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ON THE FOURIER-VILENKIN COEFFICIENTS*

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Abstract In this article, we prove the following statement that is true for both unbounded and bounded Vilenkin systems: for any $\varepsilon \in (0,1)$, there exists a measurable set $E \subset [0,1)$ of measure bigger than $1-\varepsilon$ such that for any function $f \in L^1[0,1)$, it is possible to find a function $g \in L^1[0,1)$ coinciding with f on E and the absolute values of non zero Fourier coefficients of g with respect to the Vilenkin system are monotonically decreasing.

Key words Vilenkin system; expansions; Fourier coefficients

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1 Introduction

Recall the definition of Vilenkin (multiplicative) systems of functions (see [1]). Consider the arbitrary sequence of natural numbers $P \equiv \{p_1, p_2, \cdots, p_k, \cdots\}$, where $p_j \geq 2$ for all $j \in \mathbb{N}$. We set

$$m_0 = 1, \ m_k = \prod_{j=1}^{k} p_j \ , \quad k \in \mathbb{N}. \tag{1.1}$$

It is not difficult to notice that for each point $x \in [0,1)$ and for any $n \in [m_{k-1}, m_k) \cap \mathbb{N}, \ k \in \mathbb{N}$, there exist numbers $x_j, \alpha_j \in \{0, 1, \cdots p_j - 1\}$ such that

$$n = \sum_{j=1}^{k} \alpha_j m_{j-1} \text{ and } x = \sum_{j=1}^{\infty} \frac{x_j}{m_j}, \ (P\text{-order expansions}). \tag{1.2}$$

Note that all points of type $\frac{l}{m_k}$ with $l, k \in \mathbb{N}; \ 0 \leq l \leq m_k - 1$, have two different expansions: finite and infinite, and to have only unique expansions we take only finite expansions for such points. As a result, we get the correspondences

$$n \to \{\alpha_1, \alpha_2, \cdots, \alpha_k\}, \ x \to \{x_1, x_2, \cdots, x_k, \cdots\}. \tag{1.3}$$

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The Vilenkin system for sequence P is defined as follows:

$$W_0(x) \equiv 1;\ W_n(x) = \exp\left(2\pi \mathrm{i} \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right). \tag{1.4}$$

The expression (1.4) we can change to the form

$$W_n(x) = \exp\left(2\pi \mathrm{i} \sum_{j=1}^{k} \alpha_j \frac{x_j}{p_j}\right) = \prod_{j=1}^{k} \left(\exp\left(2\pi \mathrm{i} \frac{x_j}{p_j}\right)\right)^{\alpha_j}.$$

From (1.4), it follows,

$$W_{m_{j-1}}(x) = \exp\left(2\pi \mathrm{i} \frac{x_j}{p_j}\right),$$

and for the n-th function, we obtain the expression

$$W_n(x) = \prod_{j=1}^{k} (W_{m_{j-1}}(x))^{\alpha_j}.$$

It is not difficult to notice that

$$\int_0^1 W_n(t)\overline{W}_k(t)\mathrm{d}t = \begin{cases} 1, & \text{if } k = n; \\ 0, & \text{if } k \neq n, \end{cases} \quad \text{where } \overline{W}_k(t) \text{ is the complex conjugate of } W_k(t).$$

It is obvious that systems corresponding to different sequences $\{p_k\}$, differ from each other (in case $P \equiv \{2, 2, \cdots, 2, \cdots\}$ Vilenkin system coincides with the Walsh system (see [1])). In case $\sup\{p_k\} = \infty$ ($\sup\{p_k\} < \infty$), the system $\{W_n(x)\}$ is said to be unbounded (accordingly bounded).

The theory of such systems have been introduced by N. Ja. Vilenkin in 1946 (see [2, 3]). Then, there are interesting results for Vilenkin system (see [4–7]).

