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Topology optimization for elastic base under rectangular plate subjected to moving load

SAMVEL H. JILAVYAN and ASATUR ZH. KHURSHUDYAN

Distribution optimization of elastic material under elastic isotropic rectangular thin plate subjected to concentrated moving load is investigated in the present paper. The aim of optimization is to damp its vibrations in finite (fixed) time. Accepting Kirchhoff hypothesis with respect to the plate and Winkler hypothesis with respect to the base, the mathematical model of the problem is constructed as two–dimensional bilinear equation, i.e. linear in state and control function. The maximal quantity of the base material is taken as optimality criterion to be minimized. The Fourier distributional transform and the Bubnov–Galerkin procedures are used to reduce the problem to integral equality type constraints. The explicit solution in terms of two–dimensional Heaviside‘s function is obtained, describing piecewise–continuous distribution of the material. The determination of the switching points is reduced to a problem of nonlinear programming. Data from numerical analysis are presented.

Key words: distributed system, moving load, Kirchhoff plate, Winkler base, topology optimization, problem of moments, L^∞-optimal control, distributions, Bubnov–Galerkin procedure.

1. Introduction

The optimal design problems are traditionally considered in order to minimize or maximize some parameters (weight, volume, load capacity and etc.) of a design with given structure [1]. In the monograph [1] a wide range of construction optimization problems of three main kinds: optimization of size, form and structure, are investigated. Recently, the so–called design topology optimization problems have begun to be investigated in order to minimize a specific functional describing material distribution in given domain, retaining or even improving desired properties of structures. The solution of topology optimization problems, unlike problems of structural optimization, where necessary conditions of optimality to be solved, are generally reduced to problem of nonlinear programming (see, for example, [2, 3, 4, 5, 6, 7, 8, 15, 16, 18, 17] and the references therein). Nevertheless, the explicit analytical form determination for unknown controls in such problems is connected with significant difficulties, and the numerical

The Authors are with Faculty of Mathematics and Mechanics, Yerevan State University. 1 Alex Manoogian str., 0025, Armenia. Correspondence Author is As. Zh. Khurshudyan, e-mail: khurshudyan@ysu.am
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solution requires high computational costs. Topology optimization of a construction can lead not only to an advantageous distribution of material in its volume, which is of direct practical importance, preserving or even optimizing its target properties; it also can directly affect the processes in which it is involved. It opens further opportunities as in engineering practice, as well as in manufacturing.

Such problems have not only practical importance, but stand out with complexity of investigation, because mathematically they are formulated as bilinear control systems, i.e. systems containing linear product of control and state function or their derivatives. Such systems can be used to model a wide range of physical, chemical, biological processes that cannot be effectively modelled under the assumption of linearity (see [19] and the references therein).

From the other hand side, vibration reduction and damping in particular systems present a very important engineering problem. Besides the literature cited above we refer to [9, 12, 14, 10, 13, 11] and the references therein. The investigation is devoted to vibration reduction or damping for structures with actuators or absorbers having fixed placements and configurations under it. In [8, 15, 16] we have attempted and here we are going to attempt to damp bending vibrations of elastic structures (beam and plate) by optimizing the way of placing the subgrade under those structures.

In general statement topology optimization problems require minimization of given criterion

$$\kappa[u] \underset{u}{\rightarrow} \min, \quad u \in \mathcal{U},$$

describing as usual the distribution of material in design, by choosing u bounded by our resources, under certain geometrical and characteristic constraints.

If, for example, the control problem is considered for a deformable design, equation of its motion

$$\mathcal{D}_u[w] = f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in O \times [0,T], \tag{1}$$

(or static equilibrium) may be considered as characteristic (differential) constraints, whereby $\boldsymbol{x} \in O \subset \mathbb{R}^3$ will be the geometrical constraint. Besides, the solution of (1) satisfies given conditions on boundary:

$$\mathcal{B}[w] = w_\partial(t), \quad (\boldsymbol{x},t) \in \partial O \times [0,T]. \tag{2}$$

In dynamical problems some initial conditions at $t = 0$ moment are also considered. Main purpose of control problem may be, for instance, ensuring given terminal conditions at $t = T$ moment.

The bilinear differential operator $\mathcal{D}_u[w]$ is defined in $O \times [0,T]$, $\mathcal{B}[\cdot]$ is a linear or nonlinear operator, representing the boundary conditions, $f(\boldsymbol{x},t)$ is a given function satisfying certain conditions.

Examples of operators $\mathcal{D}_u[\cdot]$ and $\mathcal{B}[\cdot]$ may be found, for instance, in the above cited literature and in [22, 23, 24, 20, 21].

