ON SOME SYSTEMS FOR ŁUKASIEWICZ'S MANY-VALUED LOGIC AND ITS PROPERTIES

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Abstract

This work is focused on the problem of constructing of several proof systems (Hilbert style, sequent systems and systems, which are dual to resolution systems) for Łukasiewicz's many valued propositional logic. The generalization of Kalmar's proof of deducibility for two valued tautologies inside classical propositional logic gives us a possibility to suggest some method for defining of proof systems for *k*-valued (k≥3) logics, completeness of which is easy proved direct, without of usually loading into two valued logic. For some class of many valued tautologies simultaneously optimal bounds for different proof complexity measures are obtained in some of considered systems .

Keywords: Many-valued logics, Hilbert style proof systems, sequent proof systems, elimination systems, completeness of formal system, proof complexity measures .

1. Introduction

Many-valued logic (MVL) as a separate subject was created and developed first by Łukasiewicz [1]. His intention was to use a third, additional truth value for "possible" (or "unknown"). In the earlier years of development, this caused some doubts about the use fullness of MVL. In the mean time, however, many interesting applications were found in such fields as logic, mathematics, hardware design, artificial intelligence and some other area soft information technologies, therefore the investigations in area of MVL are very actual.

This work is focused on the problem of constructing of several proof systems (Hilbert style, sequent systems and systems, which are dual to resolution systems) for some version of many valued propositional logic with implication and negation, introduced by Łukasiewicz. The generalization of Kalmar's proof of deducibility for two valued tautologies inside classical propositional logic [2] gives us a possibility to suggest some method for defining of proof systems for mentioned version of *k*-valued (k≥3) logic, completeness of which is easy proved direct, without of usually loading into two valued logic [3]. For some class of many valued tautologies simultaneously optimal bounds for different proof complexity measures are obtained in considered systems . The analogous results for some other version of MVL (with Godel's implication and CyclicallyNegation) are given in [5,6].

2. Main Definitions

Here we give some of well-known notions and notations in area of MVL

2.1. k-Valued Logics

Let E_k be the set $\left\{0, \frac{1}{k-1}, \ldots, \frac{k-2}{k-1}, 1\right\}$. We use the well-known notions of propositional formula, which defined as usual from *k*-valued propositional variables p, q, p_i, p_{ij} $(i\geq 1, j\geq 1)$ with values from E_k, may be also propositional constants,parentheses (,),and logical connectives $\&, \vee, \supset, \sim$ defined as follow:

$\boldsymbol{p} \vee \boldsymbol{q} = max(p,q)$, $\boldsymbol{p} \& q = min(p,q)$,

$$\boldsymbol{p} \supset \boldsymbol{q} = \begin{cases} 1, for\ p \leq q \\ 1-p+q, for\ p > q \end{cases}$$

and negation, defined as follow:

$\sim\boldsymbol{p} = 1 - p$ (Łukasiewicz's negation).

For propositional variable $\boldsymbol{p}$ and $\boldsymbol{\delta} = \frac{i}{k-1}$ $(0\leq i\leq k-1)$ we define additionally:

$\boldsymbol{p}^{\boldsymbol{\delta}}$ as $(p \supset \delta)\&(\delta \supset p)$ (exponent function).

Note, that exponent is no new logical function and note also that **exponent for $\boldsymbol{\delta}$=1 is $\boldsymbol{p}$, and for $\boldsymbol{\delta}$=0 is $\sim\boldsymbol{p}$.**

In considered logics we fix **1** as designated value, so a formula φ with variables $p_1, p_2, \ldots p_n$ is called ***k*-tautology** if for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_k^n$ assigning δ_j $(1\leq j\leq n)$ to each p_j gives the value **1** of φ.

Some notions and definitions will be given additionally further.

Our investigations will be focus on k-valued (k≥3) logics, but sometimes we demonstrate the main results only for the 3-valued logics.

2.2. Łukasiewicz's proof system L for 3-Valued Logic

For every formula *A, B, C* of 3-valued logic the following formulas are axioms schemes of ***L*** [3]:

1. $A \supset (B \supset A)$ **2.** $(A \supset (B \supset C)) \supset (B \supset (A \supset C))$ **3**. $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$

4. $(A \supset (A \supset B)) \supset ((\sim B \supset (\sim B \supset \sim A)) \supset (A \supset B))$

5. $(A \supset B) \supset (\sim B \supset \sim A)$ **6**. $A \supset \sim\sim A$ **7**. $\sim\sim A \supset A$

8. $A\&B \supset B$ **9**. $A\&B \supset A$ **10**. $(C \supset A) \supset ((C \supset B) \supset (C \supset A\&B))$

11. $A \supset A \vee B$ **12**. $B \supset A \vee B$ **13**. $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

Inference rule is modus ponens /m.p./$\frac{A, A\supset B}{B}$.