In 1957, C. Watari [8] proved that the bounded Vilenkin system is basis in L^r when $r > 1$. Then, in 1976, W.S. Young [9] for arbitrary sequence $\{p_k\}$ (that is, both for bounded and unbounded Vilenkin systems) established the basicity of Vilenkin system in L^r when $r > 1$. For any function $f \in L^1[0,1)$ and for all $n \in \mathbb{N}$ and $y \in (0, \infty)$, he also proved the inequality

$$\mathrm{mes}\{x : |S_n(x, f)| > y\} \leq \frac{C\|f\|_{L^1[0,1)}}{y} \text{ where } C - \text{ is constant.}$$

Note that the following problem remains open: Is the Fourier series of function from $L^2[0,1)$ with respect to the unbounded Vilenkin systems convergent almost everywhere or not?

Note also that in [10] P. Billard established that this problem has a positive answer for the Walsh system.

Let $f(x)$ be a real valued function from $L^r[0,1),\ r \geq 1$ and $c_n(f)$ be the Fourier-Vilenkin coefficients of function f, that is,

$$c_n(f) = \int_0^1 f(x)\overline{W}_n(x)\mathrm{d}x. \tag{1.5}$$

Let $\mathrm{spec}(f)$ be the spectrum of $f(x)$ that is, the set of integers k for which $c_k(f) \neq 0$.

In this article, we prove the following:

Theorem 1.1 Let $\{W_k(x)\}_{k=0}^{\infty}$ — be either unbounded or bounded Vilenkin system. Then, for each $0 < \epsilon < 1$, there exists a measurable set $E \subset [0,1)$ of measure $|E| > 1 - \varepsilon$ such that for any function $f \in L^1[0,1)$, there exists a function $g \in L^1[0,1)$ such that $f(x) = g(x)$ if $x \in E$ and the elements of sequence $\{|c_k(g)|\ ,\ k \in \mathrm{spec}(g)\}$ are monotonically decreasing.

The idea of modifying a function in order to improve its properties dates back to Luzin (see [11]) and it was substantially developed later on. For instance, in 1939, Men'shov [12] proved the following fundamental theorem.

Theorem (Men'shov's C-strong property) Let $f(x)$ be an a.e. finite measurable function on $[0, 2\pi]$. Then, for each $\epsilon > 0$, there is a continuous function $g(x)$ coinciding with $f(x)$ on a subset E of measure $|E| > 2\pi - \epsilon$, such that its Fourier series with respect to the trigonometric system converges uniformly on $[0, 2\pi]$.

For further results in this direction, we refer the reader to articles [13–19].

Note that in Men'shov's theorem, the set on which the function f is corrected depends on f.

Then, one can pose the following questions, which we can not answer.

Question 1 Is it possible in Theorem 1.1 to choose the modified function $g(x)$ such that also its Fourier series converges in $L^1[0,1)$?

Question 2 Is the Theorem 1.1 true for trigonometric system?

2 Proof of Main Lemmas

Let

$$\Delta_j^{(k)} = \left[\frac{j}{m_k}, \frac{j+1}{m_k}\right) \text{ where } j = 0, 1, \cdots, m_k - 1;\ k \in \mathbb{N}, \tag{2.1}$$

and $\chi_E(x)$ be the characteristic function of set E, that is,

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

We consider a set $\{\gamma, \Delta\}$, which depends on two parameters, with γ running through the set of all real numbers, and Δ running trough the set of all intervals of type $\Delta_j^{(k)}$ (see (2.1)), and a set of functions

$$B = \left\{ f(x) : f(x) = \sum_{k=1}^{\nu_0} \gamma_k \chi_{\Delta_k};\ (\gamma_k, \Delta_k) \in \{\gamma, \Delta\},\ \Delta_k \bigcap \Delta_{k'} = \emptyset,\ k \neq k' \right\}. \tag{2.2}$$

In this article, the following properties of Vilenkin system are used (see [1]):

$$W_n(x) = W_n\left(\frac{j}{m_k}\right) \text{ for all } x \in \Delta_j^{(k)} \text{ if } n, j \in [0, m_k), \tag{2.3}$$

$$\int_{\Delta_j^{(k)}} W_n(x)\mathrm{d}x = 0 \text{ for all } j \in [0, m_k)\ ,\ \text{if } n \geq m_k \tag{2.4}$$

$$W_{lm_k+\beta}(x) = W_{lm_k}(x) W_\beta(x) \text{ for all } \beta \in [0, m_k),\ l \in \mathbb{N}. \tag{2.5}$$