2. Notations and abbreviations

We use the definitions

$$\theta(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \qquad \theta(x,y) = \theta(x)\theta(y) = \begin{cases} 1, & x > 0 \text{ and } y > 0, \\ 0, & \text{otherwise}, \end{cases}$$

for Heaviside‘s one and two-dimensional function, and $\delta = \mathcal{D}\theta$ for Dirac‘s delta function. The key technique we use to deal with distributions is their Fourier transform [26, 27], defined in terms of direct

$$\mathcal{F}_t[\eta] := \overline{\eta}(\sigma) = \int\limits_{-\infty}^{\infty} \eta(t) \exp[i\sigma t]\, dt,$$

and inverse operators

$$\mathcal{F}_t^{-1}[\overline{\eta}] = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \overline{\eta}(\sigma) \exp[-i\sigma t]\, d\sigma = \eta(t),$$

$\sigma \in \mathbb{R}$ is the transform parameter. We use also the operators $\mathcal{F}_s^{-1}[\cdot]$ and $\mathcal{F}_c^{-1}[\cdot]$ for Fourier inverse sine and cosine transforms

$$\mathcal{F}_s^{-1}[\overline{\eta}] = \frac{1}{\pi} \int\limits_0^{\infty} \overline{\eta}(\sigma) \sin(\sigma t) d\sigma, \;\; \mathcal{F}_c^{-1}[\overline{\eta}] = \frac{1}{\pi} \int\limits_0^{\infty} \overline{\eta}(\sigma) \cos(\sigma t) d\sigma.$$

The Fourier transform of Dirac‘s delta $\mathcal{F}[\delta(t-t_0)] = \exp[i\sigma t_0]$ is used during computations below.

We use the filtering property of Dirac‘s delta

$$\int\limits_{-\varepsilon}^{\varepsilon} \delta(t-t_0)\eta(t)dt = \eta(t_0), \;\; t_0 \in (-\varepsilon, \varepsilon), \;\; \varepsilon > 0,$$

for all test functions η [27].

The following properties are also used throughout the paper without special reference [27]:

$$\delta(x-x_0) = \frac{1}{|x_0|}\delta\left(\frac{x}{x_0} - 1\right), \;\; \theta(x-x_0) = \begin{cases} \theta\left(\frac{x}{x_0} - 1\right), & x_0 > 0, \\ 1 - \theta\left(\frac{x}{x_0} - 1\right), & x_0 < 0, \end{cases} \qquad x_0 \neq 0,$$

in the sense of distributions [27].

D is the bending stiffness of the plate, Δ is the two–dimensional Laplacian,

$$\delta_m^\mu = \delta_\mu^m = \int_{-\pi}^{\pi} \sin(\pi m x)\sin(\pi \mu x)dx = \begin{cases} 1, & m=\mu, \\ 0, & m\neq\mu, \end{cases} \quad \delta_{mn}^{\mu\nu} = \delta_m^\mu \delta_n^\nu,$$

are the Kronecker one and two-dimensional symbols. For short we write $\{1;N\}$ instead of $\{1,2,\ldots,N\}$.

The operator $\mathcal{A}_{[a,b]}[\cdot]$, introduced in [8, 15, 16, 25, 28], is defined as follows

$$\mathcal{A}_{[a,b]}[\eta] = \begin{cases} \eta(t), & t\in[a,b], \\ 0, & t\notin[a,b]. \end{cases}$$

It obviously may be represented in terms of characteristic function of $[a,b]$

$$\chi_{[a,b]}(t) = \begin{cases} 1, & t\in[a,b], \\ 0, & t\notin[a,b], \end{cases}$$

as follows

$$\mathcal{A}_{[a,b]}[\eta] = \chi_{[a,b]}(t)\eta(t) := \eta_1(t),$$

but it is more convenient to use the representation [8, 15, 16, 25, 28]

$$\mathcal{A}_{[a,b]}[\eta] = [\theta(t-a)-\theta(t-b)]\,\eta(t).$$

$\operatorname{supp}\eta = \overline{\{x\in\mathbb{R};\ \eta(x)\neq 0\}}$ denotes the support of η. It is obvious that $\operatorname{supp}\mathcal{A}_{[a,b]}[\eta] = [a,b]$.

Remark 1 The above introduced operator $\mathcal{A}_{[a,b]}[\cdot]$ is a linear continuous mapping from the space of ordinary functions into the Sobolev space of functions, which are concentrated on $[a,b]$ and has distributional derivatives of any order.

The derivatives of η may be expressed in terms of distributional derivatives of $\eta_1 = \mathcal{A}_{[a,b]}[\eta]$:

$$\mathcal{D}^n\eta_1(t) = \mathcal{A}_{[a,b]}[\mathcal{D}^n\eta] + \sum_{\nu=1}^{n-1} C_\nu^n \left[\mathcal{D}^\nu\eta(a)\mathcal{D}^{n-\nu-1}\delta(t-a) - \mathcal{D}^\nu\eta(b)\mathcal{D}^{n-\nu-1}\delta(t-b)\right],$$

(C_m^n are the binomial coefficients).

3. The method of solution

In order to solve the bilinear control problems when $\mathcal{B}[\cdot]$ is *linear* we suggest [15] to use the Bubnov–Galerkin procedure [29]. Suppose, that we have constructed a complete system of basis (approximating) functions $\{\varphi_n(\boldsymbol{x})\}_{n=0}^N$, the first one of which satisfies non–homogeneous, and the rest– to homogeneous boundary conditions (2). Then the residual, obtained as a result of substitution of approximate solution

$$w_N(\boldsymbol{x},t) = \varepsilon_0(t)\varphi_0(\boldsymbol{x}) + \sum_{n=1}^{N} \varepsilon_n(t)\varphi_n(\boldsymbol{x}), \quad (\boldsymbol{x},t) \in \overline{O} \times [0,T], \tag{3}$$

where the coefficients $\varepsilon_n(t)$ should be determined, t is considered as a parameter, into (1.1), will be

$$\mathcal{R}_N(\boldsymbol{x},t) = \mathcal{D}_u[w_N] - f(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \overline{O} \times [0,T]. \tag{4}$$