The completeness of this system is proved as usually by "hard" loading into two valued logic (see for example in [3]). We will show the method of direct proof of completeness ***L*** further.

3.Definitions and Properties of new Hilbert style System for Łukasiewicz's logic

Here some new Hilbert style System LN_k for Łukasiewicz's logic is described. The axioms of this system are generalizations of formulas, using in Kalmar's proof of deducibility for two valued tautologies. The completeness of defined system is proved also. For simplification of main results we demonstrate them for 3-valued logic.

For every formulas *A, B, C* of 3-valued logic and each σ_1, σ_2 from the set {0, ½,1} the following formulas are axioms schemes of ***LN3*** (Łukasiewicz Negation3-Valued). Note that here we use only constant ½.

1. $A \supset (B \supset A)$
2. $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$
3. $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A \supset B)^{\sigma_1 \supset \sigma_2})$
4. $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A \vee B)^{\sigma_1 \vee \sigma_2})$
5. $A^{\sigma_1} \supset (B^{\sigma_2} \supset (A \& B)^{\sigma_1 \& \sigma_2})$
6. $A^{\sigma} \supset (\sim A)^{\sim\sigma}$
7. $(A^1 \supset B) \supset ((A^{1/2} \supset B) \supset ((A^0 \supset B) \supset B))$

Inference rule is modus ponens /m.p./ $\frac{A, A \supset B}{B}$.

Note that every of schemes 3.-5. presents at 9 schemes in its turn, and the scheme 6. presents 3 schemes.

For example $A^1 \supset (B^0 \supset (A \supset B)^{1 \supset 0})$ is the scheme $A \supset (\sim B \supset (\sim(A \supset B))$ and $A^{1/2} \supset (B^0 \supset (A\&B)^{1/2\&0})$ is the scheme ((A⊃ 1/2)& (1/2⊃A) ⊃ (~B⊃(~ (A&B))).

It is not difficult to verify that for $* \in \{\&, \vee, \supset\}$ thanks to “exponents” $\sigma_1 * \sigma_2$ all of axioms are 3-tautologies.

The notions of derivation and derivation from premises are defined as usually.

It is easy to prove the following statement for ***LN3***.

Deduction theorem. Let Γ be a set of some formulas and *A* and *B* be some formulas. If the formula *B* is derived in the system ***LN3*** from the premises Γ and *A* ($\Gamma, A \vdash_{LN3} B$) then the formula$A \supset B$ is derived in the system ***LN3*** from the premises Γ ($\Gamma \vdash_{LN3} A \supset B$).

Really, for proving this theorem it must be use the axioms 1. and 2. , and also the formula $A \supset A$, which in his turn can be derived using the formulas 1. and 2.

Further we will omit the abbreviation ***LN3*** from the denotion $\vdash_{LN3}$ sometimes.

Main Lemma. Let $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of any formula *A*, then for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_3^n$

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A^{A(\delta_1, \delta_2, \ldots, \delta_n)}.$$

This statement is generalization of corresponding Lemma for 2-valued logic (see for example [2]).

Proof is given by induction on number *n* of logical connectives in the formula *A*. For *n=1* we have by $\delta = 0 \sim p \vdash \sim p$, by $\delta = 1/2\ p^{1/2} \vdash p^{1/2}$ and by $\delta = 1\ p \vdash p$. Suppose that statement is valid for number of logical connectives <n. If the number of logical connectives is n, then formula *A* can be in the one of following forms:

1. $A = A_1 * A_2$, where $* \in \{\&, \vee, \supset\}$,
2. $A = \sim A_1$

For the case1. $A_1(\tilde{\delta}) = \sigma_1, A_2(\tilde{\delta}) = \sigma_2 => A(\tilde{\delta}) = \sigma_1 * \sigma_2$

By induction hypothesis

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A_1^{\sigma_1}$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A_2^{\sigma_2}$$

Use one of the axiom schemes 3.–5. we have

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A_1^{\sigma_1} \supset (A_2^{\sigma_2} \supset (A_1 * A_2)^{\sigma_1 * \sigma_2})$$

And after two modus ponens derives$(A_1 * A_2)^{\sigma_1 * \sigma_2}$.

