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NON-IDEMPOTENT PLONKA FUNCTIONS AND WEAKLY PLONKA SUMS

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We introduce concepts of a non-idempotent Plonka function and a weakly Plonka sum and establish an appropriate correspondence between these objects. In the last part of the paper we construct examples of non-idempotent Plonka functions and the corresponding weakly Plonka sums. In particular, we prove that every weakly idempotent quasilattice is a weakly idempotent lattice or a weakly Plonka sum of weakly idempotent lattices.

Keywords: *weakly idempotent semilattice, weakly idempotent lattice, hyperidentity, non-idempotent Plonka function, weakly Plonka sum, weakly idempotent quasilattice.*

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РУССКАЯ АННОТАЦИЯ
В статье вводятся понятия неидемпотентной функции Плонка и слабой функции Плонка, а так же устанавливается связь между этими объектами. В последней части работы нами построены примеры неидемпотентных фукций Плонка и соответствующих им сумм Плонка. В частности, доказано, что каждая слабо идемпотентная кваирешетка является либо слабо идемпотентной решеткой, либо суммой Плонка слабо идемпотентных решеток.

КЛЮЧЕВЫЕ СЛОВА: слабо идемпотентная полурешетка, слабо идемпотентная решетка, сверхтождество, неидемпотентная функция Плонка, слабая сумма Плонка, слабо идемпотентная квазирешетка.

Introduction

There exist various extensions of the concept of lattice. For example, in works [1], [2] weakly associative lattices were introduced. In [3] an algebra with a system of identities was introduced , which we call weakly idempotent lattices (see also [4], [5]). In [6] the concepts of p-function and of direct sum of algebras with l.u.b.-property were introduced. And there it was proved that there is a bijective correspondence between every p-function of the algebra U and its representation as a direct sum of algebras with l.u.b.-property. In this paper we introduce concepts of a non-idempotent Plonka function and a weakly Plonka sum and establish an appropriate correspondence between these objects. In the last part of the work we construct examples of non-idempotent Plonka functions and the corresponding weakly Plonka sums. In particular, we prove that every weakly idempotent quasilattice is a weakly idempotent lattice or a weakly Plonka sum of weakly idempotent lattices.

Definition 1 *The algebra* $(L;\wedge)$ *with one binary operation is called weakly idempotent* *semilattice, if it satisfies the following identities:*

$$a \wedge b = b \wedge a, (commutativity) \quad (1)$$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c), (associativity) \quad (2)$$

$$a \wedge (b \wedge b) = a \wedge b. (weakly \ \ idempotency) \quad (3)$$

The operation $\wedge$ is called product. Adding the idempotent identity $a \wedge a = a$ to it, we obtain a semilattice. The element $a \in L$ is called idempotent of the weakly idempotent semilattice $(L;\wedge)$, if $a \wedge a = a$. The set of the idempotent elements of each weakly idempotent semilattice forms a semilattice, i.e. the product of any two idempotent elements in a weakly idempotent semilattice is an idempotent element.

Definition 2 *The algebra $(L;\wedge,\vee)$ with two binary operations is called weakly idempotent lattice if the reducts $(L;\wedge)$ and $(L;\vee)$ are weakly idempotent semilattices and the following identities are valid:*

$$a \wedge (b \vee a) = a \wedge a, a \vee (b \wedge a) = a \vee a, (weakly \ absorption) \quad (4)$$
$$a \wedge a = a \vee a. (equalization) \quad (5)$$

For example, the following algebra $(Z \setminus \{0\};\wedge,\vee)$, where $x \wedge y = (|x|,|y|)$ and $x \vee y = [|x|,|y|]$,for which $(|x|,|y|)$ and $[|x|,|y|]$ are the greatest common divisor (gcd) and the least common multiple (lcm) of $|x|$ and $|y|$, is a weakly idempotent lattice, which is not a lattice, since $x \wedge x \neq x$ and $x \vee x \neq x$ for negative x.

