PROBABILISTIC CHARACTERISTICS OF CONVEX BODIES DEPENDING ON ORIENTATION

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The methods of form analysis are based on analysis of the objects as subsets of the plane. The study of chord length distributions for convex bodies is a classical problem of stochastic geometry. Random lines generate chords of random length in convex domains. The corresponding distribution function is called the *chord length distribution function* that we denote by F(y). In the initial stage of investigation mathematicians tried to find explicit expressions of the chord length distribution (or density) functions for concrete domains **D** in the terms of elementary functions. Till recently explicit expressions for the chord length distribution functions have been known in the case where **D** is a disc, a regular triangle and a rectangle.

The form of the chord length density function is related to certain features of the corresponding **D**. For example, poles of this function are related to parallel pieces of the contour and the form of density function for argument close to its maximum is essentially related to smaller details of the contour of a domain **D** (see [1] - [3]). Assume now to have the information about the distribution of the chord lengths not in the ``completely mixed'' form, but separated direction by direction. In the last years have been introduced the notion of oriented-dependent chord length distribution function F(φ, y), while F(y) is called mixed orientation, i.e. if we have orientation-dependent chord length distribution function, then mixed oriented is equal to average of F(φ, y)) in all directions. These questions are connected with Covariogram Problem: Does the covariogram determine a convex domain, among all convex domains, up to translations and reflections? G. Matheron conjectured a positive answer for this problem. This hypothesis is known as Matheron's conjecture (see [1] and [2]). G. Bianchi and G. Averkov confirmed Matheron's conjecture for $n = 2$ (see [2] and [4]). G. Bianchi has also proved that for n>4 the hypothesis is false. Very little is known regarding the covariogram problem when the space dimension is greater than 2. It is known that centrally symmetric convex bodies in any dimension, are determined by their covariogram up to translations. For n=3 the problem is open. Nevertheless, for the case of bounded convex polyhedron for n=3 Matheron's conjecture received a positive answer. In fact, the covariogram problem is equivalent to the problem of determining a convex domain from all orientation-dependent chord length distributions (see [3], [4]).

Let $\boldsymbol{R^n}$ be the n-dimensional Euclidean space, $\mathbf{D} \subset \mathbf{R^n}$ be a bounded convex body, S^{n-1} be the (n-1)-dimensional unit sphere centered at the origin and $\boldsymbol{L_n}(.)$ be the n-dimensional Lebesgue measure in $\boldsymbol{R^n}$. Thus, investigation of the covariogram of three dimensional convex bodies becomes an important first step in the study of Matheron's conjecture in $\boldsymbol{R}^3$. Note that the explicit form for the covariogram of three dimensional convex bodies is known only in the case of a ball. The function $C_{\boldsymbol{D}}(x) = L_n(\boldsymbol{D} \cap \{\boldsymbol{D} + x\}),\ x \in \boldsymbol{R}^n$ is called the covariogram of the body **D**. Let G be the space of all lines in the Euclidean plane $\boldsymbol{R}^2$, $g \in G$ and (p, φ) are the polar coordinates of the foot of the perpendicular to g from the origin, $p \geq 0,\ \varphi \in S^1$. For a bounded convex domain $\boldsymbol{D} \subset R^2$ we denote by $b_{\boldsymbol{D}}(\varphi)$ the breadth function in direction $\varphi \in S^1$, that is, the distance between two support lines to the boundary of **D** that are perpendicular to φ. For a bounded convex domain **D** the chord length distribution function in direction φ, denoted by $F_{\boldsymbol{D}}(x, \varphi)$, is defined to be the probability of having chord $\chi(g) = g \cap \boldsymbol{D}$ with length at most x in the bundle of lines parallel to φ. A random line which is parallel to φ and intersects **D** has an intersection point (denoted by y) with the line l_{φ}. The intersection point y is uniformly distributed on the segment $[0, b_{\boldsymbol{D}}(\varphi)]$. The orientation dependent chord length distribution function and the covariogram for n=2 are known only in the cases of a disc, a triangle, a regular polygon, a parallelogram and an ellipse (see [7] and [9]). Denote by Γ the space of lines γ in $\boldsymbol{R}^3$. Let $\Pi_{\mathbf{D}}(\omega)$ denote the projection of a bounded convex body $\boldsymbol{D} \subset \boldsymbol{R}^3$ in direction $\omega \in S^2$ and let $s_{\boldsymbol{D}}(\omega)$ be its area. Every line which is parallel to ω and intersects **D** has an intersection with $\Pi_{\boldsymbol{D}}(\omega)$. Denote that point by y and that line by $l_\omega + y$. The intersection point y is uniformly distributed on $\Pi_{\boldsymbol{D}}(\omega)$. The chord length distribution function of D in direction $\omega \in S^2$ is defined by $F_{\boldsymbol{D}}(x,\omega) = \frac{L_2\{y:\chi(l_\omega+y)\leq x\}}{s_{\boldsymbol{D}}(\omega)}$.

THEOREM. Let D be a convex planar polygon which has m pairs of parallel sides $(a_{i_1}, a_{j_1}), \dots, (a_{i_m}, a_{j_m})$. The distances of the parallel lines which carry these segments are $d_1, \dots, d_m$, respectively, and $|\pi a_{i_k} \cap \pi a_{j_k}|$ denotes the length of the intersection of the orthogonal projections of both segments onto one of the carrying lines, $k = 1, \dots, m$. Then for $k \in \{1, \dots, m\}$ for which $|\pi a_{i_k} \cap \pi a_{j_k}|$ >0, the derivative of the chord length distribution function has a discontinuity at d_k, and the limit from above at d_k is infinite.

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