For the case2. $A_1(\tilde{\delta}) = \sigma => A(\tilde{\delta}) = \sim\sigma$ and we must use the axiom scheme_ 6. □

Corollary. If A is 3-valued tautology, then for every $\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_3^n$

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A.$$

Theorem 3.1. Any formula is derived in LN3 iff it is 3-valued tautology.

Proof. It is obvious that every formula, which is derived in LN3, is 3-valued tautology.

Let $P = \{p_1, p_2, \ldots, p_n\}$ (*n≥1*) be the set of all variables of any tautology *A*. For every$\tilde{\delta} = (\delta_1, \delta_2, \ldots, \delta_n) \in E_3^n$ by above corollary we have $p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_n^{\delta_n} \vdash A$.

For every $\delta_1, \delta_2, \ldots, \delta_{n-1}$we take into consideration the following 3 truth values

$$\begin{cases} \delta_1, \delta_2, \ldots, \delta_{n-1}, 0 \\ \delta_1, \delta_2, \ldots, \delta_{n-1}, 1/2, \text{ for which we have} \\ \delta_1, \delta_2, \ldots, \delta_{n-1}, 1 \end{cases}$$

$$\begin{cases} p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^0 \vdash A \\ p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^{1/2} \vdash A \cdot \\ p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}}, p_n^1 \vdash A \end{cases}$$

By deduction theorem we have

$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash p_n^0 \supset A$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash p_n^{1/2} \supset A$$
$$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash p_n^1 \supset A.$$

Then adding

$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash (A^1 \supset B) \supset ((A^{1/2} \supset B) \supset ((A^0 \supset B) \supset B))$/axiom7./

$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash (p_n^{1/2} \supset A) \supset ((p_n^0 \supset A) \supset A)$/m.p./

$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash (p_n^0 \supset A) \supset A$/m.p./

$p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_{n-1}^{\delta_{n-1}} \vdash A$/m.p./

So, the number of premises is now *n-1*. Repeating above steps, we obtain finally the derivation of tautology *A* in ***LN3.*** □

Note, that this proof is the full analogy to proof of corresponding theorem for the 2-valued logic. Note also, that after proving by analogy the corresponding Main Lemma for any *k*-valued logic for *k≥4,* the proof of the theorem, corresponding above, can also give by analogy.

As corollary from this Theorem we can prove the following statements.

Theorem. Proof system *L is complete.*

Really at first we can deduce in ***L*** the axioms of system ***LN3*** and then can use Theorem 3.1.

4. Elimination systems

Here we introduce the systems, based on the notions of determinative conjunct and determinative disjunctive normal form (dDNF), introduced by A.Chubaryan for two-valued Boolean functions in [4] and generalized in [5,6] for some version of 3-valued logic.

At first we recall some notions.

4.1. Determinative Disjunctive Normal Form for 2-Valued Logic

Here we will use the current concepts of the unit Boolean cube (E^n) for E={0,1}, a propositional formula and a classical tautology. The particular choice of a language for presented propositional formulas is immaterial in this consideration. However, because of some technical reasons we assume that the language contains the propositional 2-valued variables$p_i (i \geq 1)$and (or)$p_{i_j} (i \geq 1; j \geq 1)$, logical connectives $\neg, \&, \vee, \supset$ and parentheses (,). Following the usual terminology we call the variables and negated variables *literals* for2-valued logic. The conjunct K(term) can be represented simply as a set of literals (no conjunct contains a variable and its negation simultaneously), and disjunctive normal form (*DNF*) can be represented as a set of conjuncts.

In [4] the following notions were introduced.

We call a *replacement-rule* each of the following trivial identities for a propositional formula ψ :

$$0\&\psi = 0, \quad \psi\&0 = 0, \quad 1\&\psi = \psi, \quad \psi\&1 = \psi,$$
$$0\vee\psi = \psi, \quad \psi\vee 0 = \psi, \quad 1\vee\psi = 1, \quad \psi\vee 1 = 1,$$
$$0\supset\psi = 1, \quad \psi\supset 0 = \overline{\psi}, \quad 1\supset\psi = \psi, \quad \psi\supset 1 = 1,$$
$$\overline{0} = 1, \quad \overline{1} = 0, \quad \overline{\overline{\psi}} = \psi.$$

Application of a replacement-rule to some word consists in replacing some its subwords, having the form of the left-hand side of one of the above identities, by the corresponding right-hand side.

