

ELEMENTARY PARTICLES AND FIELDS
Theory

Lobachevsky Geometry in TTW and PW Systems*

T. Hakobyan^{1),2)}, A. Nersessian^{1),2)***}, and H. Shmavonyan^{1)****}**

Received May 18, 2016

Abstract—We review the classical properties of Tremblay–Turbiner–Winternitz and Post–Winternitz systems and their relation with N -dimensional rational Calogero model with oscillator and Coulomb potentials, paying special attention to their hidden symmetries. Then we show that combining the radial coordinate and momentum in a single complex coordinate in a proper way, we get an elegant description for the hidden and dynamical symmetries in these systems related with action–angle variables.

DOI: 10.1134/S1063778817030085

1. PRELIMINARIES

The rational Calogero model and its generalizations, based on arbitrary Coxeter root systems, are highlighted among the non-trivial unbound superintegrable systems. Recall that the superintegrability of N -dimensional integrable system means that it possesses $2N - 1$ functionally independent constants of motion. This property was established for the classical [1] and quantum [2, 3] rational Calogero model, which is described by the Hamiltonian [4]

$$H_0 = \sum_{i=1}^N \frac{p_i^2}{2} + \sum_{i < j} \frac{g^2}{(x_i - x_j)^2}. \quad (1)$$

Its generalization, associated with an arbitrary finite Coxeter group, is defined by the Hamiltonian [5]

$$H_0 = \sum_{i=1}^N \frac{p_i^2}{2} + \sum_{\alpha \in \mathcal{R}_+} \frac{g_\alpha^2 (\alpha \cdot x)^2}{2(\alpha \cdot x)^2}. \quad (2)$$

Let us mention that the Coxeter group is described as a finite group generated by a set of orthogonal reflections across the hyperplanes $\alpha \cdot x = 0$ in the N -dimensional Euclidean space, where the vectors α from the set \mathcal{R}_+ (called the system of positive roots) uniquely characterize the reflections. The coupling constants g_α form a reflection-invariant discrete function. The original Calogero potential in (1) corresponds to the A_{N-1} Coxeter system with the

positive roots, defined in terms of the standard basis by $\alpha_{ij} = e_i - e_j$ for $i < j$. The reflections become the coordinate permutations in this particular case.

An important feature of all the rational Calogero models is the dynamical conformal symmetry $so(1, 2) \equiv sl(2, R)$, defined by the Hamiltonian H_0 together with the dilatation D and conformal boost K generators,

$$\begin{aligned} \{H_0, D\} &= 2H_0, & \{H_0, K\} &= D, \\ \{K, D\} &= -2K. \end{aligned} \quad (3)$$

This symmetry separates the radial and angular parts as follows:

$$\begin{aligned} H_0 &= \frac{p_r^2}{2} + \frac{\mathcal{I}(u)}{r^2}, & r &\equiv \sqrt{2K}, \\ p_r &\equiv \frac{D}{\sqrt{2K}}, \end{aligned} \quad (4)$$

where

$$\begin{aligned} \{p_r, r\} &= 1, & \{p_r, u^\alpha\} &= \{r, u^\alpha\} = 0, \\ \{u^\alpha, u^\beta\} &= \omega^{\alpha\beta}(u). \end{aligned} \quad (5)$$

Hence, the whole information about the rational Calogero model (and, more generally, any conformal mechanics) is encoded in its “spherical part”, given by the Hamiltonian $\mathcal{I}(u)$ (which corresponds to the Casimir element of the conformal algebra). The “angular Calogero model” given by the Hamiltonian \mathcal{I} was studied from the various viewpoints in [6–8]. Apart from its inherent interest, this system provides the rational Calogero models with an elegant explanation of maximal superintegrability [9]. This property is retained for other root systems, and persists in the presence of an additional oscillator potential (Calogero–oscillator system). Moreover, by the separation of the radial and angular parts, we have recently established the superintegrability for the rational Calogero model with Coulomb

*The text was submitted by the authors in English.

¹⁾Yerevan State University, 1 Alex Manoogian St., Yerevan, 0025, Armenia.

²⁾Tomsk Polytechnic University, Lenin Ave. 30, 634050 Tomsk, Russia.

**E-mail: tigran.hakobyan@ysu.am

***E-mail: arnerses@ysu.am

****E-mail: hovhannes.shmavonyan@ysuamail.am

potential (Calogero–Coulomb system), as well as for the Calogero–oscillator and Calogero–Coulomb systems on sphere and hyperboloid. Similar statements are valid for the Calogero-like models associated with arbitrary root systems. In case of A_N Calogero–Coulomb model, we have presented an explicit expression for the analog of Runge–Lenz vector [10], and revealed its integrable generalizations for the two-center Coulomb (two-center Calogero–Coulomb) and Stark (Calogero–Coulomb–Stark) potentials [11].