According to the Bubnov–Galerkin procedure, the unknown coefficients $\varepsilon(t)$ are determined from orthogonality conditions of the residual (4) and basis functions $\{\varphi_n(\boldsymbol{x}_)\}_{n=0}^N$

$$\int_{\overline{O}} \mathcal{R}_N(\boldsymbol{x},t)\varphi_n(\boldsymbol{x})d\boldsymbol{x} = 0, \quad t \in [0,T], \; n \in \{0;N\}. \tag{5}$$

Remark 2 If for some $N_0 \in \mathbb{N}$ the residual (4) is identically zero: $\mathcal{R}_{N_0}(\boldsymbol{x},t) \equiv 0$, then the corresponding function $w_{N_0}(\boldsymbol{x},t)$ (3) will be the exact solution of boundary–value problem (1), (2). Otherwise, increasing the number N of approximating functions $\{\varphi_n(\boldsymbol{x})\}_{n=0}^N$ we may increase the accuracy of approximation (3).

After determining the unknown coefficients $\varepsilon_n(t)$ from the system of linear equations (5) and substituting them into (3), and taking into account, that at T given terminal conditions must be satisfied, we will derive the system

$$w_N(\boldsymbol{x},T) = \varepsilon_0(T)\varphi_0(\boldsymbol{x}) + \sum_{n=1}^{N} \varepsilon_n(T)\varphi_n(\boldsymbol{x}), \quad \boldsymbol{x} \in \overline{O}.$$

Then, the unknown function u has to be determined from

$$w_{NT}^n = \varepsilon_n(T), \quad n \in \{0;N\} \tag{6}$$

in which

$$w_{NT}^n = \int_O w_N(x,T)\varphi_n(\boldsymbol{x})d\boldsymbol{x}.$$

Remark 3

i) Naturally, we should put the subscript N to the unknown u to underline that it is an approximation, but we omit it implying the dependence obvious.

ii) The complexity of the technique application depends on complexity of operator $\mathcal{D}_u[\cdot]$ and, especially, its component with respect to t: it defines the type (algebraic, differential, integral, integro-differential etc.) of (5) for $\varepsilon_n(t)$.

4. The problem statement

In this section we are going to apply the technique described in the previous section for a two–dimensional bilinear system. We consider an elastic, isotropic, solid rectangular plate of sufficiently small constant thickness $2h$, the middle plane of which in Cartesian system of coordinates occupies the domain

$$O_* = \{(x_*, y_*);\ x_* \in [-l_1, l_1],\ y_* \in [-l_2, l_2]\} \subset \mathbb{R}^2,\ \ \min(l_1, l_2) \gg 2h.$$

Let the plate lies on a linear–elastic one–parametric base with substrate ratio α_*^2, distributed in O_* by a controlled law $u = u(x, y)$. The plate is supposed to be simply supported by edges $x_* = \pm l_1$ and $y_* = \pm l_2$, as well as subjected to a constant normal load P_* distributed on the upper surface of the plate by given law $r_* = r_*(y_*)$, supp $r_* \neq \varnothing$, and moving along the upper surface of the plate in x_* direction with constant velocity v_*.

We accept the Kirchhoff hypothesis with respect to the plate and the Winkler hypothesis with respect to the elastic base. Then, the vertical displacements of the plate middle plane satisfy

$$D\Delta\Delta w_*(x_*, y_*, t_*) + \alpha_*^2 u(x_*, y_*) w_*(x_*, y_*, t_*) + 2\rho h \frac{\partial^2 w_*(x_*, y_*, t_*)}{\partial t_*^2} = f_*(x_*, y_*, t_*), \quad (7)$$

$$(x_*, y_*, t_*) \in O_* \times (0, T),$$

and the boundary conditions of simply supported edges

$$\begin{aligned} w_*(\pm l_1, y_*, t_*) &= \left.\frac{\partial^2 w_*(x_*, y_*, t_*)}{\partial x_*^2}\right|_{x_* = \pm l_1} = 0, \quad (y_*, t_*) \in [-l_2, l_2] \times [0, T], \\ w_*(x_*, \pm l_2, t_*) &= \left.\frac{\partial^2 w_*(x_*, y_*, t_*)}{\partial y_*^2}\right|_{y_* = \pm l_2} = 0, \quad (x_*, t_*) \in [-l_1, l_1] \times [0, T]. \end{aligned} \quad (8)$$

Above u is the law of the elastic base distribution in O_*, $f_*(x_*, y_*, t_*)$ characterizes the impact of the moving load on the plate:

$$f_*(x_*, y_*, t_*) = P_* \mathcal{A}_{[0, \tau_*]}[\delta(x_* + l_1 - v_* t_*)] r_*(y_*). \quad (9)$$

The initial state of the plate is known:

$$w_*(x_*, y_*, 0) = w_{0*}(x_*, y_*), \quad \left.\frac{\partial w_*(x_*, y_*, t_*)}{\partial t_*}\right|_{t_* = 0} = w_{0*}^1(x_*, y_*), \quad (x_*, y_*) \in O_*. \quad (10)$$