Let ϕ be a propositional formula, $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of ϕ, and $P' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\}$ $(1 \leq m \leq n)$ be some subset of P.

Definition 4.1. Given $\sigma = \{\sigma_1, \ldots, \sigma_m\} \subset E^m$, the conjunct $K^\sigma = \left\{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\right\}$ [2]is called $\phi-1$-determinative ($\phi-0$-determinative) if assigning σ_j $(1 \leq j \leq m)$ to each p_{i_j} and successively using replacement-rules we obtain the value of ϕ(1 or 0) independently of the values of the remaining variables.

$\phi-1$-determinative conjunct and $\phi-0$-determinative conjunct are called also ϕ-determinative or determinative for ϕ.

A *DNF* $D = \{K_1, K_2, \ldots, K_l\}$ is called determinative *DNF* (*dDNF*) for φ if φ and D are semantically equivalent and every conjunct K_j ($1 \leq i \leq j$) is 1-determinative for φ.

4.2. Determinative Disjunctive Normal Form for3-Valued Logic

Here we introducel the notion of determinative disjunctive normal form for Łukasiewicz's 3-valued logic. All above mentioned *replacement-rules* are valid (note only, that we must everywere use for negation the notation ~). For ½ we add the replacement rule ~½=½ .

For the other cases we have introduced the following *auxiliary relations for replacement*

$$½\&\psi = \psi\&½ \leq ½, ½ \vee \psi = \psi \vee ½ \geq ½, ½\supset \psi = \psi \supset ½ \geq ½,$$

For every propositional variable p in 3-valued logic p^0, $p^{1/2}$ and p^1 are the literals.

Let ϕ be a propositional formula of 3-valued logic, $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all variables of ϕ ,and $P' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_m}\}$ $(1 \leq m \leq n)$ be some subset of P.

Definition 4.2. Given$\tilde{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m) \in E_3^m$, the conjunct $K^\sigma = \left\{p_{i_1}^{\sigma_1}, p_{i_2}^{\sigma_2}, \ldots, p_{i_m}^{\sigma_m}\right\}$ [3] is called $\phi-1$-determinative ($\phi-0$-determinative, φ-1/2-determinative), if assigning σ_j $(1 \leq j \leq m)$ to each p_{i_j} and successively using replacement-rules and, if it is neccesary, the auxiliary relations for replacement also, we obtain the value of ϕ(1,0 or 1/2) independently of the values of the remaining variables.

Note that using the notion of determinative conjuncts, we can give some of axioms schemes for the systems LNk and CNk more simple.

Definition of *dDNF* for3-valued logic gives by analogy.

Remark. It is easy proved that 1) if for some tautology ϕ the minimal number of literals, containing in ϕ-determinative conjunct, is m ,then ϕ-determinative *DNF* has at least 3^m conjuncts; 2) if for some tautology ϕ there is such m that every conjunct with m literals is ϕ-determinative, then there is ϕ-determinative *DNF* with no more than 3^m conjuncts.

By analogy can be define determinative conjuncts and *dDNF* for *k*-valued logic with the mentioned properties also. For *k*-valued logic we must introduce the corresponding replacement-rules and auxiliary relations for replacement.

4.3.Definitions and Properties of Elimination Systems ELN_k

The axioms of $\boldsymbol{ELN_k}$ aren't fixed, but for every formula φ each conjunct from some *dDNF* of φ can be considered as an axiom.

For 3-valued logic the *elimination rule* (ε-rule) infers conjunct $K' \cup K'' \cup K'''$ from conjuncts $K' \cup \{p^0\}$, $K'' \cup \{p^{1/2}\}$ *and* $K''' \cup \{p^1\}$, where *K', K'' and K'''* are

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[2] As usual, given a propositional variable p and $\sigma \in E^1$, by p^σ we denote the function $p^\sigma = \begin{cases} p, & if \ \sigma = 1, \\ \overline{p}, & if \ \sigma = 0. \end{cases}$

[3] for p^σ we use here the definition of the point 2.1.

conjuncts and p is a variable. It is obvious, that this rule can be eazy generalized for k-valued logic.