Let us recall that a hyperidentity is a second-order formula of the following type:

$$\forall X_1, \ldots, X_m \forall x_1, \ldots, x_n (w_1 = w_2),$$

where $X_1, \ldots, X_m$ are functional variables, and $x_1, \ldots, x_n$ are object variables in the words (terms) of w_1, w_2. Hyperidentities are usually written without quantifiers: $w_1 = w_2$. We say that in the algebra $(Q;F)$ the hyperidentity $w_1 = w_2$ is satisfied if this equality is valid, when every object variable and every functional variable in it is replaced by any element from Q and by any operation of the corresponding arity from F (supposing the possibility of such replacement) [7]-[5].

On characterization of hyperidentities of varieties of lattices, modular lattices, distributive lattices, Boolean and De Morgan algebras see in [4] - [10]. About hyperidentities in thermal (polynomial) algebras, see in [17]-[19].

Note that every weakly idempotent lattice $(L;\wedge,\vee)$ satisfies the following hyperidentity (interlaced hyperidentity):

$$X(Y(X(x,y),z),Y(x,z)) = Y(X(x,y),z).$$

Definition 3 *A binary algebra $\mathfrak{A} = (U,\Sigma)$ is called a weakly idempotent quasilattice, if* *it satisfies the following hyperidentities:*

$$X(x,x)=Y(x,x), \tag{6}$$
$$X(x,y)=X(y,x), \tag{7}$$
$$X(x,X(y,z))=X(X(x,y),z), \tag{8}$$
$$X(x,X(y,y))=X(x,y), \tag{9}$$
$$X(Y(X(x,y),z),Y(x,z))=Y(X(x,y),z). \tag{10}$$

Note that every weakly idempotent lattice satisfies the hyperidentities of this definition.

Non-idempotent Plonka functions, weakly Plonka sums

Definition 4 *An algebra* $\mathfrak{A}=(U;\Sigma)$ *is called a weakly Plonka sum of its subalgebras* $(U_i;\Sigma)$, *where* $i\in I$, *if the following conditions are valid* (cf. [5], [6], [20], [12])*:*

i) $U_i\cap U_j=\varnothing$, for all $i,j\in I, i\neq j$;

ii) $U=\bigcup_{i\in I}U_i$;

iii) On the set of the indices I there exists a relation "$\leq$" such that $(I;\leq)$ is an upper semilattice with the following properties;

iv) If $i\leq j$; then there exists a homomorphism $\varphi_{i,j}:(U_i,\Sigma)\mapsto(U_j,\Sigma)$, where $\varphi_{i,j}\cdot\varphi_{j,k}=\varphi_{i,k}$, for $i\leq j\leq k$ and $\varphi_{i,i}(x)=F(x,\ldots,x)$ for any $F\in\Sigma$ and $x\in U_i$;

v) for all $A\in\Sigma$ and for all $x_1,\ldots,x_n\in Q$ the following equality is valid:

$$A(x_1,\ldots,x_n)=A(\varphi_{i_1,i_0}(x_1),\ldots,\varphi_{i_n,i_0}(x_n)),$$

where the arity $|A|=n$, $x_1\in U_{i_1},\ldots,x_n\in U_{i_n}, i_1,\ldots,i_n\in I, i_0=sup\{i_1,\ldots,i_n\}$.

If an algebra $\mathfrak{A}=(U;\Sigma)$ is a weakly Plonka sum of its subalgebras $(U_i;\Sigma)$, where $i\in I$, then we write $\mathfrak{A}=Sum(U_i)$.

The identity (hyperidentity) $\sigma_1=\sigma_2$ is said to be regular, if the set of object variables occurring in σ_1 is equal to the set of object variables occurring in σ_2.

Theorem 5 *Let* $\mathfrak{A}=(U;\Sigma)$ *be a weakly Plonka sum of its subalgebras* $(U_i;\Sigma)$, *where* $i\in I$ *and* $|I|\geq 2$. *Then in the algebra* $Sum(U_i)$ *are satisfied all regular identities (hyperidenties) that are satisfied in each algebra* $(U_i;\Sigma)$ *and moreover, every other identity (hyperidentity) is false in* $Sum(U_i)$.