Then, we need the following elementary result:

Lemma 2.1 Let k, ν be the arbitrary natural numbers. Then, for any $l \in \{1, \cdots, \frac{m_{k+\nu}}{m_k} - 1\}$ and $j \in \{0, 1, \cdots, m_k - 1\}$, the following is true:

$$W_{lm_k}(x) = 1 \text{ for all } x \in \Theta_j = \left[\frac{j}{m_k}, \frac{j}{m_k} + \frac{1}{m_{k+\nu}}\right).$$

Proof Let x be an arbitrary point from $\Theta_j=[\frac{j}{m_k},\frac{j}{m_k}+\frac{1}{m_{k+\nu}})$, $n=lm_k$, and let $\{x_s\}$, $\{\alpha_s\}$ be the coefficients of P-order expansions of numbers x and n (see (1.2), (1.3)). It is easy to notice that $x_s=0$ for all $s\in\{k+1,\cdots,k+\nu\}$ and $\alpha_s=0$ for all $s\in\{1,\cdots,k\}\bigcup\{k+\nu+1,\cdots\}$. Hence, $\alpha_s x_s=0$ for any $s\in\mathbb{N}$. Taking into consideration (1.4), we get $W_{lm_k}(x)=1$. □

Lemma 2.2 Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system. Then, for all $\gamma\neq 0$, $\varepsilon>0$, $N_0\in\mathbb{N}$ and $\Delta_a^{(k_0)}=[\frac{a}{m_{k_0}},\frac{a+1}{m_{k_0}}):=\Delta$, there exist a measurable set $E\subset[0,1)$ and a polynomial of Vilenkin system

$$Q(x)=\sum_{n=N_0}^{N}c_nW_n(x),$$

such that

1. non zero coefficients in $\{|c_n|\}_{n=N_0}^{N}$ are equal to $|\gamma||\Delta|$,
2. $|E|>|\Delta|(1-\varepsilon)$,
3. $Q(x)=\begin{cases}\gamma & \text{on } E;\\ 0 & \text{outside } \Delta,\end{cases}$
4. $\int_0^1|Q(x)|\mathrm{d}x<2|\gamma||\Delta|$.

Proof Let $k_1=[\log_2 N_0]+1+k_0$, where $[x]$ denotes the largest integer not greater than x. Then,

$$N_0\leq 2^{k_1-k_0}<m_{k_1}\ \text{(see (1.1))}. \tag{2.6}$$

We take ν such that

$$\frac{1}{\varepsilon}<\frac{m_{k_1+\nu}}{m_{k_1}}. \tag{2.7}$$

Let $\Theta_j=[\frac{j}{m_{k_1}},\frac{j}{m_{k_1}}+\frac{1}{m_{k_1+\nu}})$. Obviously, $\Theta_j=\Delta^{(k_1+\nu)}_{jm_{k_1+\nu}/m_{k_1}}$, where $j=0,\ 1,\ 2,\cdots,m_{k_1}-1$.

Set

$$I_{k_1}^{(\nu)}(k_0,x)=\begin{cases}1-\frac{m_{k_1+\nu}}{m_{k_1}}\chi_{\Theta_j} & \text{if } x\in\Delta_j^{(k_1)}\subset\Delta;\\ 0 & \text{if } x\notin\Delta,\end{cases} \tag{2.8}$$

where $\Delta=\bigcup\limits_{j\in A}\Delta_j^{(k_1)}$,

$$A=\left[\frac{am_{k_1}}{m_{k_0}},\frac{(a+1)m_{k_1}}{m_{k_0}}\right)\subset[0,m_{k_1}). \tag{2.9}$$

Let

$$G(x)=\gamma I_{k_1}^{(\nu)}(k_0,x),\quad c_n=\int_0^1 G(t)\overline{W}_n(t)\mathrm{d}t,\ \text{where } n=0,\ 1,\ 2,\cdots. \tag{2.10}$$