The proof in ***ELNk*** is a finite sequence of conjuncts such that every conjunct in the sequence is one of the axioms of ***ELNk***, or is inferred from earlier conjuncts in the sequence by ε-rule.

A DNF $D = \{K_1, K_2, \ldots, K_l\}$ is tautological if using ε-rule can be proven the empty conjunct ($\varnothing$) from the axioms $\{K_1, K_2, \ldots, K_l\}$.

Completeness of this systems is obvious.

4.4.The System LN3-Cut-Free

This system is defined by analogy to the cut-free Frege system F^-, which is described and investigated in [4]. Here we use the connectives, which are used in the system ***LN3.***

The schematic axioms of the system ***LN3-cut-free*** are the following

1. $\alpha_1 \& (\alpha_2 \& \ldots \& (\alpha_{m-1} \& \alpha_m) \ldots) \supset \alpha_i, m \geq 1$, $1 \leq i \leq m$,

2.a) $(K \supset \alpha^{\sigma_1}) \supset ((K \supset \beta^{\sigma_2}) \supset (K \supset (\alpha \supset \beta)^{\sigma_1 \supset \sigma_2}))$

b) $(K \supset \alpha^{\sigma_1}) \supset ((K \supset \beta^{\sigma_2}) \supset (K \supset (\alpha \vee \beta)^{\sigma_1 \vee \sigma_2}))$

c) $(K \supset \alpha^{\sigma_1}) \supset ((K \supset \beta^{\sigma_2}) \supset (K \supset (\alpha \& \beta)^{\sigma_1 \& \sigma_2}))$,

d) $(K \supset \alpha^{\sigma}) \supset (K \supset {\sim}\alpha)^{\sim\sigma})$

3.(a) $(\delta \& K \supset \varphi) \supset ((\delta^{1/2} \& K \supset \varphi) \supset ((\sim\delta \& K \supset \varphi) \supset (K \supset \varphi)))$

(b) $(\gamma \supset \varphi) \supset ((\gamma^{1/2} \supset \varphi) \supset ((\sim\gamma \supset \varphi) \supset \varphi))$,

where

1. φ is provable formula,

2. α_i $(1 \leq i \leq m)$ and γ are literals, α, β, δ are arbitrary formulas,

3. $K = \beta_1 \& (\beta_2 \& \ldots \& (\beta_{l-1} \& \beta_l) \ldots)$ $(l \geq 1)$ for arbitrary literals β_i $(1 \leq i \leq l)$,

4. for every $\beta_1 \& (\beta_2 \& \ldots (\beta_{l-1} \& \beta_l) \ldots) \supset \psi$ style subformula from some axiom of second group conjunct $\{\beta_1, \ldots, \beta_l\}$ is ψ-determinable,

5. if $K^{set} = \{\beta_1, \beta_2, \ldots, \beta_n\}$ for some subformula $K = \beta_1 \& \beta_2 \& \ldots \& \beta_k$ from first axiom of third group, then $\delta \notin K^{set}$ and $\{\delta\} \cup K^{set}$ is subset of some ϕ-determinative conjunct, but K^{set} is not ϕ-determinative.

Rule of inference is modus ponens $\dfrac{A \quad A \supset B}{B}$.

Note that in spite of rule modus ponens, the restrictions1.-5."insist on repeats" steps of *Main Lemma* for derivation of any 3-tautolgy in the system LN3-cut-free.

It is obvious that this system is complet.

Note also that on the base of the systems LNk for *k≥4* can be constructed the corresponding systems LNk-cut-free.

5.Sequent systems

We will use the current concepts of sequent systems, which operate with sequent $\Gamma \to \Delta$ where Γ (antecedent) and Δ (succedent) are sequences (may be empty) of k-valued formulas.

In the systemm ***SLN3 f***or every formula *A, B, C* of 3-valued logic, $* \; from \; the \; set$ $\{\&, \vee, \supset\}$, and each σ, σ_1, σ_2 from the set {0,1/2,1} the following formulas are

$A \to A$ axiom scheme

and schemes of logic inference rules are

1. $\dfrac{\Gamma \to A^{\sigma_1}, \Delta \quad \Gamma \to B^{\sigma_2}, \Delta}{\Gamma \to (A * B)^{\sigma_1 * \sigma_2}, \Delta}$

2. $\dfrac{\Gamma \to A^{\sigma}}{\Gamma \to (\sim A)^{\sim\sigma}}$

3. $\dfrac{\Gamma, B^0 \to A \quad \Gamma, B^{1/2} \to A \quad \Gamma, B^1 \to A}{\Gamma \to A}$

4. $\dfrac{\Gamma \to A \supset B}{\Gamma, A \to B}$

Structural rule is **5.** $\dfrac{\Gamma \to \Delta}{\Gamma^* \to \Delta^*}$, where $\Gamma'(\Delta')$ contains $\Gamma(\Delta)$ as a set.

