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Spin chain Hamiltonians with affine $U_q g$ symmetry

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Abstract

We construct the family of spin chain Hamiltonians, which have affine $U_q g$ quantum group symmetry. Their eigenvalues coincides with the eigenvalues of the usual spin chain Hamiltonians which have non-affine $U_q g_0$ quantum group symmetry, but have the degeneracy of levels, corresponding to affine $U_q g$. The space of states of these chains are formed by the tensor product of the fully reducible representations.

1. Introduction

Quantum group symmetry plays great role in integrable statistical models [1-3] and conformal field theory [4-6].

It is well known that many integrable Hamiltonians have a quantum group symmetry. For example, the XXZ Heisenberg Hamiltonian with particular boundary terms [5] is $U_q sl_2$-invariant. The infinite XXZ spin chain has a larger symmetry: affine $U_q \widehat{sl}_2$ [7]. The single spin site of most considered Hamiltonians forms an irreducible representation of the Lie algebra of its quantum deformation.

Here we construct the family of spin chain Hamiltonians, which have affine quantum group symmetry. The space of states of these chains are formed by the tensor product of the fully reducible representations. We show that the model, considered in [8], which corresponds to some generalization of the Habbard Hamiltonian in the strong repulsion limit, is a particular case of our general construction. The affine quantum group symmetry leads to a high degeneracy of energy levels.

The energy levels of these spin chains are formed on the states, constructed from the highest weight vectors of quantum group representations. In particular cases the restriction of the considered spin chain on these states gives rise to Heisenberg spin chain or the Haldane-Shastry long range interaction spin chain.

It is difficult in a moment to name a set of physical problems with which the constructed Hamiltonians directly relate (besides the above mentioned). However it is essential to point out that affine symmetries appear in 2D physics when matter fields interact with gravity (in a noncritical string theory).

2. Definitions

Let us recall the definition of the quantum Kac-Moody group $U_q g$. It is generated by the generators e_i, f_i, h_i satisfying the relations

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$$[h_i, e_j] = c_{ij} e_j, \quad [h_i, f_j] = -c_{ij} f_j$$

$$[e_i, f_j] = \delta_{ij} [h]_q$$

and q-deformed Serre relations, which we do not write here. Here q is a deformation parameter, $[x]_q := (x^q - x^{-q})/(q - q^{-1})$, c_{ij} is a Cartan matrix of the corresponding Kac-Moody algebra g.

On $U_q g$ there is a Hopf algebra structure:

$$\Delta(e_i) = k_i \otimes e_i + e_i \otimes k_i^{-1}, \quad \Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}$$

$$\Delta(f_i) = k_i \otimes f_i + f_i \otimes k_i^{-1}$$

where $k_i := q^{h_i/2}$. This comultiplication can be extended to the L-fold tensor product by

$$\Delta^{L-1}(e_i) = \sum_{l=1}^{L} k_i \otimes \ldots \otimes k_i \otimes \underbrace{e_i}_{l} \otimes k_i^{-1} \otimes \ldots \otimes k_i^{-1}$$

$$\Delta^{L-1}(f_i) = \sum_{l=1}^{L} k_i \otimes \ldots \otimes k_i \otimes \underbrace{f_i}_{l} \otimes k_i^{-1} \otimes \ldots \otimes k_i^{-1}$$

$$\Delta^{L-1}(k_i^{\pm 1}) = k_i^{\pm 1} \otimes \ldots \otimes k_i^{\pm 1}$$

Let g be an affine algebra and g_0 is the underlying finite algebra: $g = \hat{g}_0$. Then for any complex x there is the q-deformation of loop homomorphism ρ_x: $U_q g \to U_q g_0$, which is given by

$$\rho_x(e_0) = x f_\theta, \quad \rho_x(f_0) = x^{-1} e_\theta, \quad \rho_x(h_0) = -h_\theta$$

$$\rho_x(e_i) = e_i, \quad \rho_x(f_i) = f_i, \quad \rho_x(h_i) = h_i, \qquad (1)$$

where $i = 1, \ldots, n$ and θ is a maximal root of $U_q g$. Using ρ_x one can construct the spectral parameter dependent representation of $U_q g$ from the representation of $U_q g_0$.

Let $V_1(x_1)$ and $V_2(x_2)$ be constructed in such a way that they allow irreducible finite dimensional representations of $U_q g$ with parameters x_1 and x_2 correspondingly. The $U_q g$-representations on $V_1(x_1) \otimes V_2(x_2)$ constructed by means of Δ and $\bar{\Delta}$ are both irreducible, in general, and equivalent

$$R(x_1, x_2) \Delta(g) = \bar{\Delta}(g) R(x_1, x_2), \quad g \in U_q g \qquad (2)$$

The R-matrix $R(x_1, x_2)$ depends only on x_1/x_2 and is a Boltzmann weight of some integrable statistic mechanical system.

