + \cdots + x_k - kx_{k+1}),$$
$$1 \le k \le N-1. \quad (2.32a)$$

The first coordinate describes the center of mass, while the other coordinates characterize the relative motion. It can be verified that the above transformation is *orthogonal*:

$$y_k = \sum_{l=1}^N O_{kl}x_l, \qquad x_k = \sum_{l=0}^{N-1} O_{lk}y_l. \quad (2.32b)$$

In order to express the Hamiltonian and Dunkl momentum in terms of the Jacobi coordinates, we note that the permutation operators are just orthogonal reflections across the hyperplanes $\boldsymbol{\alpha}\cdot\boldsymbol{x} = 0$ in the $N$-dimensional coordinate space,

$$s_{ij}\boldsymbol{x} = s_\alpha \boldsymbol{x} := \boldsymbol{x} - \frac{2(\boldsymbol{\alpha}\cdot\boldsymbol{x})}{\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}}\boldsymbol{\alpha} \quad \text{with} \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_{ij} = \boldsymbol{e}_i - \boldsymbol{e}_j, \quad (2.33)$$

where $\boldsymbol{e}_i$ form the standard basis in $\mathbb{R}^N$: $\boldsymbol{x} = \sum_i x_i \boldsymbol{e}_i$. The operators $S_{ij}$ are expressed in a similar manner. Note that this representation admits the generalization of the exchange operators (2.8a) to an arbitrary finite reflection group known also as a Coxeter group [7]. The vector $\alpha$ is called a root of the Coxeter system. A particular choice of roots in Eq. (2.33) defines the $A_{N-1}$ Coxeter system. They separate the center-of mass coordinate from the relative motion.

We rewrite the reflections (2.33) in the Jacobi coordinates using the vector notations:

$$s_{ij}\boldsymbol{y} = s_{\boldsymbol{\beta}}\boldsymbol{y} = \boldsymbol{y} - \frac{2(\boldsymbol{\beta}\cdot\boldsymbol{y})}{\boldsymbol{\beta}\cdot\boldsymbol{\beta}}\boldsymbol{\beta}. \quad (2.34a)$$

Here the vectors $\boldsymbol{\beta}$ are images of the corresponding root vectors $\boldsymbol{\alpha}$ under the Jacobi transformation (2.32b):

$$\beta_k = \sum_{k'} O_{kk'}\alpha_{k'}, \qquad \beta_{ij,k} = O_{ki} - O_{kj}. \quad (2.34b)$$

Their explicit form (2.32a) ensures the relation $\beta_{ij,0} = 0$. In fact, the reflection (2.34a) is restricted to the relative coordinates (see Eq. (2.40) below).

In Jacobi coordinates, the inverse-square Calogero potential in the Hamiltonian (2.6) reads

$$V(\boldsymbol{y}) = \sum_{i<j}\frac{g(g-1)}{(\boldsymbol{\alpha}_{ij}\cdot\boldsymbol{x})^2} = \sum_{i<j}\frac{g(g-1)}{(\boldsymbol{\beta}_{ij}\cdot\boldsymbol{y})^2}. \quad (2.35)$$

Evidently, the Dunkl momenta along the new coordinate directions are related to the usual Dunkl momenta $\pi_{x_i} = \pi_i$ by the vector transformation

$$\tilde{\pi}_i = \pi_{y_i} = \sum_{i'=1}^N O_{ii'}\pi_{i'}. \quad (2.36)$$

Its zeroth component describes the center-of-mass momentum (2.31): $\tilde{\pi}_0 = \tilde{p}_0 = p_0$. In order to ensure the invariance of the defining relations (2.9a), the operators $S_{ij}$ must behave as a second-order tensor in an orthogonal transformation:

$$[\tilde{\pi}_i, \tilde{\pi}_j] = 0,$$
$$[\tilde{\pi}_i, y_j] = -\imath\tilde{S}_{ij} = -\imath\sum_{i',j'=1}^N O_{ii'}O_{jj'}S_{i'j'}. \quad (2.37)$$

In Jacobi coordinates, the Cherednik algebra decouples into two commuting parts describing the center-of-mass ($y_0 = x_0$, $p_0$) and relative ($y_i$, $\tilde{\pi}_i$ with $1 \le i \le N-1$) motions, respectively. Indeed, exploiting the defining relations (2.9b) and the orthogonality of the Jacobi transformation (2.32b), we obtain

$$\tilde{S}_{i0} = \frac{1}{\sqrt{N}}\sum_{j'=1}^N S_{i'j'}\sum_{i'=1}^N O_{ii'} = \sum_{i'=1}^N O_{ii'}O_{0i'} = \delta_{i0}. \quad (2.38)$$

It implies

$$[\tilde{\pi}_i, y_0] = [\tilde{\pi}_i, x_0] = [p_0, y_i] = -\imath\delta_{i0}. \tag{2.39}$$

Note that the constraint (2.38) is equivalent in old coordinates to the condition $\sum_j S_{ij} = 1$, which follows from the defining relation (2.9b).

Below we use the following vector notations for the description of the relative coordinates and momenta:

$$\boldsymbol{y} = (y_1, \ldots, y_{N-1}), \qquad \tilde{\boldsymbol{\pi}} = (\tilde{\pi}_1, \ldots \tilde{\pi}_{N-1}). \tag{2.40}$$

Note that the operators $\tilde{S}_{ij}$ do not permute the coordinates any more. However, the tensors $[\tilde{S}_{ij}, y_k]$ and $[\tilde{S}_{ij}, \tilde{\pi}_k]$ still remain symmetric so that the relations (2.10) are preserved:

$$[\tilde{S}_{ik}, y_j] = [\tilde{S}_{jk}, y_i], \qquad [\tilde{S}_{ik}, \tilde{\pi}_j] = [\tilde{S}_{jk}, \tilde{\pi}_i]. \tag{2.41}$$

This property persists for any Coxeter system [8].