Cut-rule is **5.** $\dfrac{\Gamma \to A, \Delta \quad A, \Gamma \to B, \Delta}{\Gamma \to B, \Delta}$

The corresponding system ***SLN3-cut-free*** is the system ***SLN3*** without cut-rule. It is not difficult to prove that a ***formula A is deduced in LN3 iff the sequent → A is deduced in SLN3-cut-free, therefore in SLN3 also.***

6. Proof complexity measures, polynomial equivalence of proof systems.

In the theory of proof complexity two main characteristics of the proof are: $t-complexity$, defined as the number of proof steps (lenght) and $l-complexity$, defined as total number of proof symbols (size). We consider two measures (space and width) also: $s-complexity$ (space), informal defined as maximum of minimal number of symbols on blackboard, needed to verify all steps in the proof and $w-complexity$ (width), defined as the maximum of widths of proof formulas.The formal definitions of mentioned proof complexity measures see in [7].

Let Φ be a proof system and φ be a tautology. We denote by $t_\varphi^\Phi(l_\varphi^\Phi, s_\varphi^\Phi, w_\varphi^\Phi)$ the minimal possible value of $t-complexity(l-complexity, s-complexity, w-complexity)$ for all proofs of tautology φ in Φ.

By $|\varphi|$ we denote the size of a formula φ, defined as the number of all logical signs entries. It is obvious that the full size of a formula, which is understood to be the number of all symbols is bounded by some linear function in $|\varphi|$.

The ***polynomial equivalence*** of two proof systems by some proof complexity measure means, that transformation of any proof in one system into a proof in the other system can be done with no more than polynomial increase of proof complexity measure.

Proposition. The systems ***ELN3, LN3-Cut-Free*** and ***SLN3-cut-free*** are polynomially equivalent by all proof complexity measures.

Proof, It is not difficult to see that every proof of empty conjunct from dDNF of any formula φ in the system ***ELN3*** can be easily transformed into proof of φ in the system ***LN3-Cut-Free*** with no more than linear $(*|\varphi|)$ increase of *t*, *s* and *w* complexity and with no more than $(*|\varphi|^2)$ increase of size. Really we can derive formula φ from every φ-determinative conjunct, using the axioms of first and second groups, and then, using the axioms of last group, derive φ. Reverse transformation is more simpler. Really we can take as axioms every φ-determinative conjunct from first occurrences of axioms 3(a) in proof. It is obvious, that in such reverse transformation we have no increase.

The polynomial equivalence of the systems ***ELN3*** and ***SLN3-cut-free*** can be proved by analogy to the proof of polynomial equivalence between the corresponding systems of 2-valued classical logic, which is given in [4]. □

7. Proof complexities measures of some class of k-valued tautologies.

Before we'll prove the main theorem of this point, we must give some auxiliary results.

For given $\tilde{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in E_k^m$ and $\boldsymbol{\delta} = \frac{i}{k-1}$ $(0 \le i \le k-1)$ we call **δ**-topsy-turvy-result the cortege $\widetilde{\sigma\delta}$, which contains every σ_j $(1 \le j \le m)$ with exponent **δ**.

For given $\tilde{\sigma} = (\sigma_1, \sigma_2, \dots, \sigma_m) \in E_k^m$ and $\boldsymbol{\delta} = \frac{i}{k-1}$ $(0 \le i \le k-1)$ we denote by $|\tilde{\sigma}(\boldsymbol{\delta})|$ the number of **δ** occurrence in $\tilde{\sigma}$.

Lemma 1. In given 3-valued **0,1/2,1**-matrix of order n x m can do "topsy-turvy" some strings such, that each column will contain at least one **1**, iff m is no more than f(n), where f(n) defined as follow: f(1)=1, f(n+1)=f(n)+[f(n)/2]+1.