The Trembley–Turbiner–Wintenz (TTW) system, invented a few years ago [12], is a particular case of the Calogero–oscillator system. It is defined by the Hamiltonian of two-dimensional oscillator, with the angular part replaced by a Pöschl–Teller system on circle:

$$\mathcal{H}_{\text{TTW}} = \frac{p_r^2}{2} + \frac{\mathcal{I}_{\text{PT}}}{r^2} + \frac{\omega^2 r^2}{2}, \tag{6}$$

$$\mathcal{I}_{\text{PT}} = \frac{p_\varphi^2}{2} + \frac{k^2 \alpha^2}{\sin^2 k\varphi} + \frac{k^2 \beta^2}{\sin^2 k\varphi}, \tag{7}$$

where k is an integer. It coincides with the two-dimensional rational Calogero–oscillator model associated with the dihedral group D_{2k} [7] and was initially considered as a new superintegrable model. The superintegrability was observed by numerical simulations. Later an analytic expression for the additional constant of motion was presented [13].

The two-dimensional Calogero–Coulomb system, associated with dihedral group, is known as a Post–Wintenz (PW) system. It was constructed from the TTW system by performing the well-known Levi–Civita transformation, which maps the two-dimensional oscillator into the Coulomb problem [14]. The PW system was also suggested as a new (independent from Calogero) superintegrable model. It is also expressed via the Pöschl–Teller Hamiltonian (7),

$$\mathcal{H}_{\text{PW}} = \frac{p_r^2}{2} + \frac{\mathcal{I}_{\text{PT}}}{r^2} - \frac{\gamma}{r}. \tag{8}$$

In [15], the superintegrability of the TTW system was explained from the viewpoint of action–angle variable formulation, while in [16], using the same (action–angle) arguments, the superintegrable generalizations of the TTW and PW systems on sphere and hyperboloid were suggested. Below we briefly describe the constructions.

Consider an integrable N -dimensional system with the following Hamiltonian in action–angle variables:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}(nI_1 + mI_2, I_3, \dots, I_N), \\ \{I_i, \Phi_j\} &= \delta_{ij}, \quad \Phi_i \in [0, 2\pi), \end{aligned} \tag{9}$$

where n and m are integers. The Liouville integrals are expressed via the action variables I_i . The system has a hidden symmetry, given by the additional constant of motion

$$K_{\text{hidden}} = \text{Re}A(I_i)e^{i(m\Phi_1 - n\Phi_2)}, \tag{10}$$

where $A(I_i)$ is an arbitrary complex function on Liouville integrals. Respectively, for the Hamiltonian

$$\mathcal{H} = \mathcal{H}(n_1I_1 + n_2I_2 + \dots + n_NI_N), \tag{11}$$

where n_1, \dots, n_N are integer numbers, all the functions

$$K_{ij} = \text{Re}A_{ij}(I)e^{i(n_j\Phi_i - n_i\Phi_j)} \tag{12}$$

are constants of motion, which are distinct from the Liouville integrals. The Liouville integrals together with the additional integrals I_{i+1} with $i = 1, \dots, N - 1$ constitute a set of $2N - 1$ functionally independent constants of motion, ensuring the maximal superintegrability.

In [16] the integrable deformations of the N -dimensional oscillator and Coulomb systems have been proposed on Euclidean space, sphere, and hyperboloid by replacing their angular part by an $(N - 1)$ -dimensional integrable system, formulated in action–angle variables:

$$\begin{aligned} H &= \frac{p_r^2}{2} + \frac{\mathcal{I}(I_a)}{r^2} + V(r), \\ \{p_r, r\} &= 1, \quad \{I_a, \Phi_b^0\} = \delta_{ab}, \end{aligned} \tag{13}$$

where $a, b = 1, \dots, N - 1$ and

$$V_{\text{osc}}(r) = \frac{\omega^2 r^2}{2}, \quad V_{\text{Coul}}(r) = -\frac{\gamma}{r}. \tag{14}$$

In other words, we obtain the deformation of the N -dimensional oscillator and Coulomb systems by replacing the $SO(N)$ quadratic Casimir element \mathbf{J}^2 , which defines the kinetic part of the system on sphere \mathbb{S}^{N-1} , with the Hamiltonian of some $(N - 1)$ -dimensional integrable system written in terms of the action–angle variables.

Next, we have performed a similar analyses for the systems on N -dimensional sphere and (two-sheet) hyperboloid with the oscillator and Coulomb potentials. These models were introduced, respectively, by Higgs [17] and Schrödinger [18],

$$\begin{aligned} \mathbb{S}^N : \quad H &= \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{I}}{r_0^2 \sin^2 \chi} + V(\tan \chi), \\ \{p_\chi, \chi\} &= 1, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathbb{H}^N : \quad H &= \frac{p_\chi^2}{2r_0^2} + \frac{\mathcal{I}}{r_0^2 \sinh^2 \chi} + V(\tanh \chi), \\ \{p_r, r\} &= 1, \end{aligned} \tag{16}$$

with \mathcal{I} depending on the (angular) action variables. The exact forms for the potential are:

$$\begin{aligned} \mathbb{S}^N : \quad V_{\text{Higgs}}(\tan \chi) &= \frac{r_0^2 \omega^2 \tan^2 \chi}{2}, \\ V_{\text{Sch-Coul}}(\tan \chi) &= -\frac{\gamma}{r_0} \cot \chi, \end{aligned} \quad (17)$$

$$\begin{aligned} \mathbb{H}^N : \quad V_{\text{Higgs}}(\tanh \chi) &= \frac{r_0^2 \omega^2 \tanh^2 \chi}{2}, \\ V_{\text{Sch-Coul}}(\tanh \chi) &= -\frac{\gamma}{r_0} \coth \chi. \end{aligned} \quad (18)$$

