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Mathematics

WEIERSTRASS AND BLASCHKE TYPE FUNCTIONS

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In the present paper the Weierstrass multipliers are generalized and the theorem of convergence of corresponding infinite products is proved. Representations of Blaschke type functions are established through Weierstrass type functions and Blaschke function. A way construction of new Blaschke type products is shown, and a method of proof of their convergence is developed. Some relations between Blaschke type products are established.

***Keywords*:** infinite products, Weierstrass products, Blaschke and Djrbashyan type products.

1. The Weierstrass and Blaschke type infinite products play an important role in the theory of classes factorization of functions, meromorphic in various domains. In [1] it is shown, that quite wide classes of such products have identical structure. Let $D=\{z:|z|<1\}$ be a unit disc, and $G=\{z:\operatorname{Im} z<0\}$ be the bottom half-plane. Denote

$$b_0^{(1)}(z,\varsigma)=\frac{\varsigma-z}{1-\overline{\varsigma}z}\overline{\varsigma}, \quad z,\varsigma\in D;$$

$$b_0^{(2)}(z,\varsigma)=b_0^{(3)}(z,\varsigma)=\frac{z-\varsigma}{z-\overline{\varsigma}}, \quad z,\varsigma\in G.$$

In [1] it is established, that for elementary factors $b_\alpha^{(l)}=b_\alpha^{(l)}(z,\varsigma)$ $(l=1,2,3)$ of Blaschke–Djrbashyan type infinite products, where $\alpha\in(-1,+\infty)$, the following integral representations are true:

$$b_\alpha^{(1)}(z,\varsigma)=\exp\left\{\int_1^{b_0^{(1)}}(1-t)^\alpha\frac{dt}{t}\right\}, \qquad z,\varsigma\in D, \tag{1}$$

$$b_\alpha^{(2)}(z,\varsigma)=\exp\left\{\int_1^{b_0^{(2)}}\left(\frac{1-t}{1+t}\right)^\alpha\frac{dt}{t}\right\}, \qquad z,\varsigma\in D, \tag{2}$$

$$b_\alpha^{(3)}(z,\varsigma)=\exp\left\{\int_1^{b_0^{(3)}}(1-t)^\alpha\frac{dt}{t}\right\}, \qquad z,\varsigma\in G, \tag{3}$$

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where integrals are taken along any contours laying inside the disc D and passing through points 1 and $b_0^{(l)} = b_0^{(l)}(z,\varsigma)$ $(l = 1,2,3)$, and not passing through the beginning of coordinates at $z \neq \varsigma$.

In [2], [3] it is proved, that if the sequence of complex numbers $Z = \{z_n\}_1^\infty \subset D$ satisfies the condition

$$\sum_{n=1}^{\infty} \left(1-|z_n|\right)^{1+\alpha} < +\infty, \qquad \alpha \in (-1,+\infty), \tag{4}$$

then the infinite product

$$B_\alpha^{(1)}(z,\{z_n\}) = \prod_{n=1}^{\infty} b_\alpha^{(1)}(z,z_n) \tag{5}$$

converges absolutely and uniformly inside D and represents an analytical function with zeros Z.

In [4], [5] and [6] it is proved, that if the sequence of complex numbers $Z = \{z_n\}_1^\infty \subset G$ satisfies the condition

$$\sum_{n=1}^{\infty} (\operatorname{Im} z_n)^{1+\alpha} < +\infty, \qquad \alpha \in (-1,+\infty), \tag{6}$$

then the infinite product

$$B_\alpha^{(l)}(z,\{z_n\}) = \prod_{n=1}^{\infty} b_\alpha^{(l)}(z,z_n) \quad (l = 2,3) \tag{7}$$

converges absolutely and uniformly inside G and represents an analytical function with zeros Z. In [1] the following way of construction of new Blaschke–Djrbashyan type functions is shown. Denote by Φ a class of analytic functions inside the unit disc *D*, for which $\varphi(0) = 1$, and the integrals $\int_1^z \frac{\varphi(t)}{t} dt$, $0 < |z| < 1$, taken along contours, laying inside the disc *D*, connecting the points 1 and *z* and not passing through zero, converge. For $\varphi \in \Phi$ introduce the functions

$$b_\varphi^{(l)}(z,\varsigma) = \exp\left\{ \int_1^{b_0^{(l)}} \frac{\varphi(t)}{t} dt \right\} \ (l = 1,2), \tag{8}$$

where $z,\varsigma \in D$ when $l = 1$, and $z,\varsigma \in G$ when $l = 2$, connecting the points 1 and $b_0^{(l)} = b_0^{(l)}(z,\varsigma)$ $(l = 1,2,3)$ and not passing through zero at $z \neq \varsigma$. Obviously, in (8) instead of functions $b_0^{(l)}(z,\varsigma)$ one may take other functions, analytic in *D* or *G*.