Proof. The first part of theorem obviously follows from the definition 1.1. Let us prove the second part. Let $\sigma_1=\sigma_2$ be a non regular identity satisfied in $Sum(U_i)$ and hence is satisfied in each subalgebra $(U_i;\Sigma)$. Let us denote the set of variables occurring in σ_1 by $Var(\sigma_1)$ and that of σ_2 -

by $Var(\sigma_2)$. So there exists a variable $x_j \in Var(\sigma_1) \setminus Var(\sigma_2)$. Since $|I| \geq 2$, then there is $i_1, i_2 \in I$ such that $i_1 \leq i_2$. Let us choose $a \in A_{i_1}$ and $b \in A_{i_2}$ and substitute $x_i = \begin{cases} a, if \ i \neq j, \\ b, if \ i = j. \end{cases}$ in the equation $\sigma_1 = \sigma_2$. Then obviously, $\sigma_1 \in A_{i_1}$, $\sigma_2 \in A_{i_2}$ and since $A_{i_1} \cap A_{i_1} \neq \varnothing$ we have: $\sigma_1 \neq \sigma_2$.

Definition 6 *Let* $\mathfrak{A} = (U; \Sigma)$ *be an algebra. The binary operation* $f : U \times U \mapsto U$ *is called a non-idempotent Plonka function of* U*, if it satisfies the following identities* (cf. [6], [11], [12]):

1. $f(f(x, y), z) = f(x, f(y, z))$;
2. $f(x, x) = F_t(x, \ldots, x)$, for any operation $F_t \in \Sigma$;
3. $f(x, f(y, z)) = f(x, f(z, y))$;
4. $f(F_t(x_1, \ldots, x_{n(t)}), y) = F_t(f(x_1, y), \ldots f(x_{n(t)}, y))$, for any operation $F_t \in \Sigma$;
5. $f(y, F_t(x_1, \ldots, x_{n(t)})) = f(y, F_t(f(y, x_1), \ldots f(y, x_{n(t)})))$, for any operation $F_t \in \Sigma$;
6. $f(F_t(x_1, \ldots, x_{n(t)}), x_i) = F_t(x_1, \ldots, x_{n(t)})$, for any $1 \leq i \leq n(t)$ and for any operation $F_t \in \Sigma$;
7. $f(F_t(x_1, \ldots, x_{n(t)}), F_t(x_1, \ldots, x_{n(t)})) = F_t(x_1, \ldots, x_{n(t)})$, for any operation $F_t \in \Sigma$;
8. $f(x, f(x, y)) = f(x, y)$.

To obtain a non-idempotent Plonka function different from Plonka function one is to assume that no operation of the algebra $\mathfrak{A}$ is idempotent.

Theorem 7 *Let* $\mathfrak{A} = (U; \Sigma)$ *be an algebra with a non-idempotent Plonka function. Then* $\mathfrak{A}$ *is a weakly Plonka sum of its subalgebras.*

Proof. Define on the set U the relation $\alpha \subseteq U \times U$ in the following way:

$$a \alpha b \leftrightarrow f(a, b) = f(a, a),\ f(b, a) = f(b, b),$$

where f is a non-idempotent Plonka function for $\mathfrak{A}$.

Let us show that α is an equivalence on U. Indeed, reflexivity and symmetricity immediately follow from the definition. Show transitivity: let $a\alpha b$ and $b\alpha c$, then $f(a, b) = f(a, a),\ f(b, a) = f(b, b),\ f(b, c) = f(b, b),\ f(c, b) = f(c, c)$. Hence:

$$f(a, c) \overset{8}{=} f(a, f(a, c)) \overset{1}{=} f(f(a, a), c) = f(f(a, b), c) \overset{1}{=} f(a, f(b, c)) = f(a, f(b, b)) =$$

$$f(a, f(b, a)) \overset{3}{=} f(a, f(a, b)) \overset{8}{=} f(a, b) = f(a, a);$$

$$f(c, a) \overset{8}{=} f(c, f(c, a)) \overset{1}{=} f(f(c, c), a) = f(f(c, b), a) \overset{1}{=} f(c, f(b, a)) = f(c, f(b, b)) \overset{1}{=}$$

$$f(f(c,b),b) = f(f(c,c),b) \overset{1}{=} f(c, f(c,b)) \overset{8}{=} f(c,b) = f(c,c).$$

Thus, $a\alpha c$. Denoting the corresponding equivalence classes by $U_i, i \in I$, we obtain a partition of U: $\{U_i \subseteq U, i \in I\}$.