3. Quantum group invariant Hamiltonians for reducible representations

Let $V = \oplus_{i=1}^{N} V_{\lambda_i}$ be a direct sum of the finite dimensional irreducible representations of $U_q g$. We denote by $V(x_1, \ldots, x_N)$ the corresponding affine $U_q g$ representation with spectral parameters x_i:

$$V(x_1, \ldots, x_N) = \bigoplus_{i=1}^{N} V_{\lambda_i}(x_i)$$

We consider the intertwining operator

$$H(x_1, \ldots, x_N) : V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N) \to V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N),$$

$[H(x_1, \ldots, x_N), \Delta(a)] = 0$, for all $a \in U_q g$. If $V = V_\lambda$ consists of one irreducible component then H is a multiple of identity, because the tensor product is irreducible in this case. To carry out the general case let us gather all equivalent irreps together[3]:

$$V(x_1, \ldots, x_N) = \bigoplus_i N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i),$$

where all $V_{\lambda_i}(x_i)$ are nonequivalent and $N_{\lambda_i} \simeq \mathbf{C}^{n_i}$ have a dimension equal to the multiplicity of $V_{\lambda_i}(x_i)$ in $V(x_1, \ldots, x_N)$. By the hat over the tensor product we mean that $U_q g$ does not act on $N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i)$ by means of Δ but acts as $\mathrm{id} \otimes g$.

So, we have

$$\begin{aligned} V(x_1, \ldots, x_N) &\otimes V(x_1, \ldots, x_N) \\ &= (\oplus_i N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i)) \bigotimes (\oplus_i N_{\lambda_i} \hat{\otimes} V_{\lambda_i}(x_i)) \\ &= \bigoplus_{i,j} N_{\lambda_i} \hat{\otimes} N_{\lambda_j} \hat{\otimes} (V_{\lambda_i}(x_i) \otimes V_{\lambda_j}(x_j)) \qquad (3) \end{aligned}$$

Now, $V_{\lambda_i}(x_i) \otimes V_{\lambda_j}(x_j)$ is equivalent only to itself and to $V_{\lambda_j}(x_j) \otimes V_{\lambda_i}(x_i)$ (for $i \neq j$) by the operator $\hat{R}(x_i/x_j) = PR(x_i/x_j)$, where P is tensor product permutation: $P(v_1 \otimes v_2) = v_2 \otimes v_1$. So, the commutant $H(x_1, \ldots, x_N)$ of $U_q g$ on $V(x_1, \ldots, x_N) \otimes V(x_1, \ldots, x_N)$ has the following form:

$$\begin{aligned} H \Big|_{\oplus_{i,j} N_{\lambda_i} \hat{\otimes} N_{\lambda_j} \hat{\otimes} V_{\lambda_i} \otimes V_{\lambda_j}} \\ = A_{ij} \hat{\otimes} \mathrm{id}_{V_{\lambda_i} \otimes V_{\lambda_j}} + B_{ij} \hat{\otimes} \hat{R}_{V_{\lambda_j} \otimes V_{\lambda_i}}(x_i/x_j) \qquad (4) \end{aligned}$$

[3] The $U_q g$-equivalence of $V_{\lambda_i}(x_i)$ requires that the spectral parameters x_i and the highest weights λ_i are the same.

where A_{ij} and B_{ij} are any operators on $N_{\lambda_i}\hat{\otimes}N_{\lambda_j}$.

Let us consider some particular cases of this general construction.

(i) Let $V(x) = V(x,x) = V_\lambda(x) \oplus V_\lambda(x)$. The second term in (4) is absent in this case and H has the factorized form:

$$H = A\hat{\otimes}\mathrm{id}_{V_\lambda\otimes V_\lambda}, \quad A = a^{\alpha\delta}_{\beta\gamma}$$

where $\alpha, \beta, \gamma, \delta = \pm$ are indexes, corresponding to each V_λ.