Of course, similarly to the operators $S_{ij}$, the Dunkl angular momentum transforms as a second-order tensor under an orthogonal transformation:

$$\tilde{L}_{ij} = L_{y_i y_j} = \sum_{i',j'=1}^{N} O_{ii'} O_{jj'} L_{i'j'}. \tag{2.42a}$$

The rotated components obey the commutation relations (2.12) with the permutation generators $S_{ij}$ replaced by their shifted counterparts $\tilde{S}_{ij}$. They acquire the standard form (2.11) in the Jacobi coordinates and momenta,

$$\tilde{L}_{ij} = y_i \tilde{\pi}_j - y_j \tilde{\pi}_j \quad \text{with} \quad 0 \le i, j \le N-1, \tag{2.42b}$$

$$\tilde{L}_{0i} = x_0 \tilde{\pi}_i - y_j p_0. \tag{2.42c}$$

Using (2.38), we can rewrite them in the form which splits up the center-of-mass and relative degrees of freedom:

$$[\tilde{L}_{ij}, \tilde{L}_{kl}] = \imath\tilde{L}_{ik}\tilde{S}_{lj} + \imath\tilde{L}_{jl}\tilde{S}_{ki} - \imath\tilde{L}_{il}\tilde{S}_{kj} - \imath\tilde{L}_{jk}\tilde{S}_{li}, \tag{2.43a}$$

$$[\tilde{L}_{ij}, \tilde{L}_{0l}] = \imath\tilde{L}_{0j}\tilde{S}_{li} - \imath\tilde{L}_{0i}\tilde{S}_{lj}, \tag{2.43b}$$

$$[\tilde{L}_{0j}, \tilde{L}_{0l}] = \imath\tilde{L}_{jl} \quad \text{with} \quad 1 \le i, j, k, l \le N-1. \tag{2.43c}$$

So the generators (2.42b) form the deformed analog of the $so(N-1)$ angular momentum algebra consisting of the relative degrees of freedom.

Usually, in the Coulomb problem, the separation of variables is performed in spherical coordinates. However, for the (partial) separation of variables in spherical coordinates, we should first perform the *orthogonal* transformation to the Jacobi coordinates in order to extract the center of mass [10].

### F. Separation of variables in spherical coordinates

Let us extract now the coordinates of the center of mass from the Calogero-Coulomb Hamiltonian (2.6),

$$\mathcal{H}_\gamma = \frac{p_0^2}{2} - \frac{\gamma}{\sqrt{x_0^2 + y^2}} + \tilde{\mathcal{H}}_0 \quad \text{with} \quad y = \sqrt{\boldsymbol{y}^2}. \tag{2.44}$$

The last term is just the Calogero Hamiltonian with the excluded center of mass, which depends on the $N-1$ relative degrees of freedom. In the Jacobi coordinates, it reads

$$\tilde{\mathcal{H}}_0 = \frac{\tilde{\boldsymbol{p}}^2}{2} + \sum_{i<j} \frac{g(g-1)}{(\boldsymbol{\beta}_{ij} \cdot \boldsymbol{y})^2}. \tag{2.45}$$

For the simplest $N = 2$ case, the relative Calogero model is reduced to a one-dimensional conformal mechanical system with the Hamiltonian

$$\tilde{\mathcal{H}}_0 = \frac{\tilde{p}^2}{2} + \frac{g(g-1)}{2y^2}. \tag{2.46a}$$

In the $N = 3$ case, it takes a more complicated form:

$$\tilde{\mathcal{H}}_0 = \frac{\tilde{p}_1^2 + \tilde{p}_2^2}{2} + \frac{g(g-1)}{2y_1^2} + \frac{2g(g-1)}{(\sqrt{3}y_2 + y_1)^2} + \frac{2g(g-1)}{(\sqrt{3}y_2 - y_1)^2}. \tag{2.46b}$$

In the general case, due to the conformal symmetry, the relative Hamiltonian in (2.44) can be split further into its radial and angular parts,

$$\tilde{\mathcal{H}}_0 = \frac{\tilde{\boldsymbol{p}}^2}{2} - \frac{\tilde{\mathcal{I}}(\varphi_\iota, p_{\varphi_a})}{y^2}, \tag{2.47}$$

where $\varphi_\iota$ are the angular coordinates, parametrizing the $(N-2)$-dimensional sphere, and $\tilde{\mathcal{I}}$ is the Hamiltonian of the angular Calogero model with the excluded center of mass (relative angular Calogero model). The latter system has been investigated in detail in the series of papers [9–11]. For the $N = 3$ case, it is reduced to a one-dimensional exactly solvable system on circle, introduced by Jacobi in the middle of the 19th century [21]:

$$\tilde{\mathcal{I}} = \frac{p_\varphi^2}{2} + \frac{9g(g-1)}{2\cos^2 3\varphi}. \tag{2.48}$$

For higher dimensions, the relative angular Calogero Hamiltonian has more complicate form (see Refs. [10,22]).

Now we are ready to transition to spherical coordinates. The azimuthal angle $\theta$ is chosen to measure the projection of the position vector to the relative coordinate's hyperplane:

$$x_0 = r\cos\theta, \qquad \boldsymbol{y} = r\sin\theta\tilde{\boldsymbol{n}}(\varphi_\iota). \tag{2.49}$$

Here the $(N-1)$-dimensional unit vector $\tilde{\boldsymbol{n}}$ is parametrized by the remaining angles $\varphi_\iota$ with $1 \le \iota \le N-2$. In these coordinates, the Hamiltonian (2.28) looks as follows:

$$\mathcal{H}_\gamma = -\frac{1}{2r^{N-1}}\frac{\partial}{\partial r}r^{N-1}\frac{\partial}{\partial r} + \frac{\mathcal{I}(\theta,\varphi_\iota,\partial_\theta,\partial_{\varphi_\iota})}{r^2} - \frac{\gamma}{r}. \tag{2.50}$$

The angular Calogero Hamiltonian $\mathcal{I}$ is related with the relative angular Hamiltonian $\tilde{\mathcal{I}}$ by the expression

$$\mathcal{I}(\theta,\varphi_\iota,\partial_\theta,\partial_{\varphi_\iota}) = \frac{\tilde{\mathcal{I}}(\varphi_\iota,\partial_{\varphi_\iota})}{\sin^2\theta} - \frac{1}{2\sin^{N-2}\theta}\frac{\partial}{\partial\theta}\sin^{N-2}\theta\frac{\partial}{\partial\theta}. \tag{2.51}$$