Let us prove that subsets U_i are subalgebras. Indeed, let $a_1,\ldots,a_{n(t)} \in U_i, i \in I$, then for any $F_t \in \Sigma, (\text{the arity} |F_t| = t)$ we get:

$$f(F_t(a_1,\ldots,a_{n(t)}),a_1) \overset{6}{=} F_t(a_1,\ldots a_{n(t)}) \overset{7}{=} f(F_t(a_1,\ldots a_{n(t)}), F_t(a_1,\ldots,a_{n(t)}));$$

$$f(a_1, F_t(a_1,\ldots,a_{n(t)})) \overset{5}{=} f(a_1, F_t(f(a_1,a_1),\ldots, f(a_1,a_{n(t)}))) \overset{2}{=}$$

$$f(a_1, F_t(F_t(a_1,\ldots,a_1),\ldots, F_t(a_1,\ldots,a_1))) \overset{2,7}{=} f(a_1, F_t(a_1,\ldots,a_1)) \overset{2,8}{=} f(a_1,a_1),$$

i.e. $F_t(a_1,\ldots,a_{n(t)}), a_1 \in U_i$.

Note that for every $a,b \in U$ we have:

$$f(f(a,b), f(a,b)) = f(a,b). \tag{11}$$

Indeed:

$$f(f(a,b), f(a,b)) \overset{3}{=} f(f(a,b), f(b,a)) \overset{1}{=} f(f(f(a,b),b),a) \overset{8}{=} f(f(a,b),a) \overset{1,8}{=} f(a,b)$$

Let us also note that from the identity $f(f(a,b), f(a,b)) = f(f(a,b), f(b,a))$ it follows that

$$f(a,b)\alpha f(b,a). \tag{12}$$

Furthermore, if $a\alpha a'$ and $b\alpha b'$, then $f(a,b)\alpha f(a',b')$. Indeed:

$$(a, f(b,b)), a') \overset{8}{=} f(f(a,b), f(a',b')) \overset{3}{=} f(f(a,b), f(b',a')) \overset{1,3}{=} f(f(a, f(b',b')), a') =$$

$$f(f(f(a,b),a') \overset{1,3}{=} f(f(a,a'),b) = f(f(a,a),b) \overset{1,8}{=} f(a,b) = f(f(a,b), f(a,b)).$$

In the same way we get that: $f(f(a',b'), f(a,b)) = f(f(a',b'), f(a',b'))$.

Moreover, from identity 8 of definition 5.2 it immediately follows that $a\alpha f(a,a)$, for any $a \in U$.

On the set of indices I define an order "$\leq$" by the following rule: $i_1 \leq i_2$ iff there exist such $a \in U_{i_1}$, $b \in U_{i_2}$ that $f(b,a) = f(b,b)$. This order makes the set I a structure of a semilattice. Indeed, reflexivity immediately follows from the definition. Let us show that "$\leq$" is antisymmetric:

Let $i_1 \leq i_2$ and $i_2 \leq i_1$, then there exist $a,a' \in U_{i_1}, b,b' \in U_{i_2}$ such that

$f(b,a)=f(b,b)$ and $f(a^{'},b^{'})=f(a^{'},a^{'})$. Hence:

$$f(f(a^{'},a^{'}),f(b,b))=f(f(a^{'},a^{'}),f(b,b^{'}))\overset{3}{=}f(f(a^{'},a^{'}),f(b^{'},b))=f(f(a^{'},a^{'}),f(b^{'},b^{'}))\overset{1}{=}$$
$$f(f(a^{'},f(a^{'},b^{'})),b^{'})=f(f(a^{'},f(a^{'},a^{'})),b^{'})\overset{3}{=}f(f(a^{'},a^{'}),b^{'})\overset{3}{=}f(a^{'},a^{'})=f(f(a^{'},a^{'}),f(a^{'},a^{'})).$$

In the similar way we get that $f(f(b,b),f(a^{'},a^{'}))=f(f(b,b),f(b,b))$.

So, $f(a^{'},a^{'})\alpha f(b,b)$, hence $F(b,\ldots,b)=f(b,b)\in U_{i_i}$, thus $F(b,\ldots,b)\in U_{i_i}\cap U_{i_2}$ consequently $i_1=i_2$.

