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C^*-Simplicity of n-Periodic Products

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Abstract—The C^*-simplicity of n-periodic products is proved for a large class of groups. In particular, the n-periodic products of any finite or cyclic groups (including the free Burnside groups) are C^*-simple. Continuum-many nonisomorphic 3-generated nonsimple C^*-simple groups are constructed in each of which the identity $x^n = 1$ holds, where $n \geq 1003$ is any odd number. The problem of the existence of C^*-simple groups without free subgroups of rank 2 was posed by de la Harpe in 2007.

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1. INTRODUCTION

{ssec1:u643}

Let $\lambda_G\colon G \to U(l_2(G))$ be a left regular representation of a group G. By definition, a *reduced C^*-algebra of G* is the closure of the linear span of the set $\{\lambda_G(g) \mid g \in G\}$ in the operator norm. It is denoted by $C_{\mathrm{red}}(G)$. A nonzero C^*-algebra is said to be a *simple* if it contains no proper nontrivial two-sided closed ideals. A group G is said to be *C^*-simple* if the algebra $C_{\mathrm{red}}(G)$ is simple. The C^*-simplicity of a group implies the triviality of its amenable radical (see, e.g., [1]). In particular, if a given group is C^*-simple and amenable, then it is trivial. We recall that the *amenable radical* of a group is a maximal amenable normal subgroup of this group. Day showed in [2] that any group has an amenable radical.

In the 2014 paper [3], it was proved that the amenable radical of a group G is trivial if and only if the C^*-algebra $C_{\mathrm{red}}(G)$ of G has a unique trace. A *trace* of a C^*-algebra A is any positive linear functional $T\colon A \to C$ such that $T(1) = 1$ and $T(ab) = T(ba)$ for all $a, b \in A$.

The question of whether these three properties of a group (of being a C^*-simple group, having a unique trace, and having trivial amenable radical) are equivalent has remained open for a long time (see, e.g., [1, Question 4]). In 2015, examples of non-C^*-simple groups with trivial amenable radical were constructed [4].

In the 1975 paper [5], Powers proved the C^*-simplicity of free groups of rank 2. After this, various authors described other interesting classes of C^*-simple groups. For example, the free products of two groups [6], the outer automorphism groups of free groups of rank ≥ 3 [7], and relatively hyperbolic groups without nontrivial finite normal subgroups [8] are C^*-simple.

In the survey paper [1], the following curious Question 15 (b) was asked: Are the free Burnside groups of sufficiently large odd period C^*-simple?

In [9], Ol'shanskii and Osin gave a positive answer to this question. A little later, in [3], yet another proof of the C^*-simplicity of the free Burnside groups $B(m, n)$ of sufficiently large odd period was given, which used properties of free Burnside groups obtained previously by these authors in [10] and [11]. This proof is based on the following C^*-simplicity criterion.

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Lemma 1 (see [3, Theorem 1.3]). *A discrete group with countably many amenable subgroups is C^*-simple if and only if its amenable radical is trivial.*

The objective of this paper is the proof of the C^*-simplicity of n-periodic products of groups in a fairly large class, in particular, of the n-periodic products of any cyclic groups without involutions. Since the free Burnside group $B(m, n)$ is isomorphic to an n-periodic product of m cyclic groups of order n, this result can be regarded as a substantial strengthening and simplification of the above-mentioned answers to Question 15(b) of [1] given in [9] and [3].

The notion of a periodic product of given period n (i.e., an n-periodic product) of a given family of groups $\{G_i\}_{i\in I}$ was introduced in Adyan's paper [12]. This operation, denoted by $\prod_{i\in I}^{\mathbf{n}} G_i$, is defined on the class of all groups for each odd $n \geq 665$ as taking the quotient of the free product $F = \prod_{i\in I}{}^*G_i$ of a given family of groups $\{G_i\}_{i\in I}$ by a special normal subgroup determined by a certain system of defining relations of the form $A^n = 1$. This system of relations is constructed by complex joint induction on a natural parameter, called rank, on the basis of Novikov–Adyan theory expounded in detail in the monograph [13]. A strengthening of the main result of this monograph was recently published in [14].

In [15], it was proved that an n-periodic product of a family $\{G_i\}_{i\in I}$ of groups without involutions is the quotient group F/M of the free product $F = \prod_{i\in I}^* G_i$ by the normal subgroup M uniquely determined by the following conditions:

(a) the intersections of M with all components G_i are trivial;

(b) the subgroup M is the normal closure of a certain set $\mathcal{R}$ of words of the form $C^n \in F$; moreover, if an element $X \in F$ is conjugate in F/M to no elements of the components G_i, then $X^n = 1$ in the quotient group F/M.

This description of n-periodic products was used in [15] to obtain the following result.