Our main purpose is the determination of an admissible control $u^o \in \mathcal{U} = \{0 \leqslant u \in L^\infty(O_*);\ |u| \leqslant 1,\ \operatorname{supp} u \subseteq O_*\}$ providing the following terminal state in required (fixed) time T:

$$w_*(x_*, y_*, T) = 0, \ \left.\frac{\partial w_*(x_*, y_*, t_*)}{\partial t_*}\right|_{t_*=T} = 0, \ (x_*, y_*) \in O_*, \tag{11}$$

as well as minimizing intensity of the elastic base distribution under the plate described by the functional

$$\kappa[u] = \max_{(x_*,y_*)\in O_*} |u(x_*, y_*)|, \quad u \in \mathcal{U}. \tag{12}$$

Remark 4 It is obvious that the plate vibrations may "vanish" solely after the load detaches from it. Thence we suppose that the load detaches from the plate at a given moment $\tau_* < T$, such that $v_*\tau_* = 2l_1$. Naturally, we have to restrict us by condition $T < T_0$, where T_0 is the time in which the vibrations will "vanish" in the case of $u(x, y) \equiv 1$.

We realize that, in general, terminal conditions (11) may not be provided by choice of any distribution u (even in the case $u \equiv 1$) exactly, because there will remain some residual stresses of very small amplitude inversely proportional to the base stiffness. Thus, we deal with a problem of *approximate* controllability, i.e. equality signs in (11) are, actually, approximate equalities.

We additionally suppose, that the consistency conditions concerning boundary, initial and terminal data are satisfied:

$$w_{0*}(\pm l_1, y_*) = w_{0*}(x_*, \pm l_2) = 0, \ w^1_{0*}(\pm l_1, y_*) = w^1_{0*}(x_*, \pm l_2) = 0,$$

$$\left.\frac{\partial^2 w_{0*}(x_*, y_*)}{\partial x_*^2}\right|_{x_*=\pm l_1} = \left.\frac{\partial^2 w_{0*}(x_*, y_*)}{\partial y_*^2}\right|_{y_*=\pm l_2} = 0.$$

Before proceeding to the control problem solution, let us transform (7)-(12) introducing the dimensionless variables and functions

$$w = \frac{w_*}{h},\ r = \frac{r_*}{l_2},\ x = \frac{x_*}{l_1},\ y = \frac{y_*}{d},\ t = \frac{2t_* - T}{T}\vartheta,\ \tau = \frac{2\vartheta}{T}\tau_*,\ v = \frac{v_*}{l_1}\frac{T}{2\vartheta},$$

$$\alpha^2 = \frac{\alpha_*^2 l_1^4}{D},\ \beta^2 = 2\rho h \frac{l_1^4}{D}\left(\frac{2\vartheta}{T}\right)^2,\ \gamma = \frac{l_1}{l_2},\ P_* = \frac{P d l_1^3}{Dh}.$$

Then, we will obtain

$$\mathcal{D}[w] + \alpha^2 u(x, y) w(x, y, t) + \beta^2 \frac{\partial^2 w(x, y, t)}{\partial t^2} = f(x, y, t), \quad (x, y, t) \in O \times (-\vartheta, \vartheta), \tag{13}$$

$$\begin{aligned} w(\pm 1, y, t) &= \left.\frac{\partial^2 w(x, y, t)}{\partial x^2}\right|_{x=\pm 1} = 0, \quad (y, t) \in [-1, 1] \times [-\vartheta, \vartheta], \\ w(x, \pm 1, t) &= \left.\frac{\partial^2 w(x, y, t)}{\partial y^2}\right|_{y=\pm 1} = 0, \quad (x, t) \in [-1, 1] \times [-\vartheta, \vartheta], \end{aligned} \tag{14}$$

$$\mathcal{D}[w]=\frac{\partial^4 w(x,y,t)}{\partial x^4}+2\gamma^2\frac{\partial^4 w(x,y,t)}{\partial x^2\partial y^2}+\gamma^4\frac{\partial^4 w(x,y,t)}{\partial y^4},$$
$$O:=(-1,1)\times(-1,1).$$

As a result of the variables change, the expression (9) transforms to

$$f(x,y,t)=P\mathcal{A}_{[-\vartheta,\vartheta-\tau]}\left[\delta(x+1-v(t+\vartheta))\right]r(y).$$

In order to deal with control problem under study we are aimed to use the Butkovskiy‘s generalized method, suggested in [25, 28]. For that purpose we apply the operator $\mathcal{A}_{[-\vartheta,\vartheta]}[\cdot]$ to (10) and (11). Taking into account that in the sense of distributions

$$\begin{aligned}[\theta(t+\vartheta)-\theta(t-\vartheta)]\frac{\partial^2 w(x,y,t)}{\partial t^2}&=\frac{\partial^2 w_1(x,y,t)}{\partial t^2}-[\delta(t+\vartheta)-\delta(t-\vartheta)]\frac{\partial w(x,y,t)}{\partial t}-\\&-[\delta'(t+\vartheta)-\delta'(t-\vartheta)]w(x,y,t)=\\&=\frac{\partial^2 w_1(x,y,t)}{\partial t^2}-w_0^1(x,y)\delta(t+\vartheta)-w_0(x,y)\delta'(t+\vartheta),\end{aligned}$$

we will arrive at

$$\mathcal{D}[w_1]+\alpha^2u(x,y)w_1(x,y,t)+\beta^2\frac{\partial^2 w_1(x,y,t)}{\partial t^2}=G(x,y,t),\quad (x,y,t)\in O\times\mathbb{R},\tag{15}$$