(ii) Let now $V(x_1,x_2) = V_{\lambda_1}(x_1) \oplus V_{\lambda_2}(x_2)$ ($V_{\lambda_i}(x_i)$ are mutually nonequivalent). Then H acquires the form

$$H(x_1,x_2) = \begin{pmatrix} a\cdot\mathrm{id} & 0 & 0 & 0 \\ 0 & c\cdot\mathrm{id} & d\cdot R_{21}(x_2/x_1) & 0 \\ 0 & e\cdot R_{12}(x_1/x_2) & f\cdot\mathrm{id} & 0 \\ 0 & 0 & 0 & g\cdot\mathrm{id} \end{pmatrix} \tag{5}$$

Here we used $R_{21} = \sum_i b_i \otimes a_i$ for $R_{12} = \sum_i a_i \otimes b_i$. Note, that we can normalize the R-matrices to satisfy the unitarity condition $R_{12}(z)R_{21}(z^{-1}) = \mathrm{id}$. This leads to

$$H(x_1,x_2)^2 = \mathrm{id}\otimes\mathrm{id} \tag{6}$$

(iii) If we choose $g = sl(2)$ and $V = V_{1/2} \oplus V_0 \oplus V_0 \oplus \ldots \oplus V_0$, where $V_{1/2}$ is the fundamental representation of $U_q sl_2$ and V_0 is the trivial one dimensional representation of one, one can obtain the Hamiltonian corresponding to a strong repulsion limit of some generalization of the Habbard model considered in [8]. The representation (5.13) there is a $U_q sl_2$-representation on V.

Following [8] from the operator H the following Hamiltonian acting on $W = V^{\otimes L}$ can be constructed [4]:

$$\bar{H} = \sum_{i=1}^{L-1} H_{ii+1} \tag{7}$$

Here and in the following for the operator $X = \sum_l x_l \otimes y_l$ on $V \otimes V$ we denote by X_{ij} its action on W defined by

$$X_{ij} = \sum_l \mathrm{id}\otimes\ldots\otimes\mathrm{id}\otimes\underbrace{x_l}_{i}\otimes\mathrm{id}\otimes\ldots\otimes\mathrm{id} \otimes\underbrace{y_l}_{j}\otimes\mathrm{id}\otimes\ldots\otimes\mathrm{id} \tag{8}$$

By construction, $\bar{H}$ is quantum group invariant:

$$[\bar{H}, \Delta^{L-1}(g)] = 0 \quad \forall g \in U_q g$$

Let V^0 be the linear space, spanned by the highest weight vectors in V: $V^0 := \oplus_{i=1}^N v^0_{\lambda_i}$, where $v_{\lambda_i} \in V_{\lambda_i}$ is a highest weight vector, and $W^0 := V^{0\otimes L}$. The space W^0 is $\bar{H}$-invariant. This follows from the intertwining property of $\bar{H}$. For general q, W is the $U_q g$-irreducible module so the action of $U_q g$ on W^0 generates all W. So, the energy levels of $\bar{H}$ are highly degenerate.

First, one can consider $\bar{H}$ on the space W^0 and determine (if it is possible) the energy levels and corresponding eigenvectors there. Then performing the quantum group on each eigenvector of some energy level one can obtain the whole eigenspace for this level. Moreover, the space W^0 itself is a direct sum of $\bar{H}$-invariant spaces, each is spanned by the tensor products of fixed number highest weight vectors from each equivalence class of irreps:

$$W^0 = \bigoplus_{\substack{p_1,\ldots,p_M \\ p_1+\ldots+p_M=L}} W^0_{p_1,\ldots,p_M}$$

$$W^0_{p_1\ldots p_M} := \{\oplus \mathbf{C} v^0_{\lambda_{i_1}} \otimes \ldots \otimes v^0_{\lambda_{i_L}} \mid \#(\lambda_i, x_i) \in \{(\lambda_1,x_1),\ldots,(\lambda_N,x_N)\} = p_i\}$$

The $\bar{H}$-invariance of $W^0_{p_1,\ldots,p_M}$ follows again from the definition of $\bar{H}$ as an intertwining operator. The energy levels are now determined on these spaces. Note that the dimension of $W^0_{p_1\ldots p_M}$ is

$$\binom{L}{p_1\ldots p_M}$$

Every Hamiltonian eigenvector $w_0 \in W^0_{p_1\ldots p_M}$ gives rise to a $U_q g$-representation space of dimension

$$\prod_{k=1}^{M} (\dim V_{\lambda_k})^{p_k} \tag{9}$$

This is the degeneracy level of its energy value. In the particular case when all V_{λ_i} are equivalent, the degeneracy level is $(\dim V_\lambda)^L$. Note that

[4] Here and in the following we omit the dependence on x_i.

$$\dim W = \sum_{\substack{p_1 \ldots p_M \\ p_1+\ldots+p_M=L}} \binom{L}{p_1 \ldots p_N} \prod_{k=1}^{N} (\dim V_{\lambda_k})^{p_k}$$
$$= (\sum_{k=1}^{N} N_{\lambda_k} \dim V_{\lambda_k})^L$$

as it must be.