{l2:u643}

Lemma 2 (see [15, Theorem 2]). *Let G be an n-periodic product of any family $\{G_i\}_{i\in I}$ of groups without involutions, where $n \geq 1003$ is any odd number; then each noncyclic subgroup H of G conjugate to no subgroups of the components G_i contains a subgroup isomorphic to the free periodic group $B(2, n)$ of rank 2.*

We use this lemma to prove the following theorem.

{th1:u643}

Theorem 1. *For any odd $n \geq 1003$, the n-periodic product of an at most countable family of any finite or countable groups without involutions, each of which contains only countably many amenable subgroups, is a C^*-simple group.*

Obviously, the class of groups containing only countably many amenable subgroups includes all finite groups, all finitely generated Abelian groups, and the free groups of finite or countable rank (according to a well-known result of von Neumann, only cyclic subgroups of free groups are amenable). This class includes also all free Burnside groups $B(m, n)$ of odd period $n \geq 1003$ and finite or countable rank m; this follows from Adyan's theorem that the groups $B(m, n)$ are nonamenable (see [10]) and Theorem 1 of [11], according to which all noncyclic subgroups of $B(m, n)$ are nonamenable as well (and even uniformly nonamenable; see [16]).

{c1:u643}

Corollary 1. *An n-periodic product of an at most countable family of any finite groups without involutions is C^*-simple for any odd $n \geq 1003$.*

{c2:u643}

Corollary 2. *An n-periodic product of a countable family of any cyclic groups without involutions is C^*-simple for any odd $n \geq 1003$.*

{c3:u643}

Corollary 3. *The free Burnside groups $B(m, n)$ are C^*-simple for any odd $n \geq 1003$.*

By using Corollaries 1 and 2, it is easy to construct continuum many nonisomorphic countably generated C^*-simple groups. Adyan's well-known criterion for the simplicity of n-periodic products (see [17]) makes it also possible to construct continuum many nonisomorphic countably generated simple groups which are in addition C^*-prime. On the other hand, the Tarski monsters of odd period $n \geq 1003$ constructed in [18] are 2-generated simple groups and, simultaneously, C^*-simple groups, which follows readily from Lemma 1. The existence of continuum many nonisomorphic Tarski monsters for any odd $n \geq 1003$ was shown in [19]. The following theorem is also valid.

{th2:u643}

Theorem 2. *For each odd $n \geq 1003$, there exist continuum many nonisomorphic C^*-simple nonsimple 3-generated groups in which the identity $x^n = 1$ holds.*

2. PROOF OF THEOREM 1

{ssec2:u643}

Our point of departure is the C^*-simplicity criterion proved in [3] and cited above as Lemma 1. According to this criterion, to prove that an n-periodic product $G = \prod_{i\in I}{}^{\mathbf{n}} G_i$ of a countable family $\{G_i\}_{i\in I}$ of any groups without involutions each of which contains only countably many amenable subgroups is C^*-simple, it suffices to show that G contains at most countably many amenable subgroups and its amenable radical is trivial. We use the following result obtained in [15].

{l2':u643}

Lemma 3 (see [15, Theorem 2]). *Let G be an n-periodic product of any family $\{G_i\}_{i\in I}$ of groups without involutions, where $n \geq 1003$ is any odd number. Then each noncyclic subgroup H of G which is not conjugate to any subgroups of the components G_i contains a subgroup isomorphic to the free periodic group $B(2, n)$ of rank 2.*

It is well known that subgroups of amenable groups are themselves amenable. Therefore, each noncyclic subgroup H of G which is not conjugate to any subgroups of the components G_i is nonamenable. Indeed, it follows from Lemma 3 that H contains a subgroup isomorphic to the free periodic group $B(2, n)$ of rank 2, and the group $B(2, n)$ is nonamenable by Adyan's theorem [10].

Since $\{G_i\}_{i\in I}$ is an at most countable family of finite or countable groups, it follows that the n-periodic product G of these groups is countable as well. According to what was said above, the only amenable subgroups of G are amenable subgroups of the components G_i, the number of which is countable, and the subgroups conjugate to them. Since the group G is countable, it follows that the number of subgroups of G conjugate to amenable ones is countable.

Thus, it remains to show that the amenable radical of an n-periodic product G is trivial.

{l3:u643}

Lemma 4. *For each i, any subgroup H of the component G_i is "antinormal," i.e., $x \notin G_i$ implies $xHx^{-1} \cap H = 1$.*

Proof. We recall that, according to the definition of a periodic product, the nontrivial elements of all components are generating, i.e., have length 1. Consider any element xhx^{-1}, where $h \in H$ and $x \notin G_i$. Choosing a sufficiently large α, we can assume that $x \in \mathscr{A}_\alpha \cap \mathscr{M}_\alpha$ and $xhx^{-1} \in \mathscr{R}_0$. Let us show that the word xhx^{-1} is absolutely reduced.