Both quantities $\mathcal{I}$ and $\tilde{\mathcal{I}}$ are constants of motion. They separate the radial $r$ and azimuthal $\theta$ coordinates, each being symmetric in the primary coordinates $x_i$, from the relative angular coordinates $\varphi_\iota$. Let us choose the factorized wave function:

$$\Psi(r,\theta,\varphi_\iota) = R(r)\Phi(\theta)\psi(\varphi_\iota). \tag{2.52}$$

It decouples the Schrödinger equation $\mathcal{H}_\gamma\Psi = E\Psi$ into the following system of differential equations:

$$\left(\frac{1}{r^{N-1}}\frac{d}{dr}r^{N-1}\frac{d}{dr} - \frac{l(l+N-2)}{r^2} + \frac{2\gamma}{r} + 2E_n\right) \times R_{n_r l}(r) = 0, \tag{2.53a}$$

$$\left(\frac{1}{\sin^{N-2}\theta}\frac{d}{d\theta}\sin^{N-2}\theta\frac{d}{d\theta} - \frac{\tilde{q}(\tilde{q}+N-3)}{\sin^2\theta} + l(l+N-2)\right)\Phi_l(\theta) = 0, \tag{2.53b}$$

$$(2\tilde{\mathcal{I}}(\varphi_\iota,\partial_{\varphi_\iota}) - \tilde{q}(\tilde{q}+N-3))\psi_{\tilde{q}}(\varphi_\iota) = 0. \tag{2.53c}$$

The last equation describes the spectrum and eigenstates of $\tilde{\mathcal{I}}$, which were recently investigated in detail in Ref. [9]. In particular, the spectrum is determined by the numbers

$$\tilde{q} = \frac{gN(N-1)}{2} + 3l_3 + \cdots + Nl_N \quad \text{with} \quad l_i = 0,1,2,\ldots. \tag{2.54}$$

For integer values of the coupling $g$, the angular energy spectrum is that of a free particle with angular momentum $\tilde{q}$ on the $(N-2)$-dimensional sphere, but it has a significantly lower degeneracy due to the restriction to the symmetric wave functions [9,22].

The first two equations above are similar to those for the $N$-dimensional Coulomb problem. The solutions of both equations can be found, for instance, in Ref. [23]. The second equation (2.53b) determines the spectrum and eigenfunction of the angular Hamiltonian $\mathcal{I}$. For integer coupling $g$, its spectrum coincides with that of a free particle with angular momentum $l$ on the $(N-1)$-dimensional sphere [9]. The first equation (2.53a) is related to the radial Schrödinger equation for the $N$-dimensional Coulomb problem.

For the positive *integer* values of the coupling constant, $g = 1,2,\ldots$, the solutions of Eq. (2.53b) exist, provided the quantum number $l$ takes the values

$$l = \tilde{q} + l_1 \quad \text{with} \quad l_1 = 0,1,2,\ldots, \tag{2.55}$$

so that $l \ge \frac{1}{2}N(N-1)g$.

Similarly, the well-defined solution of the radial equation (2.53a) exists for the energy spectrum depending on the principal quantum number $n$:

$$E_n = -\frac{\gamma^2}{2(n+\frac{N-3}{2})^2}, \qquad n = n_r + l + 1, \quad n_r = 0,1,2,\ldots. \tag{2.56}$$

Hence, the Calogero-Coulomb problem remains degenerate with respect to $q$. The energy spectrum of the system is the same as in the $N$-dimensional Coulomb problem, but with a lower degeneracy due to the restriction to the symmetric wave functions on the initial coordinates.

## III. COULOMB-CALOGERO-STARK PROBLEM

### A. Coulomb-Stark problem

It is well known that the $N$-dimensional Coulomb problem in a constant uniform electric field $\boldsymbol{F}$ remains integrable and admits the separation of variables in parabolic coordinates. Its Hamiltonian is defined by the expression

$$\mathcal{H}^0_{\gamma,F} = \frac{\boldsymbol{p}^2}{2} - \frac{\gamma}{r} + \boldsymbol{F}\cdot\boldsymbol{x}. \tag{3.1}$$

The perturbative spectrum of this system can be found in Refs. [1,23].

The Stark interaction breaks the $SO(N)$ rotation symmetry of the Coulomb problem down to the $SO(N-1)$ symmetry. The corresponding constants of motion are given by the components of the angular momentum (2.2) which are orthogonal to the direction of the electric field,

$$L^{\perp}_{ij} = L_{ij} + n_i n_k L_{jk} - n_j n_k L_{ik}. \tag{3.2}$$

Here $\boldsymbol{n}$ is the unit vector along the electric field:

$$\boldsymbol{F} = F\boldsymbol{n}, \qquad \boldsymbol{n}^2 = 1. \tag{3.3}$$

In coordinates, where the unit vector $\boldsymbol{n}$ is the $N$th basic vector, the perpendicular components (3.2) acquire the form of the $SO(N-1)$ algebra standard basis. Among them, one can choose the $N-1$ commuting constants of motion.

Apart from the angular momentum, the Coulomb-Stark system possesses a constant of motion, which is inherited from the $N$-dimensional Runge-Lenz vector (2.16). It reads (see Ref. [24] for the $N=3$ case)

$$A = \boldsymbol{n}\cdot\boldsymbol{A} - \frac{F}{2}(r^2 - (\boldsymbol{n}\cdot\boldsymbol{x})^2) \tag{3.4}$$

and complements the Liouville integrals to the full set.