The following expressions for the Hamiltonians of oscillator-like systems had been derived:

$$\begin{aligned} \mathcal{H}_{\text{osc}} &= \mathcal{H}_{\text{osc}}(2I_r + \sqrt{2\mathcal{I}}) \\ &= \begin{cases} \omega(2I_r + \sqrt{2\mathcal{I}}) \text{ for } \mathbb{R}^N, \\ \frac{1}{2}(2I_\chi + \sqrt{2\mathcal{I}} + \omega)^2 - \frac{\omega^2}{2} \text{ for } \mathbb{S}^N, \\ -\frac{1}{2}(2I_\chi + \sqrt{2\mathcal{I}} - \omega)^2 + \frac{\omega^2}{2} \text{ for } \mathbb{H}^N. \end{cases} \end{aligned} \quad (19)$$

Respectively, the Hamiltonians of the Coulomb-like systems read:

$$\begin{aligned} \mathcal{H}_{\text{Coul}} &= \mathcal{H}_{\text{Coul}}(I_r + \sqrt{2\mathcal{I}}) \\ &= \begin{cases} -\frac{\gamma^2}{2}(I_r + \sqrt{2\mathcal{I}})^2 \text{ for } \mathbb{R}^N, \\ -\frac{\gamma^2}{2}(I_\chi + \sqrt{2\mathcal{I}})^2 + \frac{1}{2}(I_\chi + \sqrt{2\mathcal{I}})^2 \text{ for } \mathbb{S}^N, \\ -\frac{\gamma^2}{2}(I_\chi - \sqrt{2\mathcal{I}})^2 - \frac{1}{2}(I_\chi - \sqrt{2\mathcal{I}})^2 \text{ for } \mathbb{H}^N. \end{cases} \end{aligned} \quad (20)$$

Thus, it is easy to deduce that for the angular Hamiltonian

$$\begin{aligned} \mathcal{I}_{\text{Sph-Calogero}} &= \frac{1}{2} \left(\sum_{a=1}^{N-1} k_a I_a + \text{const} \right)^2, \\ k_a &\in \mathbb{N}, \end{aligned} \quad (21)$$

the deformations of the oscillator and Coulomb systems become superintegrable. In particular, the Pöschl–Teller Hamiltonian has the same form [7]:

$$\mathcal{I}_{\text{PT}} = \frac{k^2(I + \alpha + \beta)^2}{2}. \quad (22)$$

Hence, choosing $N = 2$ and $\mathcal{I} = \mathcal{I}_{\text{PT}}$, we obtain the generalizations of the TTW and PW systems on sphere and hyperboloid with additional constants of motion given by

$$\begin{aligned} \mathcal{K}_{\text{TTW}} &= \text{Re}A(I)e^{i(k\Phi_r - 2\Phi_\varphi)}, \\ \mathcal{K}_{\text{PW}} &= \text{Re}A(I)e^{i(k\Phi_r - \Phi_\varphi)}. \end{aligned} \quad (23)$$

Here Φ_φ is the angle variable in the Pöschl–Teller system, and Φ_r is the angle variable associated with r and p_r . For explicit expressions, see [16].

Note that the angular part of the N -dimensional rational Calogero model has the form (21) as well. This is a reason for the superintegrability of the

Calogero–oscillator and Calogero–Coulomb problems. It also suggests their superintegrable generalizations on the N -dimensional spheres and hyperboloids [6]. Although the TTW and PW systems are particular cases of the Calogero-type models, they continue to attract enough interest due to their simplicity. In particular, a couple of years ago, Ranada suggested a specific representation for the constants of motion of the TTW and PW systems (including those on sphere and hyperboloid) [19], called a “holomorphic factorization”. For the TTW system it reads

$$\mathcal{R}_{\text{TTW}} = (\bar{M}_0)^k N^2, \quad (24)$$

where

$$M_0 = \frac{2p_r}{r} \sqrt{2\mathcal{I}_{\text{PT}}} + 2i\mathcal{H}_{\text{TTW}}, \quad (25)$$

and

$$\begin{aligned} N &= k(\beta - \alpha) + 2\mathcal{I}_{\text{PT}} \cos 2k\varphi \\ &\quad + i\sqrt{2\mathcal{I}_{\text{PT}}} p_\varphi \sin 2k\varphi. \end{aligned} \quad (26)$$

A similar expression exists in case of the (pseudo)-spherical TTW system. The additional constant of motion of PW system in Ranada’s representation reads:

$$\mathcal{M}_{\text{PW}} = (\bar{M}_0)^k N, \quad (27)$$

and N is given by Eq. (26), and

$$M_0 = p_r \sqrt{2\mathcal{I}_{\text{PT}}} + i\left(\gamma - \frac{2\mathcal{I}_{\text{PT}}}{r}\right). \quad (28)$$