Suppose that $xhx^{-1} \notin \mathscr{K}_\beta$ for some β. Choose a minimal β with this property. According to [13, IV.1.19], there exists a normalized occurrence $V \in \mathrm{Norm}\,(\beta, xhx^{-1}, n - 217)$. Suppose that $V = P * E * Q$. We can assume that $\beta \leq \alpha$. By virtue of Lemma IV.1.18 in [13], the reduced subwords x and x^{-1} cannot contain more than $(n+1)/2 + 42$ segments. Therefore, an elementary word E of rank β has the form $E = x_1 h x_2$, where the central letter h of the irreducible word xhx^{-1} is distinguished. Moreover, each of the subwords x_1 and x_2 of E contains at most $(n+1)/2 + 42$ segments and, hence, at least $2p$ segments, because $n - 217 - (n + 1/2 + 43) > 2p$. These occurrences of the elementary $2p$-powers x_1 and x_2 are compatible, because the occurrence of V is their common extension. Thus, by virtue of Lemma II.5.17 in [13], the subwords x_1 and x_2 are related. Without loss of generality, we can assume that x_1 is not longer than x_2. Then x_1^{-1} is a prefix of x_2. This means that the elementary p-powers x_1^{-1} and x_2 are related, which contradicts [13, II.5.22]. Therefore, we have $xhx^{-1} \in \bigcap_{i=0}^{\infty} \mathscr{K}_i$.

Now, suppose that $h_1 = xhx^{-1}$ in G for some $h_1 \in H$; then, according to [13, IV.2.16], the equality $h_1 = xhx^{-1}$ holds also in the free product of the family of groups $\{G_i\}_{i\in I}$, which contradicts the assumption $x \notin G_i$. This completes the proof of the lemma. □

Lemma 4 implies that none of the subgroups of the components G_i (and their conjugates) are normal subgroups of the n-periodic product G. Therefore, the amenable radical of G is trivial. This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

{ssec3:u643}

As shown in [19], for any odd $n \geq 1003$, there exists a family $\{\Gamma_i\}_{i \in I}$ of continuum many simple 2-generated nonisomorphic groups such that any maximal subgroup in each of these groups is a cyclic group of period n. Let $n \geq 1003$ be any odd number. Consider the n-periodic product $G_i = \langle a \rangle_n \overset{n}{*} \Gamma_i$ of the cyclic group $\langle a \rangle_n$ of order n with generator a and each of the groups $\Gamma_i \in \{\Gamma_i\}_{i \in I}$.

Consider any two groups $G_1 = \langle a \rangle_n \overset{n}{*} \Gamma_1$ and $G_2 = \langle a \rangle_n \overset{n}{*} \Gamma_2$, where $\Gamma_1, \Gamma_2 \in \{\Gamma_i\}_{i \in I}$ are nonisomorphic 2-generated simple groups of period n. Let us prove that G_1 and G_2 are nonisomorphic. Suppose that, on the contrary, there exists an isomorphism $\phi\colon G_1 \to G_2$. According to Adyan's Theorem 3 in [12], the groups $\langle a \rangle_n$ and Γ_i can be embedded as subgroups in G_i ($i = 1, 2$). We denote the normal closure of the element a in G_1 by N_1. Let us show that $G_1/N_1 \simeq \Gamma_1$. First, note that the group G_1/N_1 is obtained from G_1 by adding the single relation $a = 1$ to the system of defining relations. On the other hand, by the definition of a periodic product, this group is obtained from Γ_1 by adding a new relation of the form E^n to the system of defining relations. In the group Γ_1, the identity $x^n = 1$ holds; therefore, these additions yield the same group Γ_1.

Obviously, we have an isomorphism

$$G_1/N_1 \simeq \phi(G_1)/\phi(N_1).$$

Since

$$\phi(G_1)/\phi(N_1) = G_2/N_2,$$

where N_2 is the normal closure of the element $\phi(a)$ in the group G_2, it follows that $\Gamma_1 \simeq G_2/N_2$. On the other hand, since the subgroup $\Gamma_2 < G_2$ is a simple group, it follows from the assumption $N_2 \cap \Gamma_2 \neq \{1\}$ that $N_2 \supset \Gamma_2$. But $N_2 \supset \Gamma_2$ implies that the quotient group

$$G_2/N_2 = \langle a \rangle_n \overset{n}{*} \Gamma_2/N_2$$

is finite, while it is isomorphic to the infinite group Γ_1. This contradiction shows that $N_2 \cap \Gamma_2 = \{1\}$ and, therefore, the group Γ_2 is embedded in $G_2/N_2 = \Gamma_1$ as a proper subgroup ($\Gamma_1 \not\simeq \Gamma_2$). This is again impossible, because any proper subgroup of Γ_1 is a finite cyclic group. The obtained contradictions imply that the groups G_i with different i are nonisomorphic.