For example, if we choose two equivalent representations (the first case above), then $\dim V^0 = 2$ and there is one term in the decomposition (9). H now is the most general action on $V^0 \otimes V^0$. As a particular case, the XYZ Hamiltonian in the magnetic field can be obtained. This case is most trivial because the degeneracy of all energy levels is the same. So, for the statistical sum $Z_{\bar{H}}(\beta) = \sum_n \exp(-\beta E_n)$ we have

$$Z_{\bar{H}}(\beta) = (\dim V_\lambda)^L Z_{XYZ}(\beta)$$

Let us choose

$$a = g = e = d = 1, \quad c = f = 0 \tag{10}$$

for the second example. Then the restriction of $\bar{H}$ on W^0 coincides with the Bethe XXX spin chain.

$$\bar{H}|_{W^0} = H_{XXX} = \sum_i P_{ii+1} = \frac{1}{2} \sum_i (1 + \boldsymbol{\sigma}_i \boldsymbol{\sigma}_{i+1}) \tag{11}$$

The space $W^0_{p_1 p_2}$, $p_1 + p_2 = L$ corresponds to all states with the same $s_z = p_1/2$ value of spin projection $S^z = 1/2 \sum_i \sigma_i^z$. If we return to $\bar{H}$ the energy level degeneracy of each eigenstate with the same spin projection is multiplies by $(\dim V_{\lambda_1})^{2s_z} (\dim V_{\lambda_2})^{L-2s_z}$.

4. Generalization to long range interaction spin chains

Let us consider now the generalization of the above construction in case of long range interacting Hamiltonians.

Recall that the Haldane-Shastry spin chain is given by [9–11]

$$H_{\text{HS}} = \sum_{i<j} \frac{1}{d^2_{i-j}} P_{ij}\,, \tag{12}$$

Here the spins take values in the fundamental representation of sl_n. It is well known that the Hamiltonian (12) is integrable if d_i has one of the following values

$$d_j = \begin{cases} j\,, & \text{rational case} \\ (1/\alpha)\sinh(\alpha j), \alpha \in \mathrm{R}\,, & \text{hyperbolic case} \\ (L/\pi)\sin(\pi j/L)\,, & \text{trigonometric case} \end{cases} \tag{13}$$

The trigonometric model is defined on the periodic chain and the sum in (12) is performed over $1 \le i, j \le L$. Rational and hyperbolic models are defined on the infinite chain.

One can try to generalize the Hamiltonian (12) for the reducible spin representations by

$$\bar{H}_{\text{HS}} = \sum_{i<j} \frac{1}{d^2_{i-j}} H_{ij}\,, \tag{14}$$

where H is taken for the case (10) of the second example in the previous section. But it is easy to see that it is not invariant with respect to quantum group. This is because the equation

$$\hat{R}_{ij}(x_1,x_2)\Delta^{L-1}(g) = \Delta^{L-1}(g)\hat{R}_{ij}(x_1,x_2)\,, \quad g \in U_q g \tag{15}$$

is valid only for $i = j \pm 1$.

To overcome this difficulty let us substitute instead of H_{ij} the operator [5]

$$F_{[ij]} = G_{[ij]} H_{j-1j} G^{-1}_{[ij]}\,, \quad \text{where} \quad G_{[ij]} = H_{ii+1} H_{i+1i+2} \ldots H_{j-2j-1} \tag{16}$$

The "nonlocal" term like $F_{[ij]}$ appeared as a boundary term in the construction of quantum group invariant and in some sense periodic spin chains [12,13].

Note that it follows from (5), (10), (6) that H_{ii+1} satisfy

$$H_{i-1i} H_{ii+1} H_{i-1i} = H_{ii+1} H_{i-1i} H_{ii+1}\,, \quad H^2_{ii+1} = 1$$

This is a realization of permutation algebra. In contrast to the standard realization by P_{ij}, the relation

$$P_{i-1i} P_{ii+1} P_{i-1i} = P_{i-1i+1}$$

is not fulfilled. The restriction of H_{ii+1} on the highest weight space W_0 coincides with P_{ii+1}. Also it is easy to see from (16) that

$$F_{[ij]}|_{W_0} = P_{ij}$$

[5] Note that $F_{[ij]}$ and $G_{[ij]}$ act nontrivially on all indexes i, $i+1$, ..., j. So we include them into the bracket in order to avoid confusion with the definition (8).

So, the spin chain defined by

$$\bar{H}_{\mathrm{HS}} = \sum_{i<j} \frac{1}{d^2_{i-j}} F_{[ij]} , \tag{17}$$

is quantum group invariant and its restriction on the space W^0 coincides with the Haldane-Shastry spin chain (12). The energy levels of $\bar{H}_{\mathrm{HS}}$ coincides with the levels of (12). The degeneracy degree with respect to the latter is defined by (9).

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