Such forms of the hidden constants of motion have a visible relation with their expressions in terms of the action–angle variables, which will be discussed below. Hence, the TTW and PW systems possess a natural description in spherical coordinates, where the “radial” part is separated from the “angular” one. On the other hand, the radial parts are expressed via the generators of conformal algebra, which can be viewed as generators of isometries of the Kähler structure of Klein model of the Lobachevsky space [20]. Hence, we can represent phase spaces of the TTW and PW systems as a (semidirect) product of Lobachevsky space with cotangent bundle of circle, and expect that the reformulation in these coordinates will help us to extend the expressions of hidden constants of motion to higher dimensions. Similarly, phase spaces of the N -dimensional oscillator and Coulomb systems and their Calogero deformations could be represented as a semidirect product of Lobachevsky space and cotangent bundle on $(N - 1)$ -dimensional sphere [21]. One can expect, that Ranada’s representation of hidden symmetries of the TTW and PW systems in these terms will take a more transparent and elegant form. Furthermore, having in mind the

relation of the TTW and PW systems with rational Calogero models, one can expect that the hidden symmetries of the latters could be represented in a similar way.

Hence, the purpose of this paper is to provide the planar TTW and PW systems and their hidden constants of motion with such kind of formulation, as well as to discuss their extensions to higher dimensions. A similar description of their (pseudo)spherical generalizations will be presented elsewhere.

2. ONE-DIMENSIONAL SYSTEMS

Since the middle of seventies with [22] in the field-theoretical literature much attention has been paid to a simple one-dimensional mechanical system given by the Hamiltonian

$$H_0 = \frac{p^2}{2} + \frac{g^2}{2x^2}. \tag{29}$$

The reason was that it forms the conformal algebra $so(1, 2)$ (3) together with the generators:

$$D = px, \quad K = \frac{x^2}{2}. \tag{30}$$

In [20] the following formulation of this is suggested. Its phase space is parameterized by a single complex coordinate and identified with the Klein model of the Lobachevsky plane:

$$\begin{aligned} z &= \frac{p}{x} + \frac{ig}{x^2}, \quad \text{Im}z > 0 : \{z, \bar{z}\} \\ &= -\frac{i}{g}(z - \bar{z})^2. \end{aligned} \tag{31}$$

In this parametrization, the $so(1, 2)$ generators (29), (30) define the Killing potentials (Hamiltonian generators of the isometries of the Kähler structure) of Klein model:

$$\begin{aligned} H_0 &= g \frac{z\bar{z}}{i(\bar{z} - z)}, \quad D = g \frac{z + \bar{z}}{i(\bar{z} - z)}, \\ K &= g \frac{1}{i(\bar{z} - z)}. \end{aligned} \tag{32}$$

Let us remind, that the Kähler structure is

$$ds^2 = -\frac{gdzd\bar{z}}{(\bar{z} - z)^2}. \tag{33}$$

It is invariant under the discrete transformation

$$z \rightarrow -\frac{1}{z}, \tag{34}$$

whereas the Killing potentials (32) transform as follows:

$$H_0 \rightarrow K, \quad K \rightarrow H_0, \quad D \rightarrow -D. \tag{35}$$

Thus, it maps H_0 to the free one-dimensional particle system. This can be viewed as a one-dimensional analog of the decoupling transformation of the Calogero Hamiltonian, considered in [23].

In order to construct a similar construction for higher-dimensional systems, first, we introduce an appropriate “radial” coordinate and conjugated momentum, so that the higher-dimensional system looks very similar to the one-dimensional conformal mechanics. In that picture, the remaining “angular” degrees of freedom are packed in the Hamiltonian system on the $(N - 1)$ -dimensional sphere, which replaces the coupling constant g^2 in the one-dimensional conformal mechanics. The angular Hamiltonian defines the constant of motion of the initial conformal mechanics. Then we relate the radial part of the N -dimensional conformal mechanics with the Klein model of the Lobachevsky space, which is completely similar to the aforementioned one-dimensional case.

3. HIGHER-DIMENSIONAL SYSTEMS

Let us consider the N -dimensional conformal mechanics, defined by the following Hamiltonian and symplectic structure:

$$\begin{aligned} \omega &= d\mathbf{p} \wedge d\mathbf{x}, \quad \mathcal{H}_0 = \frac{\mathbf{p}^2}{2} + V(\mathbf{x}), \\ \text{where } (\mathbf{x} \cdot \nabla)V(\mathbf{x}) &= -2V(\mathbf{x}). \end{aligned} \tag{36}$$

This Hamiltonian together with the generators

$$\mathcal{D} = \mathbf{p} \cdot \mathbf{x}, \quad \mathcal{K} = \frac{\mathbf{x}^2}{2} \tag{37}$$

forms the conformal algebra $so(1, 2)$ (3). Here \mathcal{D} defines the dilatation and \mathcal{K} defines the conformal boost, $\mathbf{x} = (x^1, \dots, x^N)$, $\mathbf{p} = (p_1, \dots, p_N)$.