Now, we find the values of coefficients c_n. From (2.3), (2.4), (2.8)–(2.10) for any $0\leq n\leq m_{k_1}-1$ and $j\in A$, we get

$$\begin{aligned}\int_{\Delta_j^{(k_1)}}G(t)\overline{W}_n(t)\mathrm{d}t&=\gamma\overline{W}_n\left(\frac{j}{m_{k_1}}\right)\int_{\Delta_j^{(k_1)}}I_{k_1}^{(\nu)}(k_0,t)\mathrm{d}t\\&=\gamma\overline{W}_n\left(\frac{j}{m_{k_1}}\right)\int_{\Delta_j^{(k_1)}}\left(1-\frac{m_{k_1+\nu}}{m_{k_1}}\chi_{\Theta_j}(t)\right)\mathrm{d}t\end{aligned}$$

$$= \gamma \overline{W}_n \left(\frac{j}{m_{k_1}}\right)\left(\frac{1}{m_{k_1}} - |\Theta_j|\frac{m_{k_1+\nu}}{m_{k_1}}\right) = 0.$$

From this and from (2.8), (2.10), it follows that $c_n = 0$ when $n \leq m_{k_1} - 1$. From (2.4), for all $q \in \{0, 1, \cdots, m_{k_1+\nu} - 1\}$ and $n \geq m_{k_1+\nu}$, we obtain $\int_{\Delta_q^{(k_1+\nu)}} \overline{W}_n(t)\mathrm{d}t = 0$. Therefore (see also (2.8), (2.10)) $c_n = 0$, if $n \geq m_{k_1+\nu}$. Now, let $m_{k_1} \leq n \leq m_{k_1+\nu} - 1 := N$. It is obvious that there exist natural numbers $l_n, \beta_n \in \mathbb{N},\ 1 \leq l_n \leq \frac{m_{k_1+\nu}}{m_{k_1}} - 1,\ \ 0 \leq \beta_n \leq m_{k_1} - 1$, such that $n = l_n m_{k_1} + \beta_n$. Taking into consideration that $\overline{W}_{l_n m_{k_1}}(t) = 1$ when $t \in \bigcup_{j=0}^{m_{k_1}-1} \Theta_j$ (see Lemma 2.1) and $W_{\beta_n}(t) \equiv W_{\beta_n}(\frac{j}{m_{k_1}})$ when $t \in \Delta_j^{(k_1)}$ from (2.3)–(2.5) and (2.8)–(2.10), we have

$$\begin{aligned}
c_n &= \int_0^1 G(t)\overline{W}_n(t)\mathrm{d}t = \gamma \int_\Delta I_{k_1}^{(\nu)}(k_0, t)\overline{W}_{l_n m_{k_1}}(t)\overline{W}_{\beta_n}(t)\mathrm{d}t \\
&= \gamma \sum_{j\in A}\left\{\overline{W}_{\beta_n}\left(\frac{j}{m_{k_1}}\right)\int_{\Delta_j^{(k_1)}}\left(1 - \frac{m_{k_1+\nu}}{m_{k_1}}\chi_{\Theta_j}(t)\right)\overline{W}_{l_n m_{k_1}}(t)\mathrm{d}t\right\} \\
&= \gamma \sum_{j\in A}\left\{\overline{W}_{\beta_n}\left(\frac{j}{m_{k_1}}\right)\left(-\frac{m_{k_1+\nu}}{m_{k_1}}\int_{\Theta_j}\overline{W}_{l_n m_{k_1}}(t)\mathrm{d}t\right)\right\} \\
&= -\gamma \sum_{j\in A}\overline{W}_{\beta_n}\left(\frac{j}{m_{k_1}}\right)\frac{1}{m_{k_1}} = -\gamma\int_\Delta \overline{W}_{\beta_n}(t)\mathrm{d}t \\
&= \begin{cases} -\gamma|\Delta|\overline{W}_{\beta_n}(\frac{a}{m_{k_0}}) & \text{if } 0 \leq \beta_n \leq m_{k_0} - 1; \\ 0 & \text{if } m_{k_0} \leq \beta_n < m_{k_1}. \end{cases}
\end{aligned}$$

Finally, we get

$$|c_n| = |\gamma||\Delta| \text{ for } n \in \operatorname{spec}(G) \subset [N_0, N]. \tag{2.11}$$