### B. Integrable Hamiltonian of the Coulomb-Calogero-Stark system

The inclusion of the inverse-square Calogero potential breaks down the $SO(N)$ rotational symmetry of the initial Hamiltonian (2.1) to the discrete group containing the coordinate permutations (2.8b). Therefore, the different field directions $\boldsymbol{n}$ become nonequivalent. Among them, there is a preferable direction, which ensures the permutation invariance of the Stark term $\boldsymbol{F}\cdot\boldsymbol{x}$:

$$\boldsymbol{n} = \frac{1}{\sqrt{N}}(1,\ldots,1). \tag{3.5}$$

This is virtually the direction of the center-of-mass coordinate, so that in the Jacobi system [21], the Stark interaction takes the following form:

$$\boldsymbol{F}\cdot\boldsymbol{x} = Fx_0. \tag{3.6}$$

So the generalized Calogero-Coulomb Hamiltonian in an external electric field is set to

$$\begin{aligned}\mathcal{H}^{\rm gen}_{\gamma,F} &= \mathcal{H}^{\rm gen}_{\gamma} + Fx_0 = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma}{r} + Fx_0 \\ &= \frac{\boldsymbol{p}^2}{2} + \sum_{i<j}\frac{g(g-s_{ij})}{(x_i-x_j)^2} - \frac{\gamma}{r} + Fx_0.\end{aligned} \tag{3.7}$$

We see that the particular choice of the field direction (3.5) keeps the entire Hamiltonian invariant under the coordinate permutations:

$$[\mathcal{H}^{\rm gen}_{\gamma,F}, s_{ij}] = 0. \tag{3.8}$$

Therefore, it is well defined on the symmetric wave functions, where it is reduced to the Hamiltonian, as mentioned in the Introduction [see Eq. (1.2)]. We rewrite it here in a more compact form:

$$\mathcal{H}_{\gamma,F} = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j}^{N}\frac{g(g-1)}{(x_i-x_j)^2} - \frac{\gamma}{r} + Fx_0. \tag{3.9}$$

### C. Constants of motion for $\mathcal{H}^{\rm gen}_{\gamma}$

It turns out that the symmetry generators (3.2) and (3.4) to the Stark Hamiltonian (3.1) are extended straightforwardly to the generalized Calogero-Coulomb-Stark problem.

First, consider the invariants of the generalized Calogero Hamiltonian $\mathcal{H}^{\rm gen}_{\gamma}$. Substituting the highlighted direction (3.5) into the orthogonal angular momentum tensor (3.2), which describe the remaining $SO(N-1)$ symmetry, we immediately get

$$L^{\perp}_{ij} = L_{ij} + \frac{1}{N}\sum_k (L_{jk} - L_{ik}). \tag{3.10}$$

Of course, this definition remains valid for the Dunkl angular momentum (2.11) too. Below we will consider just this case. It is easy to verify the following commutation relations among the coordinates and Dunkl angular momenta:

$$[L_{ij}, x_l] = \imath x_j S_{il} - \imath x_i S_{jl}, \qquad [L_{ij}, x_0] = -\imath(x_i - x_j). \tag{3.11}$$

They ensure that the orthogonal components of the Dunkl angular momentum are constants of motion of the generalized Calogero-Coulomb-Stark problem:

$$[L^{\perp}_{ij}, \mathcal{H}^{\rm gen}_{\gamma,F}] = 0. \tag{3.12}$$

Alternatively, one can switch to the relative Jacobi coordinates and use the basis formed by the $\tilde{L}_{ij}$ (2.42b) for the description of the same algebra (3.10). Using the established identity $S_{0i} = \delta_{0i}$, it is easy to ensure an analog of the commutation relations (3.11):

$$[\tilde{L}_{ij}, y_l] = \imath y_j \tilde{S}_{il} - \imath y_i \tilde{S}_{jl}, \qquad [\tilde{L}_{ij}, y_0] = 0. \tag{3.13}$$

The above relation implies

$$[\tilde{L}_{ij}, \mathcal{H}^{\rm gen}_{\gamma,F}] = 0. \tag{3.14}$$

The deformed analog of the modified Runge-Lenz component along the field direction is a constant of motion, too,

$$A = \frac{1}{\sqrt{N}}\mathcal{A}_1 - \frac{F}{2}(r^2 - x_0^2) \tag{3.15}$$

$$= x_0\left(2\mathcal{H}^{\rm gen}_{\gamma,F} + \frac{\gamma}{r}\right) - \left(rp_r + \frac{N-1}{2\imath}\right)p_0 - \frac{F}{2}(r^2 + 3x_0^2), \tag{3.16}$$

where $\mathcal{A}_1 = \sum_i A_i$ according to the definition (2.26), and we have used the identity (2.30).

The proof is more complicated, and we present here some details of the derivation. Using the definitions (3.7), (3.15), and the already proven fact that the deformed Runge-Lenz vector is a constant of motion of the unperturbed Hamiltonian [see (2.18)], we obtain

$$[A, \mathcal{H}^{\rm gen}_{\gamma,F}] = \frac{F}{\sqrt{N}}[\mathcal{A}_1, x_0] + \frac{F}{4}[\boldsymbol{\pi}^2, r^2 - x_0^2]. \tag{3.17}$$

First, we calculate the second commutator from the right side of the above equation. Using a simple algebra and the relations (2.23), we get

$$[\boldsymbol{\pi}^2, r^2] = -2\imath(\boldsymbol{\pi}\cdot\boldsymbol{x} + \boldsymbol{x}\cdot\boldsymbol{\pi}) = -4\imath r p_r - 2N, \tag{3.18a}$$

$$[\boldsymbol{\pi}^2, x_0^2] = [p_0^2, x_0^2] = -4\imath x_0 p_0 - 2. \tag{3.18b}$$

Next, applying the identities (2.30) and (3.18b), the first commutator in Eq. (3.17) can be simplified as follows:

$$\frac{1}{\sqrt{N}}[\mathcal{A}_1, x_0] = x_0[\boldsymbol{\pi}^2, x_0] + \imath x_0 p_0 + \imath r p_r + \frac{N-1}{2} = \imath(r p_r - x_0 p_0) + \frac{N-1}{2}. \tag{3.19}$$

Finally, substituting the commutation relations (3.18a), (3.18b), and (3.19) into the equation (3.17), we get the desired conservation condition:

$$[A, \mathcal{H}^{\rm gen}_{\gamma,F}] = 0. \tag{3.20}$$

It is easy to check that the modified Runge-Lenz longitudinal invariant commutes with the deformed angular momentum restricted to the relative coordinates:

$$[\tilde{L}_{ij}, A] = 0 \quad \text{with} \quad 1 \le i, j \le N-1. \tag{3.21}$$

So we have proved that, in the presence of a constant uniform electric field, the generalized Calogero-Coulomb model (3.7) still remains an integrable system. The integrals of motion are deformations of those for the conventional Coulomb model with the Stark interaction by the Dunkl operators.