$$\begin{aligned}w_1(\pm1,y,t)&=\left.\frac{\partial^2 w_1(x,y,t)}{\partial x^2}\right|_{x=\pm1}=0,\quad (y,t)\in[-1,1]\times\mathbb{R},\\ w_1(x,\pm1,t)&=\left.\frac{\partial^2 w_1(x,y,t)}{\partial y^2}\right|_{y=\pm1}=0,\quad (x,t)\in[-1,1]\times\mathbb{R},\end{aligned}\tag{16}$$

where

$$G(x,y,t)=f_1(x,y,t)+\beta^2[w_0(x,y)\delta'(t)+w_0^1(x,y)\delta(t)].$$

Applying now the Fourier transform with respect to t variable to system (15), (16), we will obtain

$$\mathcal{D}[\overline{w}_1]+\left[\alpha^2u(x,y)-\sigma^2\beta^2\right]\overline{w}_1(x,y,\sigma)=\overline{G}(x,y,\sigma),\quad (x,y,\sigma)\in O\times\mathbb{R},\tag{17}$$

$$\begin{aligned}\overline{w}_1(\pm1,y,\sigma)&=\left.\frac{\partial^2 \overline{w}_1(x,y,\sigma)}{\partial x^2}\right|_{x=\pm1}=0,\quad (y,\sigma)\in[-1,1]\times\mathbb{R},\\ \overline{w}_1(x,\pm1,\sigma)&=\left.\frac{\partial^2 \overline{w}_1(x,y,\sigma)}{\partial x^2}\right|_{y=\pm1}=0,\quad (x,\sigma)\in[-1,1]\times\mathbb{R},\end{aligned}\tag{18}$$

$$\overline{G}(x,y,\sigma)=\overline{f}_1(x,y,\sigma)+\beta^2[w_{01}^1(x,y)-i\sigma w_{01}(x,y)],$$

$$\overline{f}_1(x,y,\sigma)=\frac{P}{v}[\theta(x+1)-\theta(x-1)]\exp\left[i\sigma\left(\frac{x+1}{v}+\vartheta\right)\right]r(y).$$

It is taken into account here that $v\tau = 2$.

In order to solve this problem we aimed to apply the Bubnov–Galerkin procedure. Orthonormal in O system of functions $\{\sin(\pi mx)\sin(\pi ny)\}_{m,n\in\mathbb{N}}$, is, obviously, approximating for boundary value problem (17), (18), and therefore we may look its solution for as follows:

$$\overline{w}_{1N}(x,y,\sigma) = \sum_{m,n=1}^{N} \bar{\varepsilon}_{mn}(\sigma)\sin(\pi mx)\sin(\pi ny), \quad (x,y,\sigma)\in O\times\mathbb{R}, \tag{19}$$

where $\bar{\varepsilon}_{mn}(\sigma)$ are unknown yet. Applying the procedure described above, we will derive the following system of linear algebraic equations for $\bar{\varepsilon}_{mn}(\sigma)$:

$$\sum_{m,n=1}^{N} \Lambda_{mn}^{\mu\nu}(\sigma)\bar{\varepsilon}_{mn}(\sigma) = \Omega_{\mu\nu}(\sigma), \;\; \mu,\nu\in\{1;N\}, \tag{20}$$

where

$$\Lambda_{mn}^{\mu\nu}(\sigma) = \Gamma_{mn}\delta_{mn}^{\mu\nu} + \alpha^2 J_{mn}^{\mu\nu}[u], \;\; \Gamma_{mn} = \left[(\pi m)^2 + \gamma^2(\pi n)^2\right]^2 - \sigma^2\beta^2,$$

$$J_{mn}^{\mu\nu}[u] = \int_{-1}^{1}\int_{-1}^{1} u(x,y)\sin(\pi mx)\sin(\pi\mu x)\sin(\pi ny)\sin(\pi\nu y)dxdy,$$

$$\Omega_{\mu\nu}(\sigma) = Y_\mu(\sigma)\int_{-1}^{1} r(y)\sin(\pi\nu y)dxdy + \beta^2\left[K_{\mu\nu} - i\sigma L_{\mu\nu}\right],$$

$$Y_\mu(\sigma) = \frac{(-1)^{\mu+1}\pi\mu\cdot Pv}{\sigma^2 + (\pi\mu v)^2}\left[1 - \exp\left[\frac{2i\sigma}{v}\right]\right]\exp\left[i\sigma\vartheta\right],$$

$$K_{\mu\nu} = \int_{-1}^{1}\int_{-1}^{1} w_{01}^{1}(x,y)\sin(\pi\mu x)\sin(\pi\nu y)dxdy,$$

$$L_{\mu\nu} = \int_{-1}^{1}\int_{-1}^{1} w_{01}(x,y)\sin(\pi\mu x)\sin(\pi\nu y)dxdy.$$

It is obvious, that $J_{mn}^{mn}[u] \geqslant 0$, $J_{mn}^{\mu\nu}[u] = J_{nm}^{\nu\mu}[u]$, $m,n,\mu,\nu\in\{1;N\}$.

Remark 5 Since the Chebyshev polynomials of the first kind $\{\mathrm{T}_m(x)\mathrm{T}_n(y)\}_{m,n\in\mathbb{N}}$ are orthogonal in O and provide more accuracy of approximation compare with that by trigonometric system $\{\sin(\pi mx)\sin(\pi ny)\}_{m,n\in\mathbb{N}}$ [29], it is more efficient to use them instead. But we take into account, that the aim is to show that the proposed algorithm works in dimensions two, and not the accuracy of the approximation.