Extracting the radius $r = |\mathbf{x}|$, we can present the above generators in the following form:

$$\begin{aligned} \mathcal{D} &= p_r r, \quad \mathcal{K} = \frac{r^2}{2}, \quad \mathcal{H}_0 = \frac{p_r^2}{2} + \frac{\mathcal{I}}{r^2}, \\ \mathcal{I} &\equiv \frac{\mathbf{J}^2}{2} + U, \quad U \equiv r^2 V(\mathbf{r}). \end{aligned} \tag{38}$$

Here $p_r = (\mathbf{p} \cdot \mathbf{x})/r$ is the momentum, conjugate to the radius: $\{p_r, r\} = 1$. It is easy to check that \mathcal{I} is the Casimir element of conformal algebra $so(1, 2)$:

$$\begin{aligned} 4\mathcal{H}\mathcal{K} - \mathcal{D}^2 &= 2\mathcal{I} : \{\mathcal{I}, \mathcal{H}_0\} \\ &= \{\mathcal{I}, \mathcal{K}\} = \{\mathcal{I}, \mathcal{D}\} = 0. \end{aligned} \tag{39}$$

Thus, it defines the constant of motion of the system (36) and commutes with r, p_r and, hence, does not depend on them. Instead, it depends on spherical coordinates ϕ^a and canonically conjugate momenta π_a . As a Hamiltonian, \mathcal{I} defines the particle motion

on $(N - 1)$ -sphere in the potential $U(\phi^\alpha)$. The phase space is the cotangent bundle T^*S^{N-1} .

As in one dimension [20] instead of the radial phase space coordinates r and p_r we introduce the following complex variable (for simplicity, we restrict to $\mathcal{I} > 1$):

$$z = \frac{p_r}{r} + \frac{i\sqrt{2\mathcal{I}}}{r^2} \equiv \frac{\mathcal{D} + i\sqrt{2\mathcal{I}}}{2\mathcal{K}}, \quad \text{Im}z > 0. \quad (40)$$

It obeys the following Poisson brackets:

$$\{z, \bar{z}\} = -\frac{i}{\sqrt{2\mathcal{I}(u)}}(z - \bar{z})^2, \quad (41)$$

$$\begin{aligned} \{u^\alpha, u^\beta\} &= \omega^{\alpha\beta}(u), \quad \{u^\alpha, z\} = (z - \bar{z})\frac{V^\alpha(u)}{2\mathcal{I}}, \\ \{u^\alpha, \bar{z}\} &= (z - \bar{z})\frac{V^\alpha(u)}{2\mathcal{I}}, \end{aligned} \quad (42)$$

where $V^\alpha = \{u^\alpha, \mathcal{I}(u)\}$ are the equations of motion of the angular system.

The symplectic structure of the conformal mechanics can be represented as follows:

$$\begin{aligned} \Omega &= -i\frac{\sqrt{2\mathcal{I}(u)}dz \wedge d\bar{z}}{(\bar{z} - z)^2} + \frac{(dz + d\bar{z}) \wedge d\sqrt{2\mathcal{I}(u)}}{i(\bar{z} - z)} \\ &\quad + \frac{1}{2}\omega_{\alpha\beta}du^\alpha \wedge du^\beta, \end{aligned} \quad (43)$$

while the local one-form, defining this symplectic structure, reads

$$\begin{aligned} \Omega &= d\mathcal{A}, \quad \mathcal{A} = i\sqrt{2\mathcal{I}(u)}\frac{dz + d\bar{z}}{i(z - \bar{z})} + A_0(u), \\ dA_0 &= \frac{1}{2}\omega_{\alpha\beta}du^\alpha \wedge du^\beta. \end{aligned} \quad (44)$$

Taking into account Eq. (39), we can write:

$$\begin{aligned} \mathcal{H}_0 &= \sqrt{2\mathcal{I}(u)}\frac{z\bar{z}}{i(\bar{z} - z)}, \quad \mathcal{D} = \sqrt{2\mathcal{I}(u)}\frac{z + \bar{z}}{i(\bar{z} - z)}, \\ \mathcal{K} &= \frac{\sqrt{2\mathcal{I}(u)}}{i(\bar{z} - z)}. \end{aligned} \quad (45)$$

The transformation (34) does not preserve the symplectic structure, i.e., it is not a canonical transformation for the generic conformal mechanics of dimension $d > 1$.