### D. Constants of motion of $\mathcal{H}_\gamma$

The integrals of the pure Calogero-Coulomb system (3.9), obtained by the restriction to the symmetric wave functions, must be symmetric too. Since the longitude component of the Runge-Lenz vector (3.15) obeys this condition, it remains as a correct integral for this system,

$$[A, \mathcal{H}_{\gamma,F}] = 0. \tag{3.22}$$

We should take symmetric expressions of the kinematical constants of motion, too, as in the absence of the electric field (2.25). For this purpose, it is more suitable to use the angular momentum in Jacobi coordinates (2.42a). Thus, we have

$$[\mathcal{H}_{\gamma,F}, \tilde{\mathcal{L}}_{2k}] = 0, \qquad \tilde{\mathcal{L}}_{2k} = \sum_{1\le i<j\le N-1} \tilde{L}_{ij}^{2k}. \tag{3.23}$$

The first member of this set is the square of the relative Dunkl angular momentum:

$$\tilde{\mathcal{L}}_2 = 2\tilde{\mathcal{I}} + S(S-N+2). \tag{3.24}$$

Here $\tilde{\mathcal{I}}$ is the angular part of the Calogero Hamiltonian with reduced center of mass (the relative angular Calogero Hamiltonian). It also forms a single Casimir element of the relative Dunkl angular momentum algebra [8]:

$$[\tilde{L}_{ij}, \tilde{\mathcal{I}}] = 0, \qquad \tilde{H}_0 = \frac{p_y^2}{2} - \imath\frac{N-2}{2}p_y + \frac{\tilde{\mathcal{I}}}{y^2}. \tag{3.25}$$

Thus, we have proved the integrability of the Calogero-Coulomb-Stark system.

### E. Separation of variables in parabolic coordinates

It is well known that the Coulomb-Stark system (3.1) admits separation of variables in parabolic coordinates, in which the Stark effect can be immediately calculated. Is this true for the Calogero-Coulomb-Stark system? Below we will show that the system admits complete separation of variables in parabolic coordinates for $N = 2$, 3 and partial separation for $N > 3$.

In the Jacobi coordinates (2.32a), the Calogero-Coulomb-Stark system acquires the following form,

$$\mathcal{H}_{\gamma,F} = \frac{p_0^2}{2} - \frac{\gamma}{\sqrt{x_0^2+y^2}} + F x_0 + \tilde{\mathcal{H}}_0, \tag{3.26}$$

where again the last term is the Calogero Hamiltonian with the reduced center of mass (2.45). From the spherical coordinates $(r, \theta, \varphi_\imath)$ (2.49), we move to the parabolic coordinates $(\xi, \eta, \varphi_\imath)$, using the map which leaves the relative angular variables $\varphi_\imath$ unaltered:

$$\xi = r + x_0, \qquad \eta = r - x_0 \quad \text{with} \quad x_0 = r\cos\theta. \tag{3.27}$$

The inverse transformation reads

$$x_0 = \frac{\xi-\eta}{2}, \qquad r = \frac{\xi+\eta}{2}, \qquad \boldsymbol{y} = \sqrt{\xi\eta}\,\tilde{\boldsymbol{n}}(\varphi_\imath). \tag{3.28}$$

In the new coordinates, the Hamiltonian (3.26) acquires the following form,

$$\mathcal{H}_{\gamma,F} = -\frac{2}{\xi+\eta}(\gamma + B_\xi + B_\eta) + \frac{\tilde{\mathcal{I}}}{\xi\eta} + \frac{F}{2}(\xi-\eta), \quad (3.29)$$

where

$$B_\xi = \frac{1}{\xi^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}\xi^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \quad (3.30)$$

We further proceed by extending straightforwardly the steps applied for the usual Coulomb system in an external field in Ref. [25]. Employing the following ansatz to the total wave function,

$$\Psi(\xi,\eta,\varphi_\iota) = \Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota), \quad (3.31)$$

we decouple the Schrödinger equation $\mathcal{H}_{\gamma,F}\Psi = E\Psi$ into three parts. Two of them depend on $\xi$ and $\eta$, respectively, and replace the aforementioned equations (2.53a) and (2.53b) for the Calogero-Coulomb system in the spherical coordinates,

$$\left(B_\xi + \frac{E}{2}\xi - \frac{F}{4}\xi^2 - \frac{\tilde{q}(\tilde{q}+N-3)}{4\xi} + \lambda_1\right)\Phi_1(\xi) = 0, \quad (3.32a)$$

$$\left(B_\eta + \frac{E}{2}\eta + \frac{F}{4}\eta^2 - \frac{\tilde{q}(\tilde{q}+N-3)}{4\eta} + \lambda_2\right)\Phi_2(\eta) = 0. \quad (3.32b)$$

Here the Coulomb charge is fractioned into two parts,

$$\lambda_1 + \lambda_2 = \gamma. \quad (3.32c)$$

The third equation determines the eigenstate of the algular Hamiltonian of the relative Calogero model with the eigenvalue $\tilde{I}_q$ given by (2.54). It remains unchanged and is given by Eq. (2.53c).

Evidently, in Eqs. (3.32a) and (3.32b), the entire state (3.31) can be used instead of the partial wave functions. So, substituting $\Phi_{1,2} \to \Psi$, one can eliminate $E$ by subtracting from the first equation multiplied by $\eta$, and from the second one multiplied by $\xi$. This manipulation together with Eq. (3.32c) yields the eigenstate equation for the modified Runge-Lenz invariant for the Calogero-Coulomb-Startk problem (3.15) in parabolic coordinates,

$$A\Psi = (\lambda_2 - \lambda_1)\Psi, \quad (3.33)$$

$$A = \frac{2}{\xi+\eta}(\eta B_\xi - \xi B_\eta) + \frac{\xi-\eta}{\xi\eta}\tilde{\mathcal{I}} - \frac{\xi-\eta}{\xi+\eta}\gamma - \frac{F}{2}\xi\eta. \quad (3.34)$$

Therefore, the invariant $A$ is responsible for the separation of the variables $\xi$ and $\eta$ in the parabolic coordinate system. It is used instead of the angular Hamiltonian $\mathcal{I}$, which separates the radial and azimuthal variables in the spherical coordinates. The second invariant, given by the relative angular Hamiltonian $\tilde{\mathcal{I}}$, is common in both cases and separates the relative angular degrees of freedom.