Set

$$Q(x) = \sum_{n=N_0}^{N} c_n W_n(x). \tag{2.12}$$

From (2.11) and (2.12), it follows

$$Q(x) = G(x) \text{ for all } x \in [0, 1). \tag{2.13}$$

Let

$$E \equiv \{x : Q(x) = \gamma\}. \tag{2.14}$$

From (2.8), (2.10), (2.13) and (2.14), we get

$$Q(x) = \begin{cases} \gamma & \text{if } x \in E; \\ \gamma\Big(1 - \dfrac{m_{k_1+\nu}}{m_{k_1}}\Big) & \text{if } x \in \Delta\backslash E; \\ 0 & \text{if } x \notin \Delta. \end{cases} \tag{2.15}$$

From (2.8), (2.10), (2.13), and (2.15), we have

$$E = \bigcup_{j\in A}(\Delta_j^{(k_1)}\backslash\Theta_j).$$

Hence,

$$|E| = \frac{m_{k_1}}{m_{k_0}}\left(\frac{1}{m_{k_1}} - \frac{1}{m_{k_1+\nu}}\right) = |\Delta|\left(1 - \frac{m_{k_1}}{m_{k_1+\nu}}\right). \tag{2.16}$$

From (2.7) and (2.16), we get

$$|E| \geq |\Delta|(1-\varepsilon).$$

Finally, by virtue of (2.15) and (2.16), we have

$$\int_0^1 |Q(x)|\mathrm{d}x < 2|\gamma||\Delta|.$$

□

Lemma 2.3 Let $\{W_k(x)\}_{k=0}^{\infty}$ be either unbounded or bounded Vilenkin system, and let $N_0 > 1$, $\varepsilon > 0$, $\delta > 0$. Then, for each function $f(x) \in B$, there exist a polynomial of Vilenkin system $Q(x) = \sum\limits_{k=N_0}^{N} a_k W_k(x)$ and a measurable set $E \subset [0,1)$ satisfying the following conditions:

1. $|E| > 1-\varepsilon$,
2. $Q(x) = f(x)$ for all $x \in E$,
3. $0 \leq |a_k| < \delta$ and non zero coefficients in the sequence $\{|a_k|\}_{k=N_0}^{N}$ are monotonically decreasing,
4. $\int_0^1 |Q(x)|\mathrm{d}x \leq 2\int_0^1 |f(x)|\mathrm{d}x$.

Proof Let

$$f(x) = \sum_{k=1}^{\nu_0} \gamma_k \chi_{\Delta_k}\ . \tag{2.17}$$

Without loss of generality, we can assume that

$$0 < |\gamma_{\nu_0}||\Delta_{\nu_0}| \leq \cdots \leq |\gamma_k||\Delta_k| \leq \cdots \leq |\gamma_1||\Delta_1| < \delta. \tag{2.18}$$

Successively applying Lemma 2.2, we define sets $E_k \subset \Delta_k$ and polynomials

$$Q_k(x) = \sum_{j=N_{k-1}}^{N_k-1} a_j W_j(x) \quad |a_j| = 0 \text{ or } |\gamma_k||\Delta_k| \text{ when } j \in [N_{k-1}, N_k),\ N_k \nearrow, \tag{2.19}$$

satisfying the following conditions:

$$|E_k| > (1-\varepsilon)|\Delta_k|, \tag{2.20}$$

$$Q_k(x) = \begin{cases} \gamma_k, & \text{if } x \in E_k; \\ 0, & \text{if } x \notin \Delta_k, \end{cases} \tag{2.21}$$

$$\int_0^1 |Q_k(x)|\mathrm{d}x \leq 2|\gamma_k||\Delta_k|. \tag{2.22}$$

We set

$$Q(x) = \sum_{k=1}^{\nu_0} Q_k(x) = \sum_{j=N_0}^{N} a_j W_j(x),\ (N = N_{\nu_0} - 1), \tag{2.23}$$

$$E = \{x \in [0,1) : Q(x) = f(x)\}. \tag{2.24}$$

From (2.17), (2.20), (2.21), (2.23) and (2.24), we get $|E| > 1-\varepsilon$. From (2.18), (2.19), and (2.23), it follows that $|a_n| \searrow$ for $n \in \operatorname{spec}(Q)$. Taking into consideration (2.17), (2.21)–(2.23), we obtain

$$\int_0^1 |Q(x)|\mathrm{d}x = \sum_{k=1}^{\nu_0} \int_{\Delta_k} |Q_k(x)|\mathrm{d}x \leq 2\sum_{k=1}^{\nu_0} |\gamma_k||\Delta_k| = 2\int_0^1 |f|\mathrm{d}x.$$