Since $w_1(x,y,t) \equiv \mathcal{A}_{[-\vartheta,\vartheta]}[w]$ by the definition is compactly supported in $[-\vartheta,\vartheta]$, then, according to Wiener–Paley–Schwartz theorem [26, 27], the extension $\overline{w}_1(x,\sigma+i\varsigma)$, $\varsigma\in\mathbb{R}$, is entire. It means, that $\Delta_{11}(z)=0$, as long as $\Delta_0(z)=0$, where Δ_0 and Δ_{11} are the main and (for instance) the first auxiliary determinants of (20), respectively.

Remark 6

i) It is easy to see from the expression of $\Lambda_{mn}^{\mu\nu}$, the determinant $\Delta_0(\sigma)$ is a polynomial of degree $2N$, therefore $\Delta_0(z)=0$ admits $2N$ complex roots. From the other hand side

$$\text{Re }\Delta_0(-\sigma-i\varsigma)=\text{Re }\Delta_0(\sigma+i\varsigma),\ \ \text{Im }\Delta_0(-\sigma-i\varsigma)=\text{Im }\Delta_0(\sigma+i\varsigma),$$

which means that $z_k=\sigma_k+i\varsigma_k$ and $z_k=-\sigma_k-i\varsigma_k$ satisfy $\Delta_0(z)=0$ simultaneously. Then, since

$$\text{Re }\Delta_{11}(-\sigma-i\varsigma)=\text{Re }\Delta_{11}(\sigma+i\varsigma),\ \ \text{Im }\Delta_{11}(-\sigma-i\varsigma)=\text{Im }\Delta_{11}(\sigma+i\varsigma),$$

we conclude that $\Delta_{11}(z)=0$ contains only N independent conditions.

ii) Moreover, even though

$$\text{Re }\Delta_0(-\sigma+i\varsigma)=\text{Re }\Delta_0(\sigma-i\varsigma)=\text{Re }\Delta_0(\sigma+i\varsigma)$$

$$\text{Im }\Delta_0(-\sigma+i\varsigma)=\text{Im }\Delta_0(\sigma-i\varsigma)=-\text{Im }\Delta_0(\sigma+i\varsigma),$$

nevertheless $\Delta_{11}(\sigma+i\varsigma)$ does not have the same property, therefore the number of independent equalities cannot be reduced further.

Suppose that we were able to solve $\Delta_0(z)=0$ and find the roots z_k, $k\in\{1;N\}$. Substituting them into $\Delta_{11}(z)=0$ and separating its real and imaginary parts, with respect to $J_{mn}^{\mu\nu}[u]$ we will obtain the system of restrictions

$$J_{mn}^{\mu\nu}[u]=\mathcal{M}_{mn}^{\mu\nu},\ \ m,n,\mu,\nu\in\{1;N\},$$

According to [17, 28], the solution will be

$$u^o(x,y)=\sum_{j=1}^{J}\left[\theta\left(x-x_j^o,y-y_j^o\right)-\theta\left(x-x_{j+1}^o,y-y_{j+1}^o\right)\right],\ \ (x,y)\in O. \tag{21}$$

The obtained solution describes the distribution law of the elastic base under the plate, and the set of points $\{x_j^o,y_j^o\}_{j=1}^J\in O$ underlines the domains where the base exists and depends on inner and external parameters $D,r(y),v,\tau,P,\alpha^2,\beta^2,\gamma$.

After determination of optimal solution (21), we might need also the optimal deflection of the plate. Applying the Fourier inverse transform to (19)we will obtain

$$w_{1N}(x,y,t)=\sum_{m,n=1}^{N}\varepsilon_{mn}(t)\sin(\pi mx)\sin(\pi ny),\ \ (x,y,t)\in\overline{O}\times\mathbb{R}, \tag{22}$$

Remark 7 Taking into account that

$$\Lambda^{\mu\nu}_{mn}(-\sigma) = \Lambda^{\mu\nu}_{mn}(\sigma), \;\; \text{Re}\,\Omega_{\mu\nu}(-\sigma) = \text{Re}\,\Omega_{\mu\nu}(\sigma), \quad \text{Im}\,\Omega_{\mu\nu}(-\sigma) = -\text{Im}\,\Omega_{\mu\nu}(\sigma),$$

we see that

$$\text{Re}\,\bar{\varepsilon}_{mn}(-\sigma) = \text{Re}\,\bar{\varepsilon}_{mn}(\sigma), \;\; \text{Im}\,\bar{\varepsilon}_{mn}(-\sigma) = -\text{Im}\,\bar{\varepsilon}_{mn}(\sigma).$$