In the absence of the electric field, $F = 0$, the fractional Coulomb charges $\lambda_i$ with $i = 1, 2$ take discrete values depending on the parabolic quantum numbers $n_i = 0, 1, 2, \ldots$:

$$\lambda_i = \gamma\frac{n_i + \frac{1}{2}(\tilde{q}+1)}{n + \frac{1}{2}(N-3)}. \quad (3.35)$$

They also specify the partial wave functions $\Phi_1(\xi) = \Phi_{n_1\tilde{q}}(\xi)$ and $\Phi_2(\eta) = \Phi_{n_2\tilde{q}}(\eta)$. The principal quantum number, which characterizes the spectrum (2.56), is expressed now via parabolic quantum numbers:

$$n = n_1 + n_2 + \tilde{q} + 1. \quad (3.36)$$

As in the usual Coulomb problem [1], the electric field completely removes the degeneracy in the orbital momentum but preserves the degeneracy with respect to $q$. Using the similarity between the Calogero-Coulomb-Stark and $N$-dimensional Coulomb-Stark problems, we can immediately write down the first-order energy correction caused by the weak electric field $F$ in perturbation theory:

$$E_n^{(1)} = \frac{3}{2}\left(n + \frac{N-3}{2}\right)(n_1 - n_2)F. \quad (3.37)$$

Hence, as in the spherical case, inclusion of the Calogero term preserves the perturbative spectrum of the system, but changes its degree of degeneracy.

## IV. TWO-CENTER CALOGERO-COULOMB SYSTEM

### A. Two-center Coulomb system

As is known, the two-center Coulomb problem is integrable in any dimension [12,26]. Let us locate the two central Coulomb charges $\gamma_{1,2}$ at the points $\boldsymbol{x}_1 = \boldsymbol{a}$ and $\boldsymbol{x}_2 = -\boldsymbol{a}$. The distances from them to the given space point are denoted by $r_1$ and $r_2$, respectively. Then the Hamiltonian acquires the following form:

$$\mathcal{H}^0_{\gamma_1,\gamma_2} = \frac{\boldsymbol{p}^2}{2} - \frac{\gamma_1}{|\boldsymbol{x}-\boldsymbol{a}|} - \frac{\gamma_2}{|\boldsymbol{x}+\boldsymbol{a}|} = \frac{\boldsymbol{p}^2}{2} - \frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}. \quad (4.1)$$

The Hamiltonian (4.1) possesses the same spacial symmetry as the Stark problem described by the Hamiltonian (3.1). It remains invariant under the $SO(N-1)$ rotations,

which preserve the line containing both charges. Let us denote a unit vector along this line by $\boldsymbol{n}$:

$$\boldsymbol{a} = a\boldsymbol{n}, \qquad \boldsymbol{n}^2 = 1. \tag{4.2}$$

According to the aforementioned symmetry, the orthogonal components of the angular momentum tensor $L^{\perp}_{ij}$, given by (3.2), are preserved. They provide the system with $N-1$ commuting integrals of motion.

The remaining $N$th constant of motion, like in the case of the Stark system, is provided by the suitably modified longitudinal component of the Runge-Lenz vector [12]:

$$A = \boldsymbol{L}^2 + (\boldsymbol{a}\cdot\boldsymbol{p})^2 - 2(\boldsymbol{a}\cdot\boldsymbol{x})\left(\frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}\right). \tag{4.3}$$

In three dimensions, a close expression for this integral had been obtained before [27]. (See Ref. [28] for relations between both integrals.)

It is easy to see that the above quantity commutes with $L^{\perp}_{ij}$.

### B. Two-center Coulomb system with Calogero term

Consider now this system in the presence of the inverse-square Calogero potential. In order to construct the Hamiltonian of this system, we should replace the momenta operators by the Dunkl momenta, as in previous sections, and then restrict the Hamiltonian to the symmetric wave function. In order to provide the system with permutation symmetry, like for the Stark Hamiltonian, we choose the line, connecting two Coulomb charges, to be directed along the center-of-mass coordinate (3.5). Hence, in the Jacobi coordinates (2.32a), the distances to the charges are given by

$$\begin{aligned} r_1 &= |\boldsymbol{x} - \boldsymbol{a}| = \sqrt{y^2 + (x_0 - a)^2}, \\ r_2 &= |\boldsymbol{x} + \boldsymbol{a}| = \sqrt{y^2 + (x_0 + a)^2}. \end{aligned} \tag{4.4}$$

The generalized two-center Calogero-Coulomb Hamiltonian is

$$\mathcal{H}^{\mathrm{gen}}_{\gamma_1,\gamma_2} = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}. \tag{4.5}$$

The particular choice of the direction of the vector $\boldsymbol{a}$ guarantees its permutation invariance. On the symmetric wave functions, it gives rise to the two-center Calogero-Coulomb system defined in the Introduction (1.3), which we present in a compact form,

$$\mathcal{H}_{\gamma_1,\gamma_2} = \frac{\boldsymbol{p}^2}{2} + \sum_{i<j}^{N} \frac{g(g-1)}{(x_i - x_j)^2} - \frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}. \tag{4.6}$$

Like the Calogero-Coulomb-Stark Hamiltonian, it possesses the symmetry given by the deformed angular momentum generators perpendicular to the predefined direction (3.10).

It turns out that the modified Runge-Lenz integral (4.3) of the $g = 0$ Hamiltonian (4.1) is extended straightforwardly to the current case:

$$A = \mathcal{L}_2 + a^2 p_0^2 - 2a x_0 \left(\frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}\right). \tag{4.7}$$

The proof is based on the extension of the elegant trick performed in Ref. [12], where it is shown that the Hamiltonian (4.1) preserves the modified Runge-Lenz component (4.3).

First, we define the generalized Calogero-Coulomb Hamiltonians (2.7) with the shifted central charges:

$$\mathcal{H}^{\mathrm{gen}}_{\gamma_1} = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma_1}{r_1}, \qquad \mathcal{H}^{\mathrm{gen}}_{\gamma_2} = \frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma_2}{r_2}. \tag{4.8}$$

Then we combine, in a proper way, the symmetries of both Hamiltonians, which we have already discussed, in order to establish the conservation of the element (4.7) for the Hamiltonian (4.6).