Let $i_1\le i_2$ and $i_2\le i_3$. Then there exist $a\in U_{i_1},b,c\in U_{i_2},d\in U_{i_3}$, such that $f(b,a)=f(b,b)$, $f(d,c)=f(d,d)$. So:

$$f(d,b)\overset{8}{=}f(d,f(d,b))\overset{1}{=}f(f(d,d),b)=f(f(d,c),b)\overset{1}{=}f(d,f(c,b))=f(d,f(c,c))\overset{1}{=}$$

$$f(f(d,c),c))=f(f(d,d),c)\overset{1}{=}f(d,f(d,c))=f(d,f(d,d))\overset{8}{=}f(d,d),$$

hence:

$$f(d,a)=f(d,f(d,a))=f(f(d,d),a)=f(f(d,b),a)\overset{1}{=}f(d,f(b,a))=$$
$$f(d,f(b,b))=f(f(d,b),b)=f(f(d,d),b)\overset{1}{=}f(d,f(d,b))=f(d,f(d,d))\overset{8}{=}f(d,d),$$

which proves that $i_1\le i_3$.

Thus, $(I;\le)$ is an ordered set. To show that $(I;\le)$ is a semilattice, let $a\in U_i,b\in U_j$ and $f(a,b)\in U_k$. Then:

$$f(f(a,b),a)=f(a,f(b,a))=f(a,f(a,b))=f(f(a,a),b)\overset{8}{=}f(a,b)=f(f(a,b),f(a,b))$$
.

Hence, for any $i,j\in I$ there exists an upper bound $k\in I$ such that $f(a,b)\in U_k$, for some $a\in U_i,b\in U_j$.

Let us assume that for some $l\in I$, $i\le l$ and $j\le l$, i.e. there are $a^{'}\in U_i,c\in U_l$ such that $(c,a^{'})=f(c,c)$ and there are $b^{'}\in U_j,d\in U_l$ such that $(d,b^{'})=f(d,d)$. Hence, we have:

$$f(c,f(a^{'},b^{'}))=f((c,a^{'}),b^{'})=f(f(c,c),b^{'})=$$
$$f(f(c,d),b^{'})=f(c,f(d,b^{'})=$$
$$f(c,f(d,d))=f(f(c,d),d)=f(f(c,c),d)=f(c,c).$$

Thus, $f(a',b')\alpha c$ and from the assertion $f(a,b)\alpha f(a',b')$ that is proven above, we obtain that $f(a,b)\alpha c$ which means $k \le l$ and $k = sup(i, j)$.

Define the mappings $\varphi_{i_1,i_2} : U_{i_1} \mapsto U_{i_2}$ for $i_1 \le i_2$ in the following way:

$$\varphi_{i_1,i_2}(a) = f(a,b),$$

where $a \in U_{i_1}, b \in U_{i_2}$.

First of all, let us show that $f(a,b) \in U_{i_2}$ for all $a \in U_{i_1}, b \in U_{i_2}$. Since $i_1 \le i_2$, then there exist $c \in U_{i1}, d \in U_{i_2}$ such that $f(d,c) = f(d,d)$. Thus, we obtain :

$$f(d, f(d,c)) \overset{3}{=} f(d, f(c,d)) \overset{8}{=} f(d,c) = f(d,d) \text{ and}$$
$$f(f(c,d),d) \overset{3,8}{=} (f(c, f), f(c,d)).$$

This gives $b\alpha d\alpha f(c,d)\alpha f(a,b)$, and hence, $f(a,b) \in U_{i_2}$.

The definition of mappings φ_{i_1,i_2} is consistent, i.e. it is independent from the choice of the element $b \in U_{i_2}$. Indeed, let $f(a,b_1), f(a,b_2)$ be arbitrary elements from U_{i_2} and $a \in U$. Then:

$$f(a,b_1) = f(f(a,b_1), f(a,b_1)) \overset{3}{=} f(f(a,b_1), f(b_1,a)) \overset{1}{=} f(f(a, f(b_1,b_1)),a) =$$
$$f(f(a, f(b_1,b_2)),a) \overset{3}{=} f(f(a, f(b_2,b_1)),a) = f(f(a, f(b_2,b_2)),a) \overset{1,3}{=} f(f(a,b_2), f(a,b_2)) = f(a,b_2)$$
.