It is necessary and sufficient for $\varepsilon_{mn}(t) = \mathcal{F}_t^{-1}[\bar{\varepsilon}_{mn}]$, and therefore for $w_{1N}(x,y,t)$, to be real valued. Then,

$$\varepsilon_{mn}(t) = \frac{1}{\pi}\int\limits_0^\infty [\text{Re}\,\bar{\varepsilon}_{mn}(\sigma)\cos(\sigma t) + \text{Im}\,\bar{\varepsilon}_{mn}(\sigma)\sin(\sigma t)]\,d\sigma = \mathcal{F}_c^{-1}[\text{Re}\,\bar{\varepsilon}_{mn}] + \mathcal{F}_s^{-1}[\text{Im}\,\bar{\varepsilon}_{mn}],$$

where

$$\text{Re}\,\bar{\varepsilon}_{mn}(\sigma) = \frac{\text{Re}\,\Delta_{mn}(\sigma)\text{Re}\,\Delta_0(\sigma) + \text{Im}\,\Delta_{mn}(\sigma)\text{Im}\,\Delta_0(\sigma)}{(\text{Re}\,\Delta_0(\sigma))^2 + (\text{Im}\,\Delta_0(\sigma))^2},$$

$$\text{Im}\,\bar{\varepsilon}_{mn}(\sigma) = -\frac{\text{Re}\,\Delta_{mn}(\sigma)\text{Im}\,\Delta_0(\sigma) - \text{Im}\,\Delta_{mn}(\sigma)\text{Re}\,\Delta_0(\sigma)}{(\text{Re}\,\Delta_0(\sigma))^2 + (\text{Im}\,\Delta_0(\sigma))^2}.$$

5. Numerics

To demonstrate the procedure of the switching points calculation, let us consider a numerical experiment. Let

$$w_{01}(x,y) = \sin(\pi x)\sin(\pi y), \;\; \dot{w}_{01}(x,y) = 0.$$

We additionally assume, that the moving load has a point support: $r(y) = \delta(y - y_0)$, $y_0 \in (-1,1)$. Then, limiting the consideration by $N = 3$ we have the results combined in Figures 1 and 2 and in Tables 1–3 for various values of dimensionless parameters D, α^2, β^2, γ, P, v, τ. Poisson's ratio of the plate material is taken equal to 0.25.

Since the expressions for Δ_0 and Δ_{11} are unwieldy and it is useless to bring them here, we bring the expressions for

$$J^{\mu\nu}_{mn}[u] = \sum_{j=1}^{J} \int\limits_{x^o_j}^{x^o_{j+1}} \sin(\pi m x)\sin(\pi\mu x)dx \int\limits_{y^o_j}^{y^o_{j+1}} \sin(\pi n y)\sin(\pi\nu y)dy,$$

$$\Omega_{\mu\nu}(z) = \frac{(-1)^{\mu+1}\pi\mu\cdot vP}{z^2 + (\pi\mu v)^2}\left[1 - \exp\left[\frac{2iz}{v}\right]\right]\exp[iz\vartheta]\cdot\sin(\pi\nu y_0) - iz\beta^2\delta^\mu_1\delta^\nu_1,$$

Table 2: $\beta^2 = 50$, $\gamma = 0.25$

α^2	P	v	τ	y_0	x_1^o / y_1^o	x_2^o / y_2^o	x_3^o / y_3^o	x_4^o / y_4^o
0.01	0.1	0.01	100	-0.25	0.1973 0.3307	0.2650 0.4931	0.2691 0.5330	0.3355 0.6699
0.01	0.5	0.05	150	0.5	−0.4444 −0.4444	−0.2989 −0.2989	0.3074 0.3074	0.4349 0.4349
0.05	0.1	0.01	100	0.25	−0.7264 −0.7809	−0.5139 −0.4936	−0.0475 −0.0077	0.0969 0.1864
0.05	0.5	0.1	150	0.25	−0.4167 −0.4167	−0.2703 −0.2703,	0.3270 0.3270	0.4466 0.4466
0.1	0.1	0.1	100	-0.5	−0.7557 −0.5619	0.1406 0.0554	0.5143 0.4113	0.7555 0.9799
0.1	0.5	0.1	150	0.5	−0.4371 −0.4371	−0.2963 −0.2963	0.3042 0.3042	0.4282 0.4282

Table 3: $\beta^2 = 25$, $\gamma = 1$

α^2	P	v	τ	y_0	x_1^o / y_1^o	x_2^o / y_2^o	x_3^o / y_3^o	x_4^o / y_4^o
0.01	0.1	0.01	100	0.25	−0.8376 −0.9815	−0.6615 −0.4452	0.6087 −0.3912	0.6897 0.0556
0.01	0.5	0.05	150	-0.75	−0.4146 −0.4146	−0.2677 −0.2677	0.3283 0.3283	0.4349 0.4477
0.05	0.1	0.01	100	-0.1	−0.2748 −0.3973	−0.2709 −0.2035	0.5555 0.2735	0.7504 0.3567
0.05	0.5	0.1	150	0.1	−0.4421 −0.4421	−0.2965 −0.2965	0.2926 0.2926	0.4228 0.4228
0.1	0.1	0.05	100	0.4	−0.2065 −0.2984	−0.1324 −0.2968	0.4080 0.3931	0.4781 0.3943
0.1	0.5	0.1	150	0.1	−0.4211 −0.4211	−0.2759 −0.2759	0.3234 0.3233	0.4437 0.4437