Each Hamiltonian (4.8) preserves the deformed angular generators, defined with respect to their Coulomb centers. Note that the Dunkl operators (2.8a) remain unchanged under the coordinate shift $\boldsymbol{x} \to \boldsymbol{x} \mp \boldsymbol{a}$. In turn, the deformed angular momenta are transformed under the shift of the original point according to the standard rule. For the first Hamiltonian, it reads as

$$\begin{aligned} L^{\gamma_1}_{ij} &= \left(x_i - \frac{a}{\sqrt{N}}\right)\pi_j - \left(x_j - \frac{a}{\sqrt{N}}\right)\pi_i \\ &= L_{ij} + \frac{a}{\sqrt{N}}(\pi_i - \pi_j). \end{aligned} \tag{4.9}$$

This provider the transformation rule for the deformed angular momentum square $\mathcal{L}_2$, defined in (2.25):

$$\begin{aligned} \mathcal{L}^{\gamma_1}_2 &= \sum_{i<j} (L^{\gamma_1}_{ij})^2 \\ &= \mathcal{L}_2 + \frac{a}{2\sqrt{N}} \sum_{i,j} \{L_{ij}, (\pi_i - \pi_j)\} + \frac{a^2}{2N} \sum_{i,j} (\pi_i - \pi_j)^2 \\ &= \mathcal{L}_2 - \frac{a}{\sqrt{N}} \sum_{i,j} \{L_{ij}, \pi_j\} + a^2(\boldsymbol{\pi}^2 - p_0^2). \end{aligned} \tag{4.10}$$

The second term enters also in the expression of the longitudinal component of the deformed Runge-Lenz vector $\mathcal{A}_1$ (2.30). It behaves under the coordinate shift in the following way:

$$\sum_{i,j}\{L_{ij}^{\gamma_1},\pi_j\} = \sum_{i,j}\{L_{ij},\pi_j\} + \frac{a}{\sqrt{N}}\sum_{i,j}(\pi_i-\pi_j)\pi_j = \sum_{i,j}\{L_{ij},\pi_j\} - \sqrt{N}a(\boldsymbol{\pi}^2-p_0^2). \tag{4.11}$$

Hence, the entire component behaves under the coordinate shift as

$$\mathcal{A}_1^{\gamma_1} = \frac{1}{2}\sum_{i,j}\{L_{ij},\pi_j\} - \sqrt{N}a(\boldsymbol{\pi}^2-p_0^2) - \sqrt{N}\gamma_1\frac{x_0-a}{r_1}. \tag{4.12}$$

Take now a combination of these two invariants of the Hamiltonian $\mathcal{H}_{\gamma_1}^{\text{gen}}$, in which the term with the anticommutator is eliminated:

$$\mathcal{L}_2^{\gamma_1} + \frac{2a}{\sqrt{N}}\mathcal{A}_1^{\gamma_1} = \mathcal{L}_2 - a^2(\boldsymbol{\pi}^2-p_0^2) - 2a\gamma_1\frac{x_0-a}{r_1}. \tag{4.13}$$

Substituting the expressions from (4.8), (4.10), and (4.12) into the conservation condition

$$\left[\mathcal{H}_{\gamma_1}^{\text{gen}}, \mathcal{L}_2^{\gamma_1} + \frac{2a}{\sqrt{N}}\mathcal{A}_1^{\gamma_1}\right] = 0, \tag{4.14}$$

and canceling out the terms containing the $\boldsymbol{\pi}^2$ on the right part of the commutator, we come to the equation

$$\left[\frac{\boldsymbol{\pi}^2}{2} - \frac{\gamma_1}{r_1}, \mathcal{L}_2 + a^2p_0^2 - \frac{2a\gamma_1x_0}{r_1}\right] = 0. \tag{4.15}$$

Since the square of the Dunkl momentum commutes with the deformed angular momentum (2.14), we have

$$[\boldsymbol{\pi}^2, \mathcal{L}_2] = 0. \tag{4.16}$$

This simplifies the commutator in the left side of the relation (4.15), and we can replace it by

$$\left[\frac{\boldsymbol{\pi}^2}{2}, \frac{2a\gamma_1x_0}{r_1}\right] + \left[\frac{\gamma_1}{r_1}, \mathcal{L}_2 + a^2p_0^2\right] = 0. \tag{4.17}$$

Remember now that the two Hamiltonians in Eqs. (4.8) are distinct from each other by the sign of the shift parameter and the values of the Coulomb coupling:

$$\mathcal{H}_{g,\gamma_2}^{\text{gen}} = \mathcal{H}_{g,\gamma_1}^{\text{gen}}|_{a\to-a,\gamma_1\to\gamma_2}. \tag{4.18}$$

Therefore, an analogue of the equation (4.17) for $\mathcal{H}_{g,\gamma_2}^{\text{gen}}$ would be

$$\left[\frac{\boldsymbol{\pi}^2}{2}, -\frac{2a\gamma_2x_0}{r_2}\right] + \left[\frac{\gamma_2}{r_2}, \mathcal{L}_2 + a^2p_0^2\right] = 0. \tag{4.19}$$

The sum of both equations produces the relation

$$\left[\frac{\boldsymbol{\pi}^2}{2}, 2ax_0\left(\frac{\gamma_1}{r_1}-\frac{\gamma_2}{r_2}\right)\right] + \left[\frac{\gamma_1}{r_1}+\frac{\gamma_2}{r_2}, \mathcal{L}_2 + a^2p_0^2\right] = 0. \tag{4.20}$$

Again, using the relation (4.16), we conclude that the above equation is equivalent to the conservation condition for the modified Runge-Lenz vector's component along the symmetry axis of the system (4.7):

$$[\mathcal{H}_{\gamma_1,\gamma_2}^{\text{gen}}, A] = 0. \tag{4.21}$$

### C. Separation of variables in elliptic coordinates

Now let us show that in complete analogy with the previous case, the two-center Calogero-Coulomb system (4.6) admits complete separation of variables in the elliptic coordinates for $N = 2, 3$ and partial separation for $N > 3$.