Thus, we have: $f(a,b_1) = f(a,b_2)$.

It is clear that the mappings φ_{i_1,i_2} are homomorphisms and $\varphi_{i,i}(x) = F_t(x,\ldots,x)$ for any $F_t \in \Sigma$.

Finally, we prove that for any $n(t)$-ary operation $F \in \Sigma$ and $x_1 \in U_{i_1},\ldots,x_{n(t)} \in U_{i_{n(t)}}$, $F(x_1,\ldots,x_{n(t)}) = F(\varphi_{i_1,i_0}(x_1),\ldots,\varphi_{i_1,i_{n(t)}}(x_{n(t)}))$, where $i_0 = sup\{i_1,\ldots,i_n\}$.

To make the proof easier let us denote $f = \cdot$.

We have already noticed that for $a \in U_i$ and $b \in U_j$, $a \cdot b \in U_{sup(i,j)}$. This implies that $y = x_1 \cdot \ldots \cdot x_{n(t)} \in U_{i_0}$.

By 8 for each $1 \le i \le n(t)$, $y \cdot x_i = x_1 \cdot \ldots \cdot x_{n(t)} \cdot x_i = x_1 \cdot \ldots \cdot x_{n(t)} = y$. Hence, by 5 we have:

$$y \cdot F(x_1,\ldots,x_{n(t)}) = y \cdot F(y \cdot x_1,\ldots,y \cdot x_{n(t)}) = y \cdot F(y,\ldots,y) = y \cdot y \cdot y = y \cdot y.$$

Since, by 6, for each $1 \le i \le n(t)$, $F(x_1,\dots,x_{n(t)}) \cdot x_i = F(x_1,\dots,x_{n(t)})$, we obtain that $F(x_1,\dots,x_{n(t)}) \cdot y = F(x_1,\dots,x_{n(t)}) \overset{7}{=} F(x_1,\dots,x_{n(t)}) \cdot F(x_1,\dots,x_{n(t)})$. This means that $y \alpha F(x_1,\dots,x_{n(t)})$ and in consequence, $F(x_1,\dots,x_{n(t)}) \in U_{i_0}$.

Let $x \in U_{i_0}$. Then

$$\phi_{i_0,i_0}(F(x_1,\dots,x_{n(t)})\ ) =$$

$$F(\phi_{i_1,i_0}(x_1)\ ,\dots,\phi_{i_{n(t)},i_0}(x_{n(t)})\ ) = F_t(x_1 \cdot x,\dots,x_{n(t)}) \cdot x \overset{4}{=} F_t(x_1,\dots,x_{n(t)}) \cdot x =$$

$$F(x_1,\dots,x_{n(t)}) \cdot F(x_1,\dots,x_{n(t)}) \overset{4}{=} F(x_1,\dots,x_{n(t)}),$$

which finishes the proof.

Examples

I. Let $Q = (Q;\cdot)$ be a semigroup with the identities: $x \cdot x \cdot y = x \cdot y$, $x \cdot y \cdot y = x \cdot y$ and $x \cdot y \cdot z = x \cdot z \cdot y$. The function $f : Q \times Q \mapsto Q$ defined by the equality $f(x,y) = x \cdot y$ is a non-idempotent Plonka function of Q and hence by Theorem 2.4 Q is a weakly Plonka sum of its subalgebras.

II. Let $(Q;\cdot)$ be a semigroup with the identities: $x \cdot x \cdot y = x \cdot y$, $x \cdot y \cdot y = x \cdot y$ and $x \cdot y \cdot z \cdot t = x \cdot z \cdot y \cdot t$. Then the function $f : Q \times Q \mapsto Q$ defined by the rule $f(x,y) = x \cdot y \cdot x$ is a non-idempotent Plonka function for Q. Indeed:

1. $f(x,x) = x \cdot x \cdot x = x \cdot x$.
2. $f(x, f(y,z)) = x \cdot y \cdot z \cdot y \cdot x = x \cdot z \cdot y \cdot y \cdot x = x \cdot z \cdot y \cdot x = x \cdot y \cdot z \cdot x$,

$f(f(x,y)\ ,z) = x \cdot y \cdot x \cdot z \cdot x \cdot y \cdot x = x \cdot y \cdot x \cdot x \cdot z \cdot y \cdot x = x \cdot y \cdot x \cdot z \cdot y \cdot x = x \cdot x \cdot y \cdot z \cdot y \cdot x =$

$x \cdot y \cdot z \cdot y \cdot x$, hence $f(x, f(y,z)) = f(f(x,y),z)$.