Now we introduce the following generators, which will be used in our further considerations:

$$M = \frac{z}{\sqrt{i(\bar{z} - z)}}, \quad \bar{M} = \frac{\bar{z}}{\sqrt{i(\bar{z} - z)}}. \quad (46)$$

With the generators of the conformal algebra they form a highly nonlinear algebra:

$$\{M, \mathcal{H}_0\} = \frac{i}{2}z\sqrt{i(\bar{z} - z)}, \quad \{M, \mathcal{K}\} = \frac{2z}{i(\bar{z} - z)},$$

$$\{M, \mathcal{D}\} = \frac{z}{\sqrt{i(\bar{z} - z)}} = M, \quad (47)$$

$$\{M, \bar{M}\} = \frac{z - \bar{z}}{2\sqrt{2\mathcal{I}}}. \quad (48)$$

Let us introduce the angle-like variable, conjugate with $\sqrt{2\mathcal{I}}$:

$$\Phi(u) : \{\Phi, \sqrt{2\mathcal{I}}\} = 1, \quad \Phi \in [0, 2\pi). \quad (49)$$

Using M and Φ , one can easily build a (complex) constant of motion for the conformal mechanics:

$$\mathcal{M} = Me^{i\Phi}, \quad \{\mathcal{M}, \mathcal{H}_0\} = 0. \quad (50)$$

Evidently, its real part is the ratio of Hamiltonian and its angular part and does not contain any new constant of motion. Nevertheless, such a complex representation seems to be useful not only from an aesthetical viewpoint, but also for the construction of supersymmetric extensions.

Note that we can write down the hidden symmetry generators for the conformal mechanics, modified by the oscillator and Coulomb potentials as well. The Hamiltonian of the N -dimensional oscillator and its hidden symmetry generators look as follows:

$$\begin{aligned} \mathcal{H}_{\text{osc}} &= \mathcal{H}_0 + \omega^2\mathcal{K}, \quad \mathcal{M}_{\text{osc}} = \frac{z^2 + \omega^2}{i(\bar{z} - z)}e^{2i\Phi} \\ &= (M^2 + \omega^2\mathcal{K})e^{2i\Phi} : \{\mathcal{M}_{\text{osc}}, \mathcal{H}_{\text{osc}}\} = 0. \end{aligned} \quad (51)$$

The Hamiltonian and hidden symmetry of the Coulomb problem are defined by

$$\begin{aligned} \mathcal{H}_{\text{Coul}} &= \mathcal{H}_0 - \frac{\gamma}{\sqrt{2\mathcal{K}}}, \quad \mathcal{M}_{\text{Coul}} = \left(M \right. \\ &\quad \left. - \frac{i\gamma}{(8\sqrt{2\mathcal{I}})^{3/2}} \right) e^{i\Phi} : \{\mathcal{M}_{\text{Coul}}, \mathcal{H}_{\text{Coul}}\} = 0. \end{aligned} \quad (52)$$

The absolute values of both integrals do not produce anything new:

$$\begin{aligned} |\mathcal{M}_{\text{osc}}|^2 &= \frac{\mathcal{H}_{\text{osc}}^2}{2\mathcal{I}} - \omega^2, \\ |\mathcal{M}_{\text{Coul}}|^2 &= \frac{\mathcal{H}_{\text{Coul}}}{\sqrt{2\mathcal{I}}} + \frac{\gamma^2}{2(\sqrt{2\mathcal{I}})^3}. \end{aligned} \quad (53)$$

So, the hidden symmetry is encoded in their phase, depending on the angular variables $\Phi(u)$. Assume that the angular system is integrable. Hence, the Hamiltonian and two-form are expressed in terms of the action–angle variables as follows:

$$\mathcal{I} = \mathcal{I}(I_a), \quad \Omega = \sum_a dI_a \wedge d\Phi_a.$$

Then the condition (49) implies the following local solutions for Λ :

$$\Lambda_a = \frac{\Phi_a}{\omega_a(I)}, \quad (54)$$

where

$$\omega_a = \frac{\partial\sqrt{2\mathcal{I}}}{\partial I_a}.$$

Thus, to provide the global solution for a certain coordinate a , we are forced to set $\omega_a(I) = k_a$ to a rational number:

$$k_a = \frac{n_a}{m_a}, \quad m_a, n_a \in \mathcal{N}. \quad (55)$$

Then, taking k_a th power for the locally defined conserved quantity, we get a globally defined constant of motion for the system. In this case, the hidden symmetry of the conformal mechanics reads:

$$\mathcal{M}_a = M^{n_a} e^{im_a \Phi_a}. \quad (56)$$

Similarly, for the systems with oscillator and Coulomb potentials one has:

$$\begin{aligned} \mathcal{M}_{(a)\text{osc}} &= (M^2 + \omega^2 \mathcal{K})^{n_a} e^{2im_a \Phi_a}, \\ \mathcal{M}_{(a)\text{Coul}} &= \left(M - \frac{\nu\gamma}{(8\sqrt{2\mathcal{I}})^{3/2}} \right)^{n_a} e^{im_a \Phi_a}. \end{aligned} \quad (57)$$

To find the expression(s) for Φ , let us remind that the angular part of these systems is just the quadratic Casimir element (angular momentum) of $so(N)$ algebra on $(N - 1)$ -dimensional sphere, $\mathcal{I} = L_N^2/2$. It can be decomposed by the eigenvalues of the embedded $SO(a)$ angular momenta I_a as follows:

$$\mathcal{I} = \frac{1}{2} \left(\sum_{a=1}^{N-1} I_a \right)^2. \quad (58)$$

The functions I_a and their canonically conjugates Φ_a play the role of the action–angle variables of the free particle on S^{N-1} . (For details, see Appendix in [16].) Their explicit forms are:

$$I_a = \sqrt{j_a} - \sqrt{j_{a-1}}, \quad (59)$$

where

$$j_a = p_{a-1}^2 + \frac{j_{a-1}}{\sin^2 \theta_{a-1}}, \quad a = 1, \dots, N.$$