Proof is given by induction on number n of matrix strings. For *n=1, m=1* also. Suppose that statement is valid for n strings. If the number of strings is n+1, we consider the last string. If for some $\boldsymbol{\delta} = \frac{i}{k-1}$ $(0 \le i \le k-1)$ $|\tilde{\sigma}(\boldsymbol{\delta})| \ge [(m+2)/3]$ then after **δ**-topsy-turvy we obtain in the last string at least [(m+2)/3] number of **1**, therefore we'll have at least [(m+2)/3] columns, which contain at least one **1**. For the other columns we consider cortege of n-th string and repeat the same action. Let x be the maximum columns, which we can add to matrix by order (n+1)xf(n) such, that we can do "topsy-turvy" some strings such, that each column of new matrix will contain at least one **1**. For above it is follow, that f(n+1)=f(n)+x, where x=[(f(n+1)=2)/3], therefore we have x≤ (f(n+1)=2)/3, then f(n+1)≥3x-2, so f(n)+x≥3x-2, from which f(n)/2≥x-1 and finally x=[f(n)/2]+1. □

Corollary 1. In given k-valued (k≥4) matrix of order n x m can do "topsy-turvy" some strings such, that each column will contain at least one **1**, iff m is no more than f(n), where f(n) defined as follow: f(1)=1, f(n+1)=f(n)+[f(n)/k-1]+1.

Proof can be given by analogy.

. *Corollary 2.*For every n≥1 and m≤f(n) the following formulas are k-tautologies

$$TTM_{n,m} = \bigvee_{(\sigma_1,\dots,\sigma_n)} \mathop{\&}_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}^{\sigma_i} \begin{pmatrix} where\ first\ disjunctions\ are \\ for\ all\ (\sigma_1, \sigma_2, \dots, \sigma_n) \in E_k^n \end{pmatrix}$$

Proof is obvious.

Note that $f(n+1)=f(n)+[f(n)/2]+1 \ge f(n)+f(n)/2=3f(n)/2 \ge \dots \ge 3^n/2^n > 3^{[n/3]}$.

Corollary 3. For every n≥1 and $m=3^{[n/3]}$ the following formulas are 3-tautologies

$$TTM_{n,m} = \bigvee_{(\sigma_1,\dots,\sigma_n)} \mathop{\&}_{j=1}^{m} \bigvee_{i=1}^{n} p_{i_j}^{\sigma_i} \begin{pmatrix} where\ first\ disjunctions\ are \\ for\ all\ (\sigma_1, \sigma_2, \dots, \sigma_n) \in E_3^n \end{pmatrix}.$$

The analgous formulas for k-valued logics are k-tautologies for every n≥1 and $m=k^{[n/k]}$.

Lemma 2. The bounds of minimal possible value of s-complexity for all proofs of 3-tautology φ with n variables in ***ELN3*** are: $s_\varphi = O(n^2)$ and $s_\varphi = \boldsymbol{\Omega}(n)$.

Proof. For upper bound we use the *perfect* DNF D of φ , which obviously is dDNF.

We consider the following tree like refutation of D in the system ***ELN3***, where as axioms from the left to the right are the following conjuncts:

$p_1^0, p_2^0, \dots, p_n^0$ $p_1^0, p_2^0, \dots, p_n^{1/2}$ $p_1^0, p_2^0, \dots, p_n^1$ ….. $p_1^1, p_2^1, \dots, p_n^1$

Number of conjuncts used as axioms will be 3^k .In first stage we can take first 3 axioms and make elimination rule on them, then next 3 and so on. As result we will have 3^{k-1} conjuncts without p_n variable. Then on next stage we will eliminate p_{n-1} in same way. Consequentially eliminating all variables we will have tree like proof with height n+1, where each node of tree will be one conjunct which is result of elimination rule of 3 conjuncts from previous level. Let number of levels of tree like proof be from 0 to n (all conjuncts on the level of number 0 have size n, the empty conjunct is on the last level with number n). Let c_l $(1 \le i \le n)$ be on of conjuncts on level l of tree like proof, it is result of elimination rule on 3 conjuncts c' ,c'' and c''' from level $l-1$. By proving c' ,c'' and c''' separately we will have following $s(c_l)$ space usage for proving c_l in above described tree like proof:

$$s(c_i)=s(c')+|c''|+|c'''|=|c'|+|c''|+s(c''')=|c'|+s(c'')+|c'''|$$

All conjuncts on the same level l of tree like proof have same size *n-l.* So above equation will look like this: $s(c_i)=s(c')+2(n-(l-1))$.

As all conjuncts on same level have same space usage, we denote by $S(l)$ the space used for each conjunct on level l : $S(l)=S(l-1)+2(n-l+1)$.