3. $f(x, f(y,z)) = x \cdot y \cdot z \cdot y \cdot x = x \cdot y \cdot y \cdot z \cdot x = x \cdot y \cdot z \cdot x$,
   $f(x, f(z,y)) = x \cdot z \cdot y \cdot z \cdot x = x \cdot z \cdot z \cdot y \cdot x = x \cdot z \cdot y \cdot x = x \cdot y \cdot z \cdot x$.
4. $f(x \cdot y, z) = x \cdot y \cdot z \cdot x \cdot y = x \cdot z \cdot x \cdot z \cdot y = x \cdot x \cdot y \cdot z \cdot y = x \cdot y \cdot z \cdot y =$

$f(x,z) \cdot f(y,z) = x \cdot z \cdot x \cdot y \cdot z \cdot y = x \cdot x \cdot z \cdot y \cdot z \cdot y = x \cdot z \cdot y \cdot z \cdot y = x \cdot z \cdot z \cdot y \cdot y = x \cdot y \cdot z \cdot y$.

5. $f(x, y \cdot z) = x \cdot y \cdot z \cdot x$,

$f(x, f(x,y) \cdot f(x,z)\ ) = x \cdot x \cdot y \cdot x \cdot x \cdot z \cdot x \cdot x = x \cdot y \cdot x \cdot z \cdot x \cdot x =$
$x \cdot y \cdot x \cdot x \cdot z \cdot x = x \cdot y \cdot x \cdot z \cdot x = x \cdot x \cdot y \cdot z \cdot x = x \cdot y \cdot z \cdot x$.

6. $f(x \cdot y, x) = x \cdot y \cdot x \cdot x \cdot y = x \cdot y \cdot x \cdot y = x \cdot x \cdot y \cdot y = x \cdot y \cdot y = x \cdot y$,
   $f(x \cdot y, y) = x \cdot y \cdot y \cdot x \cdot y = x \cdot y \cdot x \cdot y = x \cdot x \cdot y \cdot y = x \cdot y \cdot y = x \cdot y$.

7.

$f(x\cdot y, x\cdot y) = x\cdot y\cdot x\cdot y\cdot x\cdot y = x\cdot x\cdot y\cdot y\cdot x\cdot y = x\cdot x\cdot y\cdot x\cdot y\cdot y = x\cdot y\cdot x\cdot y = x\cdot x\cdot y\cdot y = x\cdot y$
.

8. $f(x, f(x,y)) = x\cdot x\cdot y\cdot x\cdot x = x\cdot x\cdot x\cdot y\cdot x = x\cdot y\cdot x = f(x,y)$.

Thus, by Theorem 2.4, Q is a weakly Plonka sum of its subalgebras.

III. Let $\mathfrak{A} = (U; A, B)$ be a weakly idempotent quasilattice with two operations. Using the following hyperidentities:

$$X(Y(X(Y(z,y),x), X(y,x)), Y(x, X(y,x)) = Y(X(Y(z,y),x), X(y,x)), \tag{13}$$
$$Y(X(Y(z,y),x), X(y,x)) = X(Y(X(Y(z,y),x), X(y,x)), Y(y,x)), \tag{14}$$
$$X(x, X(Y(x,y), Y(Y(x,y),z))) = X(x, Y(z, Y(x,y))), \tag{15}$$
$$X(y, Y(y,z)) = Y(y, X(y,z)), \tag{16}$$
$$Y(Y(x, X(z, Y(y,z))), y) = Y(x, Y(y,z)), \tag{17}$$
$$X(x, Y(x, X(y, Y(y,z))) = Y(x, Y(y,z)), \tag{18}$$
$$X(x, Y(x, X(z, Y(y,z))) = Y(x, Y(y,z)). \tag{19}$$

that are the consequences of the hyperidentities **Ошибка! Источник ссылки не найден.**, we show that the function $f: U\times U \mapsto U$ defined by the rule $f(x,y) = A(x, B(x,y))$ is a non-idempotent Plonka function of $\mathfrak{A}$. Indeed:

1. $f(f(x,y),z) = f(A(x,B(x,y)),z) = A(A(x,B(x,y)), B(A(x,B(x,y)),z)) \overset{(10)}{=}$

$A(A(x,B(x,y)), A(B(A(x,B(x,y)),z)), B(B(x,y),z)) \overset{(17),(7)}{=}$

$A(A(x,B(x,y)), A(B(A(x,B(x,y)),z), B(B(z,A(x,B(x,y))),z))) \overset{(15)}{=}$

$A(A(x,B(x,y)), B(B(z,A(x,B(x,y))),y)) \overset{(17)}{=} A(A(x,B(x,y)), B(B(x,y),z)) \overset{(8)}{=}$

$A(x, A(B(x,y), B(B(x,y),z))) \overset{(15)}{=} A(x, B(B(x,y),z))$.

$f(x, f(y,z)) = f(x, A(y, B(y,z))) = A(x, B(x, A(y, Y(y,z)))) \overset{(18)}{=} A(x, B(x, B(y,z))))$.

2. $f(x,x) = A(x, B(x,x)) \overset{(6)}{=} A(x, A(x,x)) \overset{(9)}{=} A(x,x)$.

3. $f(x, f(y,z)) = f(x, A(y, B(y,z))) = A(x, B(x, A(y, B(y,z))))$;

$f(x, f(z,y)) = f(x, A(z, B(z,y))) = A(x, B(x, A(z, B(z,y))))$.

From the hyperidentity it follows that $f(x, f(y,z)) = f(x, f(z,y))$.

Further, without loss of generality, we suppose that $F_t = A$.

4. $f(A(x_1,x_2),y) = A(A(x_1,x_2), B(A(x_1,x_2),y)) \overset{(10)}{=} A(A(x_1,x_2), A(B(x_1,x_2),y), B(x_1,y)) \overset{(13)}{=}$

$A(A(x_1,x_2), B(x_1,y))$.

$A(f(x_1,y), f(x_2,y)) = A(A(x_1, Y(x_1,y)), A(x_2, Y(x_2,y))) \overset{(8)}{=} A(x_1, A(B(x_1,y), A(x_2, B(x_2,y)))) \overset{(6)}{=}$

$A(x_1, A(x_2, A(B(x_1,y), B(x_2,y)))) \overset{(13)}{=} A(x_1, A(x_2, B(x_1,y))) \overset{(8)}{=} A(A(x_1,x_2), B(x_1,y))$.

5. $f(y, A(x_1,x_2)) = A(y, B(y, A(x_1,x_2))) = B(y, A(y, A(x_1,x_2)))$.

6. $f(A(x_1,x_2),x_i) = A(A(x_1,x_2),B(A(x_1,x_2),x_i)) \overset{(17)}{=} B(A(x_1,x_2),A(A(x_1,x_2),x_i)) \overset{(8),(9)}{=}$
$B(A(x_1,x_2),A(x_1,x_2)) = A(A(x_1,x_2),A(x_1,x_2)) = A(x_1,x_2)$

7. $f(A(x_1,x_2),A(x_1,x_2)) = A(B(x_1,x_2),B(A(x_1,x_2),A(x_1,x_2))) \overset{(6)}{=}$
$A(A(x_1,x_2),A(A(x_1,x_2),X(x_1,x_2))) \overset{(8),(9)}{=} A(A(x_1,x_2),A(x_1,x_2)) \overset{(8),(9)}{=} A(x_1,x_2).$

8. $f(x,f(x,y)) = f(x,A(x,B(x,y))) = A(x,B(x,A(x,B(x,y)))) \overset{(16)}{=}$
$A(x,B(x,B(x,A(x,Y)))) \overset{(8),(9)}{=} A(x,B(x,A(x,y))) \overset{(16)}{=} B(x,A(x,A(x,y))) \overset{(8),(9)}{=}$
$B(x,A(x,y)) \overset{(16)}{=} A(x,B(x,y)) = f(x,y).$

Now using Theorem 2.4, we get the following result.

Theorem 8 *Every weakly idempotent quasilattice with two binary operations is a weakly idempotent lattice or a weakly Plonka sum of weakly idempotent lattices.*

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