The related angle variables are:

$$\begin{aligned} \Phi_a &= \sum_{l=a}^{N-1} \arcsin(u_l) \\ &+ \sum_{l=a+1}^{N-1} \arctan \left(\sqrt{\frac{j_{l-1}}{j_l}} \frac{u_l}{\sqrt{1-u_l^2}} \right), \end{aligned} \quad (60)$$

where

$$u_a = \sqrt{\frac{j_a}{j_a - j_{a-1}}} \cos \theta_a, \quad \sqrt{j_a} = \sum_{m=1}^a I_m. \quad (61)$$

Hence, our expressions define the $N - 1$ functionally independent constants of motion

$$\begin{aligned} \mathcal{M}_{(a)\text{osc}} &= (M^2 + \omega^2 \mathcal{K}) e^{2i\Phi_a}, \\ \mathcal{M}_{(a)\text{Coul}} &= (M + \nu\gamma) e^{i\Phi_a}, \end{aligned} \quad (62)$$

respectively, for the N -dimensional oscillator and Coulomb problems. Since these systems have N commuting constants of motion (I_a, \mathcal{H}) , we have obtained in this way the full set of their integrals.

To clarify the origin of these generators, let us consider a particular case of two-dimensional systems. The angular part is a circle, and, respectively, $I = |p_\varphi|$, $\Phi = \varphi$ with φ being a polar angle. In this case, the oscillator Hamiltonian and its hidden constant of motion read

$$\begin{aligned} H_{\text{osc}} &= |p_\varphi| \frac{z\bar{z} + \omega^2}{i(\bar{z} - z)}, \\ \mathcal{M}_{\text{osc}} &= \frac{i}{z - \bar{z}} (z^2 + \omega^2) e^{2i\varphi}. \end{aligned} \quad (63)$$

The latter can also be presented as follows:

$$\begin{aligned} \mathcal{M}_{\text{osc}} &= \frac{H_1 - H_2 + 2iH_{12}}{|p_\varphi|}, \\ \text{with } H_{ab} &= p_a p_b + \omega^2 x_a x_b. \end{aligned} \quad (64)$$

Here H_{ab} is a standard representation of the oscillator’s hidden symmetry generators, sometimes called a Demkov tensor.

The Hamiltonian of the two-dimensional Coulomb problem and its hidden symmetry generator are of the form

$$\begin{aligned} H_{\text{Coul}} &= |p_\varphi| \frac{z\bar{z}}{i(\bar{z} - z)} - \gamma \sqrt{\frac{i(\bar{z} - z)}{2|p_\varphi|}}, \\ \mathcal{M}_{\text{Coul}} &= \left(\frac{z}{\sqrt{i(\bar{z} - z)}} - \frac{\nu\gamma}{\sqrt{2|p_\varphi|^3}} \right) e^{i\varphi}. \end{aligned} \quad (65)$$

The latter is related with the components of the two-dimensional Runge–Lenz vector $\mathbf{A} = (A_x, A_y)$ as follows

$$\mathcal{M}_{\text{Coul}} = \frac{A_y - iA_x}{\sqrt{2|p_\varphi|^3}}, \quad (66)$$

where

$$A_x = p_\varphi p_y - \gamma \cos \varphi, \quad A_y = p_\varphi p_x - \gamma \sin \varphi.$$

Now we are ready to apply this constructions to the TTW and PW systems.

3.1. TTW and PW systems

In order to formulate TTW and PW systems in the above terms, we will use the action–angle formulation of the Pöschl–Teller Hamiltonian given in [7]:

$$\mathcal{I}_{PT} = \frac{k^2 \tilde{I}^2}{2}, \quad \tilde{I} = I + \alpha + \beta, \quad (67)$$

where I is an action variable.

The angle variable is related to the initial phase space coordinates as follows:

$$a \sin(-2\Phi) = \cos(2k\varphi) + b, \\ a = \sqrt{\left(1 - \frac{2(\alpha + \beta)}{(k\tilde{I})^2} + b^2\right)}, \quad b = \frac{\beta - \alpha}{(k\tilde{I})^2}. \quad (68)$$

Using the above expressions, we can present the Hamiltonian of TTW system and its hidden symmetry generator as follows:

$$H_{TTW} = k\tilde{I} \frac{z\bar{z} + \omega^2}{i(\bar{z} - z)}, \\ \mathcal{M}_{TTW} = \left(\frac{z^2 + \omega^2}{i(\bar{z} - z)}\right)^k e^{2i\Phi}. \quad (69)$$

The Ranada's constant of motion is related with the above one:

$$K = -a^2 \frac{(2k\tilde{I})^{2k+4}}{16} \left(\frac{\bar{z}^2 + \omega^2}{z - \bar{z}}\right)^{2k} e^{-4i\Phi} \\ = -a^2 \frac{(2k\tilde{I})^{2k+4}}{16} \bar{\mathcal{M}}_{TTW}^2. \quad (70)$$