The map from the Jacobi variables $(x_0, y)$ to the elliptic coordinates $(\xi,\eta)$ looks as follows (similar to the usual hydrogen atom case [25,29]):

$$\xi = \frac{r_1+r_2}{2a}, \qquad \eta = \frac{r_1-r_2}{2a}, \tag{4.22}$$

where $r_i$ is the distance from the $i$th Coulomb charge (4.4). The relative angles $\varphi_i$ remain unchanged. The new coordinates belong to the regions $\xi \geq 1$ and $-1 \leq \eta \leq 1$. The inverse transformation reads

$$x_0 = -a\xi\eta, \qquad y = a\sqrt{(\xi^2-1)(1-\eta^2)}. \tag{4.23}$$

The two-center Calogero-Coulomb Hamiltonian (4.6) in elliptic coordinates reads

$$\mathcal{H}_{\gamma_1,\gamma_2} = \frac{1}{2a^2(\xi^2-\eta^2)}(B_\eta - B_\xi) + \frac{\tilde{\mathcal{I}}(\varphi_\iota,\partial_{\varphi_\iota})}{a^2(\xi^2-1)(1-\eta^2)} - \frac{\gamma_1}{a(\xi+\eta)} - \frac{\gamma_2}{a(\xi-\eta)}, \tag{4.24}$$

where the operator $B_\xi$ from the kinetic energy part acquires the following form:

$$B_\xi = \frac{1}{(\xi^2-1)^{\frac{N-3}{2}}}\frac{\partial}{\partial\xi}(\xi^2-1)^{\frac{N-1}{2}}\frac{\partial}{\partial\xi}. \tag{4.25}$$

Then, choosing the wave function,

$$\Psi(\xi,\eta,\varphi_\iota) = \Phi_1(\xi)\Phi_2(\eta)\psi(\varphi_\iota), \tag{4.26}$$

we can (partially) separate the variables in the Schrödinger equation $\mathcal{H}_{\gamma_1,\gamma_2}\Psi = E\Psi$ into the following parts:

$$\left(B_\xi - \frac{\tilde{q}(\tilde{q}+N-3)}{\xi^2-1} + 2a(\gamma_1+\gamma_2)\xi + 2a^2E\xi^2 - \lambda\right)\Phi_1(\xi) = 0, \tag{4.27a}$$

$$\left(B_\eta - \frac{\tilde{q}(\tilde{q}+N-3)}{\eta^2-1} + 2a(\gamma_1-\gamma_2)\eta + 2a^2E\eta^2 - \lambda\right)\Phi_2(\eta) = 0, \tag{4.27b}$$

$$(2\tilde{\mathcal{I}}(\varphi_\iota, \partial_{\varphi_\iota}) - \tilde{q}(\tilde{q}+N-3))\psi_{\tilde{q}}(\varphi_\iota) = 0. \tag{4.27c}$$

The third equation, which appears also in the spherical and parabolic cases (2.53c), describes the energy eigenstates of the relative angular Calogero Hamiltonian and its spectrum, depending on the composite quantum number $\tilde{q}$ (2.54). In the absence of the Calogero term, it determines the spectrum and energy states of a free particle system on a $(N-2)$-dimensional sphere.

Obviously, the partial states $\Phi_{1,2}$ in the first two equations depend on the energy level $E$ and the $\tilde{q}$. The parameter $\lambda$ in the first two equations separates the variables $\xi$ and $\eta$. It coincides with the eigenvalue of the slightly redefined Runge-Lenz invariant for the two-center Calogero-Coulomb system (4.7) with the Dunkl angular momentum square replaced by the doubled angular Calogero Hamiltonian,

$$A = 2\mathcal{I} + a^2p_0^2 - 2ax_0\left(\frac{\gamma_1}{r_1} - \frac{\gamma_2}{r_2}\right). \tag{4.28}$$

The redefinition neglects the nonessential permutational invariant $S$ term in Eq. (2.27). In the $g=0$ limit, both integrals (4.7) and (4.28) become identical. In elliptic coordinates, the above formula looks like

$$A = \frac{1}{\xi^2-\eta^2}(\xi^2B_\eta - \eta^2B_\xi) + \frac{2(\xi^2+\eta^2-1)}{(\xi^2-1)(1-\eta^2)}\tilde{\mathcal{I}}(\varphi_\iota, \partial_{\varphi_\iota}) + 2a\xi\eta\left(\frac{\gamma_1}{\xi+\eta} - \frac{\gamma_2}{\xi-\eta}\right). \tag{4.29}$$

We follow the steps made above for the parabolic case. First, use the total wave function $\Psi$ instead of the partial ones, $\Phi_{1,2}$, in Eqs. (4.27a) and (4.27b). Next, cancel out the energy $E$ by taking appropriate combinations of both equations. This yields the eigenstate equation for the modified Runge-Lenz invariant (4.29),

$$A\Psi = \lambda\Psi. \tag{4.30}$$

It separates the two equations (4.27a) and (4.27b) from each other, as in the absence of the Calogero potential [25].

## V. CONCLUSION

In the present paper, we suggested new integrable generalizations of the rational Calogero model extended by (i) the Coulomb potential with the Stark term and (ii) the two-center Coulomb potential. We also found explicit expressions of complete sets of their constants of motion formulated in terms of the Dunkl operators. We demonstrated that these systems admit partial separation of variables in parabolic and elliptic coordinates, similar to conventional Coulomb-Stark and two-center Coulomb problems. This allows us to conclude that the suggested Calogero extensions preserve the spectra of initial systems but change the degree of degeneracy. As an illustration, we have written the first-order energy correction of the spectrum of the Calogero-Coulomb Hamiltonian caused by an electric field.

The physical significance of the Coulomb-Stark and two-center Coulomb systems has been well known since (at least) the 19th century. The importance of the Calogero model does not raise any doubts either. The integrable synthesis, proposed in our paper, significantly extends its area of application. The study of some related systems is still in order. First of all, it is unclear whether similar integrable systems associated with an arbitrary root exist and, if so, how their hidden constants of motion look. Even so, we have no doubts that such generalizations will not admit the partial separation of variables. We intend to investigate these problems in a separate work.

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