□

3 Proof of Theorem 1.1

Proof It is not difficult to notice that

$$\exists \{f_n(x)\}_{n=1}^{\infty} \subset B \text{ (see (2.2)) dense in } L^1[0,1). \tag{3.1}$$

Then, for every $n \in \mathbb{N}$ successively, applying Lemma 2.3, we obtain a sequence of measurable sets $E_n \subset [0,1),\ n=1,2,\cdots$; and polynomials $Q_n(x) = \sum_{j=M_{n-1}}^{M_n-1} a_j^{(n)} W_{k_j}(x),\ n=1,2,\cdots;\ M_n \nearrow$ $(M_0=1);\ k_j \nearrow$ such that for each $n \in \mathbb{N}$,

$$|E_n| > 1-\varepsilon 2^{-n}, \tag{3.2}$$

$$Q_n(x) = f_n(x),\ x \in E_n, \tag{3.3}$$

$$||Q_n|| \le 2||f_n|| \text{ (where } \|\cdot\| \text{ is the norm of space } L^1[0,1)), \tag{3.4}$$

and

$$|a_{M_{k-1}-1}^{(n-1)}| > |a_{j-1}^{(n)}| \ge |a_j^{(n)}| \ (a_0^{(0)}=1) \text{ for } j \in (M_{k-1}, M_k). \tag{3.5}$$

We put

$$a_j = a_j^{(n)} \text{ where } j \in [M_{n-1}, M_n),\ n=1,2,\cdots, \tag{3.6}$$

$$E = \bigcap_{n=1}^{\infty} E_n. \tag{3.7}$$

From (3.2), (3.5)–(3.7), we have

$$|E| > 1-\varepsilon \text{ and } |a_j| \searrow. \tag{3.8}$$

Let $f(x) \in L^1[0,1)$, from (3.1) it follows that it is possible to find a subsequence $\{f_{n_\nu}(x)\}_{\nu=1}^{\infty}$ from $\{f_n(x)\}_{n=1}^{\infty}$, such that

$$\lim_{N\to\infty} \left\| \sum_{n=1}^{N} f_{n_\nu}(x) - f(x) \right\| = 0, \quad ||f_{n_\nu}(x)|| \le 2^{-\nu-1} \text{ for all } \nu \ge 2. \tag{3.9}$$

From (3.4) and (3.9), it follows that the sequence $\left\{ \sum_{\nu=1}^{N} Q_{n_\nu}(x) \right\}_{N=1}^{\infty}$ is fundamental in $L^1[0,1)$. From this and (3.3), (3.7), (3.9), then there exists $g(x) \in L^1[0,1)$ such that

$$\lim_{N\to\infty} \left\| \sum_{\nu=1}^{N} Q_{n_\nu}(x) - g(x) \right\| = 0 \tag{3.10}$$

and

$$g(x) = f(x) \text{ if } x \in E.$$

We set

$$\sum_{l=1}^{\infty} c_l W_l(x) = \sum_{\nu=1}^{\infty} Q_{n_\nu}(x), \text{ where } c_l = \begin{cases} a_j & \text{if } l \in \{n_j\}_{j=M_{n_{\nu-1}}}^{M_{n_\nu}-1}\ \nu=1,2,\cdots; \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

Let $S_n(x) = \sum_{l=0}^{n-1} c_l W_l(x)$, then from (3.10) and (3.11) it is clear that the sequence $S_{M_{n_\nu}}(x)$ converges to $g(x)$ by $L^1[0,1)$ norm, therefore,

$$c_l = \int_0^1 g(x)\overline{W}_l(x)\mathrm{d}x,\ l=1,2,\cdots.$$

Obviously, $\{|c_l| : l \in \mathrm{spec}(g)\} \searrow$ (see (3.8) and (3.11)). □

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