Total space usage will be space usage on level *n*:

$$s_\varphi \le S(n)=S(n-1)+2=S(n-2)+2(2+1)=\dots=2(1+2+\dots+n)=O(n^2).$$

Using the fact that at least 3 determinative conjunct must be in every proof, we have $s_\varphi = \boldsymbol{\Omega}(n)$. □

The analogous result for more-value logics can be proved also.

Theorem 7.1.

There exists a sequence of 3-tautologies φ_n, for the proof complexity measures of which in the systems

ELN3, LN3-Cut-Free and ***SLN3-cut-free*** are valid the following equations:

1) $\log_3(|\varphi_n|) = \theta(n)$;
2) $\log_3\log_3(t(\varphi_n)) = \theta(n)$;
3) $\log_3\log_3(l(\varphi_n)) = \theta(n)$;
4) $\log_3(s(\varphi_n)) = \theta(n)$;
5) $\log_3(w(\varphi_n)) = \theta(n)$.

Proof. As φ we take the formulas $TTM_{n,m}$ for every n≥1 and $m=3^{[n/3]}$. For upper bounds we use the *perfect* DNF of φ , and for lower bounds – the properties of φ_n –determinative conjuncts.

It is not difficult to see, that number of variables of φ_n is $n3^n3^{[n/3]}$, the minimal number of variables in every φ_n –determinative conjuncts is $3^{[n/3]}$, therefore by *Remark* in the end of point 4.2. the minimal number of φ_n –determinative conjuncts is $3^{3^{[n/3]}}$, hence the number of axioms, using in the system ***ELN3*** , must be at least $3^{3^{[n/3]}}$ also .

So, using these statements and *Lemma 2.,* we can obtain the all upper and lower bounds for the system ***ELN3*** and using the statement of *Proposition* from point 6., for the systems ***LN3-Cut-Free*** and ***SLN3-cut-free***. □

*Corollary.*If we take the analgous formulas for k-valued logics for every n≥1 and $m=k^{[n/k]}$, then the analogous results can be obtained for more valued logics:

There exists a sequence of k-tautologies (k≥4) φ_n, for the proof complexity measures of which in the systems ***ELNk, LNk-Cut-Free*** and ***SLNk-cut-free*** are valid the following equations:

1. $\log_k(|\varphi_n|) = \theta(n)$;
2. $\log_k\log_k(t(\varphi_n)) = \theta(n)$;
3. $\log_k\log_k(l(\varphi_n)) = \theta(n)$;
4. $\log_k(s(\varphi_n)) = \theta(n)$;
5. $\log_k(w(\varphi_n)) = \theta(n)$.

Proof is given as above.

8.Conclusion

As many-valued logics have many interesting aplications, then the definition of new proof systems and the proof complexities research in them are very important. We hope that analogous results can be proved for other versions of *k*-valued logics (*k≥3*) also.

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OPTICAL PROPERTIES OF SILICON SURFACE (111) AND NANOCRYSTALLINE SILICON FILMS PREPARED AT LOW TEMPERATURES

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Abstract

Crystalline silicon films with crystal phase (111) and great homogeneity were prepared by using plasma enhanced chemical vapor deposition can be surely applied as promising material. Cathodoluminescent spectroscopy was used for film quality testing. Second-harmonic generation, photoluminescence and Fourier-transformed infrared spectra from silicon films with different average size of nanocrystals were analyzed to develop new possible material for active channel in nonlinear optical devices. The volume fraction of nanocrystals was varied from 65% to 90%. It is seen the spectral peak with energy 3.26 eV is related to defects appeared in interface area silicon-silicon dioxide. For films with small silicon crystals (less than 20 nm) the nonlinear optical response contains two spectral peaks. The second peak is caused by optical response from nanocrystal grain boundary that contains oxygen atoms incorporated in silicon as dipoles inside film.

Keywords: polycrystalline silicon films, nanocrystals. second-harmonic generation, cathodeluminescence, Fourier-transformed infrared spectra**.**

In last decades polycrystalline silicon films were widely used in manufacturing of thin field transistors, solar cells, photon detectors and optoelectronic devices. Plasma-enhanced chemical vapor deposition technique was most promising to deposit silicon films with various structural properties, such as amorphous or polycrystalline, silicon, different thicknesses from 10 nm to 10 μm, various orientations of crystals, for example,