It follows directly from Theorem 1 that each of the groups

$$G_i = \langle a \rangle_n \overset{n}{*} \Gamma_i$$

is C^*-simple. On the other hand, the criterion for the simplicity of n-periodic products proved by Adyan in [17] implies that all groups G_i are nonsimple. This proves Theorem 2.

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REFERENCES

1. P. de la Harpe, “On simplicity of reduced C^*-algebras of groups,” Bull. Lond. Math. Soc. **39** (1), 1–26 (2007).
2. M. M. Day, “Amenable semigroups,” Illinois J. Math. **1**, 509–544 (1957).
3. E. Breuillard, M. Kalantar, M. Kennedy, and N. Ozawa, *$C*$-Simplicity and the Unique Trace Property for Discrete Groups*, `arXiv: 1410.2518` (2014).
4. A. Le Boudec, *Discrete Groups That are Not $C*$-Simple*, `arXiv: 1507.03452` (2015).
5. R. T. Powers, “Simplicity of the C^*-algebra associated with the free group on two generators,” Duke Math. J. **42**, 151–156 (1975).
6. W. L. Paschke and N. Salinas, “C^*-algebras associated with free products of groups,” Pacific J. Math. **82** (1), 211–221 (1979).
7. M. R. Bridson and P. de la Harp, “Mapping class groups and outer automorphism groups of free groups are C^*-simple,” J. Funct. Anal. **212** (1), 195–205 (2004).
8. G. Arzhantseva and A. Minasyan, “Relatively hyperbolic groups are C^*-simple,” J. Funct. Anal. **243** (1), 345–351 (2007).
9. A. Yu. Olshanskii and D. V. Osin, “C^*-Simple groups without free subgroups,” Groups Geom. Dyn. **8** (3), 933–983 (2014).
10. S. I. Adyan, “Random walks on free periodic groups,” Izv. Akad. Nauk SSSR Ser. Mat. **46** (6), 1139–1149 (1982) [Math. USSR-Izv. **21** (3), 425–434 (1983)].
11. V. S. Atabekyan, “On subgroups of free Burnside groups of odd exponent $n \geq 1003$,” Izv. Ross. Akad. Nauk Ser. Mat. **73** (5), 3–36 (2009) [Izv. Math. **73** (5), 861–892 (2009)].
12. S. I. Adyan, “Periodic products of groups,” in *Trudy Math. Inst. Steklov*, Vol. 142: *Number Theory, Mathematical Analysis, and Their Applications* (1976), pp. 3–21 [Proc. Steklov Inst. Math. **142**, 1–19 (1979)].
13. S. I. Adyan, *Burnside's Problem and Identities in Groups* (Nauka, Moscow, 1975) [in Russian].
14. S. I. Adyan, “New estimates of odd exponents in infinite Burnside groups,” in *Trudy Math. Inst. Steklov*, Vol. 289: *Selected Questions of Mathematics and Mechanics. Collection of Papers. On the Occasion of the 150th Birthday of Academician Vladimir Andreevich Steklov* (2015), pp. 31–82 [Proc. Steklov Inst. Math. **289**, 33–71 (2015)].
15. S. I. Adyan and V. S. Atabekyan, “Characteristic properties and uniform non-amenability of n-periodic products of groups,” Izv. Ross. Akad. Nauk Ser. Mat. **79** (6), 3–17 (2015) [Izv. Math. **79** (6), 1097–1110 (2015)].
16. V. S. Atabekyan, “Uniform nonamenability of subgroups of free Burnside groups of odd period,” Mat. Zametki **85** (4), 516–523 (2009) [Math. Notes **85** (3–4), 496–502 (2009)].
17. S. I. Adyan, “On the simplicity of periodic products of groups,” Dokl. Akad. Nauk SSSR **241** (4), 745–748 (1978) [Soviet Math. Dokl. **19**, 910–913 (1978)].
18. S. I. Adyan and I. G. Lysenok, “On groups all of whose proper subgroups are finite cyclic,” Izv. Akad. Nauk SSSR Ser. Mat. **55** (5), 933–990 (1991) [Math. USSR-Izv. **39** (2), 905–957 (1992)].
19. V. S. Atabekyan, “On periodic groups of odd period $n \geq 1003$,” Mat. Zametki **82** (4), 495–500 (2007) [Math. Notes **82** (3–4), 443–447 (2007)].