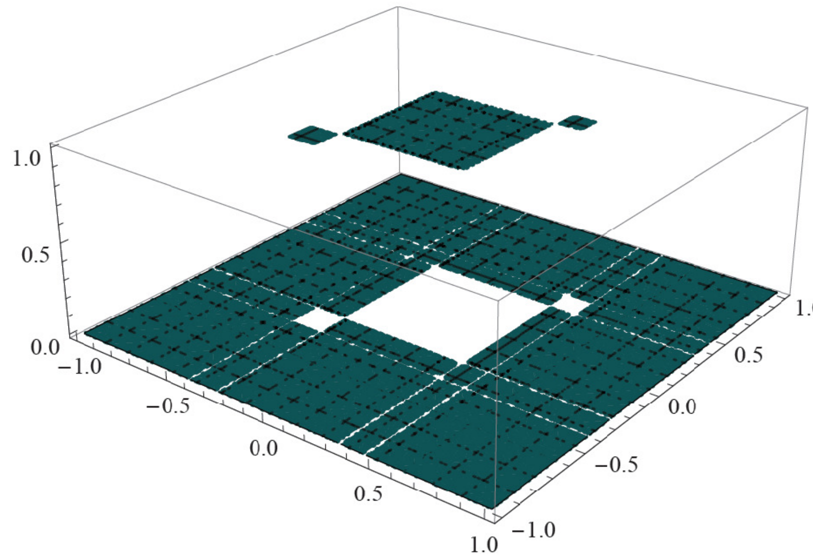

Figure 1: $u^o(x,y)$ when $\beta^2 = 50, \gamma = 0.25, \alpha^2 = 0.05, P = 0.5, v = 0.1, \tau = 150$ and $y_0 = 0.25$.

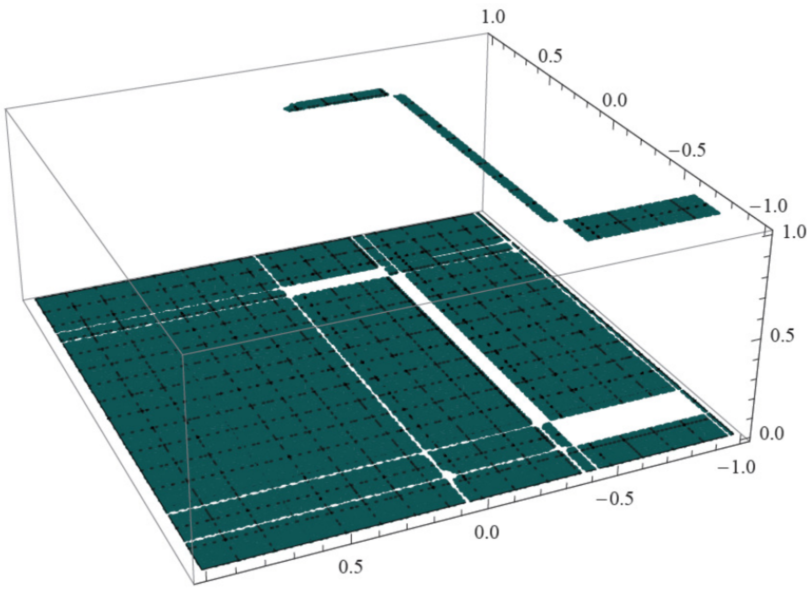

Figure 2: $u^o(x,y)$ when $\beta^2 = 25, \gamma = 1, \alpha^2 = 0.01, P = 0.5, v = 0.05, \tau = 150$ and $y_0 = -0.75$

Table 4: $\beta^2 = 20, \gamma = 2$

α^2	P	v	τ	y_0	x_1^o y_1^o	x_2^o y_2^o	x_3^o y_3^o	x_4^o y_4^o
0.01	0.1	0.01	100	0.25	−0.4258 −0.3881	−0.2114 −0.2724	0.3054 = 0.4269	0.4408, 0.4270
0.01	0.5	0.05	150	-0.75	−0.4389 −0.4146	−0.2925 −0.2925	0.3025 0.3025	0.4308, 0.4308
0.05	0.1	0.05	100	-0.4	−0.4360 −0.4360	−0.2888 −0.2888	0.3136 0.3136	0.4399, 0.4399
0.05	0.5	0.1	150	0.4	−0.4105 −0.3583	−0.1466 −0.3567	0.4508 0.4778	0.5467, 0.5478
0.1	0.1	0.01	100	0.25	−0.4456 −0.3596	−0.2521 −0.2483	0.4542 0.2450	0.4566 0.46
0.1	0.5	0.1	150	- -0.25	−0.2776 −0.1503	0.0706 −0.1494	0.4388 0.5020	0.5237, 0.5192

6. Conclusion

The solution of distribution optimization problem for elastic material in rectangular domain is obtained explicitly using the Butkovskiy‘s generalized method and the Bubnov–Galerkin procedure in turn. The aim is the minimization of the material quantity in order to damp the bending vibrations of a simply supported isotropic elastic plate caused by a moving load in required time. The mathematical model is constructed as a bilinear partial differential equation of fourth order. Introducing a new, generalized state function, the state equation is written in the class of distributions and the initial conditions are included in it. Involving the Fourier distributional transform, the governing system is reduced to a boundary–value problem for partial differential equation with two independent variables. The solution is approximated by means of Bubnov–Galerkin orthonormal sequence and with respect to approximation coefficients a system of algebraic equations is obtained. Those coefficients are found and extended in $\mathbb{C}$ as entire functions. Resolving conditions (necessary and sufficient) are the zeros of one of the auxiliary determinants extended in $\mathbb{C}$ in roots of the main determinant extended in $\mathbb{C}$. It turns out that the piecewise–continuous distribution of the material is optimal and is determined uniquely via solution of a problem of nonlinear programming. Numerical analysis is done for a particular initial state and for various values of the system internal and external parameters.

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