We repeat the same procedure for the PW system as well. Using the expressions for action–angle variables of the Pöschl–Teller Hamiltonian, we get:

$$\mathcal{H}_{PW} = ik\tilde{I} \frac{\bar{z}z}{z - \bar{z}} - \frac{\gamma}{2k\tilde{I}} \sqrt{i(\bar{z} - z)}, \\ \mathcal{M}_{PW} = \left(\frac{z}{\sqrt{i(\bar{z} - z)}} - \frac{i\gamma}{k\tilde{I}\sqrt{2k\tilde{I}}}\right)^k e^{i\Phi}. \quad (71)$$

Respectively, the Ranada's constant of motion takes the form

$$K = -ia(k\tilde{I})^2 \left(k\tilde{I}\sqrt{2k\tilde{I}} \frac{z}{\sqrt{i(\bar{z} - z)}} + i\gamma\right)^{2k} e^{2i\Phi} \\ = -ia(k\tilde{I})^{2k+2} \bar{\mathcal{M}}_{PW}^2. \quad (72)$$

ACKNOWLEDGMENTS

The authors thank Vahagn Yeghikyan for useful comments. This work was partially supported by the Armenian State Committee of Science grants

15RF-039 and 15T-1C367. It was done within program of Regional Training Network on Theoretical Physics sponsored by Volkswagenstiftung Contract no. 86 260.

REFERENCES

1. S. Wojciechowski, Phys. Lett. A **95**, 279 (1983).
2. V. Kuznetsov, Phys. Lett. A **218**, 212 (1996).
3. C. Gonera, Phys. Lett. A **237**, 365 (1998).
4. F. Calogero, J. Math. Phys. **10**, 2191 (1969); **12**, 419 (1971); J. Moser, Adv. Math. **16**, 197 (1975).
5. M. A. Olshanetsky and A. M. Perelomov, Phys. Rept. **71**, 313 (1981); **94**, 313 (1983).
6. T. Hakobyan, O. Lechtenfeld, and A. Nersessian, Nucl. Phys. B **858**, 250 (2012), arXiv:1110.5352; T. Hakobyan, D. Karakhanyan, and O. Lechtenfeld, Nucl. Phys. B **886**, 399 (2014), arXiv:1402.2288; T. Hakobyan, A. Nersessian, and V. Yeghikyan, J. Phys. A **42**, 205206 (2009), arXiv:0808.0430; T. Hakobyan, O. Lechtenfeld, A. Nersessian, and A. Saghatelian, J. Phys. A **44**, 055205 (2011), arXiv:1008.2912.
7. O. Lechtenfeld, A. Nersessian, and V. Yeghikyan, Phys. Lett. A **374**, 4647 (2010), arXiv:1005.0464.
8. M. Feigin, O. Lechtenfeld, and A. Polychronakos, JHEP **1307**, 162 (2013), arXiv:1305.5841.
9. T. Hakobyan, S. Krivonos, O. Lechtenfeld, and A. Nersessian, Phys. Lett. A **374**, 801 (2010).
10. T. Hakobyan and A. Nersessian, Phys. Rev. A **92**, 022111 (2015).
11. T. Hakobyan and A. Nersessian, arXiv:1509.01077.
12. F. Tremblay, A. V. Turbiner, and P. Winternitz, J. Phys. A **42**, 242001 (2009), arXiv:0904.0738.
13. E. G. Kalnins, K. M. Kress, and W. Miller, Jr., J. Phys. A **43**, 265205 (2010), arXiv:1002.2665.
14. S. Post and P. Winternitz, J. Phys. A **43**, 222001 (2010), arXiv:1003.5230.
15. C. Gonera, Phys. Lett. A **376**, 2341 (2012), arXiv:1010.2915.
16. T. Hakobyan, O. Lechtenfeld, A. Nersessian, et al., Phys. Lett. A **376**, 679 (2012), arXiv:1108.5189.
17. P. W. Higgs, J. Phys. A **12**, 309 (1979); H. I. Leemon, J. Phys. A **12**, 489 (1979).
18. E. Schrödinger, Proc. Roy. Irish Soc. **46**, 9, 183 (1941).
19. M. F. Ranada, J. Phys. A **47**, 165203 (2014), arXiv:1403.6266; J. Phys. A **45**, 465203 (2012); **46**, 125206 (2013).
20. T. Hakobyan and A. Nersessian, Phys. Lett. A **373**, 1001 (2009), arXiv:0803.1293.
21. C. Burdik and A. Nersessian, SIGMA **5**, 004 (2009), arXiv:0901.1644.
22. V. de Alfaro, S. Fubini, and G. Furlan, Nuovo Cimento A **34**, 569 (1976).
23. N. Gurappa and P. K. Panigrahi, Phys. Rev. B **59**, R2490 (1999); T. Brzezinski, C. Gonera, and P. Maslanka, Phys. Lett. A **254**, 185 (1999); A. Galajinsky, O. Lechtenfeld, and K. Polovnikov, Phys. Lett. B **643**, 221 (2006).