On the Number of Solutions of Systems of Equations with not everywhere defined Boolean Functions

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Introduction. Many problems of discrete mathematics, including problems which are traditionally considered to be complex, lead to the solutions of the systems of Boolean equations of the form

$$\begin{cases} f_i\left(x_1, \ldots, x_n\right) = 1 \\ i = 1, ..., l \end{cases} \tag{1}$$

or to the revealing of those conditions, under which the system (I) has a solution. In general problem of realizing whether the system (l) has a solution or not is NP-complete [1]. Therefore it is often necessary to consider special classes of the systems of equations, using their specificity, or explore a number of solutions for the "typical" case.

Give a necessary definitions. Let $\{M(n)\}_{n=1}^{\infty}$ is the collection of sets, such that $|M(n)| \to \infty$when$n \to \infty$, ($|M|$ is the power of the set$M$), and $M^s(n)$ is the subset of the all elements from $M(n)$, which have the property S. We say, that almost all the elements of the set $M(n)$ have the property$S$, if $\left|M^S(n)\right| / |M(n)| \to 1$, when $n \to \infty$.

We denote by $S_{n,l}$ the set of all the systems of the form (1), where $f_i\left(x_1, \ldots, x_n\right), i = 1, ..., l-$ pairwise different Boolean functions of variables $x_1, x_2, ..., x_n$. It is easy to see, that $|S_{n,l}| = C_{2^{2^n}}^{l}$.

Let $B = \{0, 1\}$,$B^n = \{\tilde{\alpha}/\tilde{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n), \alpha_i \in B, 1 \le i \le n\}$. The vector $\tilde{\alpha}_i = (\alpha_1, \alpha_2, ..., \alpha_n) \in B^n$is called a solution of (1), if

$$\begin{cases} f_i(\alpha_1, \alpha_2, ..., \alpha_n) = 1 \\ i = 1, ..., l \end{cases}$$

We denote by $t(S)$ the number of the solutions of the system S. In [2,3] it is shown the asymptotics of the number of the solutions $t(S)$ for almost all the systems S of the set $S_{n,l}$ the whole range of parameter l changes, when $n \to \infty$. In [4] are investigated a class of systems of Boolean equations of a special form. Determined the asymptotic estimates of the number of solutions of systems of equations.

In this paper a class of systems of equations with partial (not everywhere defined) Boolean functions is considered. Found the asymptotic behavior of the number of solutions of systems of equations for a "typical" case.

Partial Boolean function $f(x_1, \ldots, x_n)$ on the vector $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots.., \alpha_n) \in B^n$ or is not defined, or is 0 or 1. Let $Q(n)$ denote the set of all partial Boolean functions, depending on variables $x_1, x_2, ..., x_n$. Obviously, $|Q(n)| = 3^{2^n}$. Let $R(n,l)$ denote the set of all systems of l equations of the form (1), where $f_i(x_1, \ldots, x_n), i = 1, ..., l$ are pairwise differing partial Boolean functions of the variables $x_1, x_2, ..., x_n$ ($f_i \neq f_j$ if $i \neq j$condition persists). It is easy to see, that $|R_{n,l}| = C^l_{3^{2^n}}$. For the numbers of the solutions $t(S)$of almost all the systems S of the set $R(n,l)$ the following statement is true (here and further $f(n) \sim g(n)$, if $f(n)/g(n) \to 1$ when $n \to \infty$, $f(n) = o(g(n))$ if $f(n)/g(n) \to 0$ when $n \to \infty$. Everywhere under the log refers to the logarithm to the base 2).

Theorem 1.
1.If $n - \ell \log 3 \to \infty$ when $n \to \infty$, then for almost all the systems $S$of the set $R(n,l)$ occurs $t(S) \sim 2^n 3^{-l}$.
2. If$n - \ell \log 3 \to -\infty$ when $n \to \infty$, then almost all the systems $S$of the set $R(n,l)$ have no solutions.
3. If$n - \ell \log 3$is restricted when $n \to \infty$, then for almost all the systems of the set $R(n,l,m)$ the number of the solutions $t(S)$is restricted from above by an arbitrary function $\varphi(n)$, satisfying the condition $\varphi(n) \to \infty$, when $n \to \infty$.

References

[1] Geri M., Johnson D. Computers and Intractability.Moscow, Mir, 1982 (In Russian)

[2] Yeghiazaryan E.V. Metric properties of systems of Boolean equations. DAN Armenian SSR, m. 72, N2,pp.67-72, 1981 (In Russian).

[3] Yeghiazaryan E.V. Estimates related to the number of solutions of Boolean equations. Coll. Tasks of Cybernetics, combinatorial analysis and graph theory, Moscow, 1980, pp.124-130 (In Russian).

[4] Yeghiazaryan E.V. ,Tonoyan G.P. An asymptotic Estimate of the number of solutions of the System of Boolean Equations of the special form. Proceedings of the YSU,Phyzical and Mathematical Sciences,2016, N1, pp.35-39.