2. The function

$$E(z,q) = \begin{cases} 1-z, & q=0, \\ (1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^q}{q}}, & q \geq 1. \end{cases}$$

is called the Weierstrass primary multiplier. Let $\alpha = \{\alpha_k\}_1^\infty$ be a sequence of complex numbers, and the series $\sum_{k=1}^{\infty} \frac{\alpha_k}{k} z^k$ converges at $|z| < \infty$. Consider the entire function

$$E_\alpha(z) = (1-z)\exp\left\{\sum_{k=1}^{\infty}\frac{\alpha_k}{k}z^k\right\}.$$

If $\alpha_k = 1$ for $k = 1,2,\ldots,q$, and $\alpha_k = 0$ for $k = q+1, q+2,\ldots$, then $E_\alpha(z) \equiv E(z,q)$.

Theorem 1. Let $k = q \ge 1$ be the least integer, at which α_k is not equal to 1 and the sequence α is bounded: $|\alpha_k| \le M, k = 1,2,\ldots$ Then

1) at $|z| < \frac{1}{2}$ the inequality $|\log E_\alpha(z)| \le 2M|z|^{q+1}$ holds;

2) if for the sequence $Z = \{z_n\}_1^\infty$ of complex numbers the series $\sum_{n=1}^{\infty}|z_n|^{-q-1}$ converges, then the infinite product

$$\prod_{n=1}^{\infty}E_\alpha\left(\frac{z}{z_n}\right) \tag{9}$$

in each bounded part of the plane, the zeros of which coincide with Z, converges absolutely and uniformly to an entire function.

Proof. We assume that $|z| < 1 - \dfrac{1}{2(q+1)}$. Then

$$|\log E_\alpha(z)| = \left|\sum_{k=q+1}^{\infty}\frac{\alpha_k}{k}z^k\right| \le \frac{|z|^{q+1}}{q+1}\left(|\alpha_{q+1}| + |\alpha_{q+2}| + \ldots\right) \le \frac{M}{q+1}\frac{|z|^{q+1}}{1-|z|} \le 2M|z|^{q+1}.$$

Since $1 - \dfrac{1}{2(q+1)} \ge \dfrac{1}{2}$, the statement 1) of Theorem 1 is proved.

For the proof of the statement 2) note that at $|z| < \frac{1}{2}r \le \frac{1}{2}|z_n|$ the following inequalities are valid: $\left|\log E_\alpha\left(\dfrac{z}{z_n}\right)\right| < 2M\dfrac{|z|^{q+1}}{|z_n|^{q+1}} < \dfrac{Mr^{q+1}}{2^q|z_n|^{q+1}}$. Therefore, a series $\sum_{n=n_0}^{\infty}\left|\log E_\alpha\left(\dfrac{z}{z_n}\right)\right|$ converges uniformly in $|z| < \frac{1}{2}r$, if n_0 is large enough, and, therefore, the product (9) converges absolutely and uniformly.

3. Let $\varphi \in \Phi$ and $\varphi(t) = 1 + \sum_{n=1}^{\infty}\beta_n t^n$, $|t| < r$ $(r > 1)$.

Theorem 2. Functions $b_\varphi^{(l)} = b_\varphi^{(l)}(z,\varsigma)$, $l = 1,2$, can be represented as $b_\varphi^{(l)}(z,\varsigma) = E_\alpha(1 - b_0^{(l)}(z,\varsigma))$, where $\alpha_k = (-1)^k\sum_{n=k}^{\infty}\beta_n C_{n-1}^{k-1}$, $C_n^m = \dfrac{n(n-1)\cdots(n-m+1)}{m!}$.

Proof. For $|t| < r - 1$ we have

$$\frac{\varphi(t)-1}{t} = \sum_{n=1}^{\infty}\beta_n t^{n-1} = \sum_{n=1}^{\infty}\beta_n(1-(1-t))^{n-1} = \sum_{n=1}^{\infty}\beta_n\sum_{k=0}^{n-1}C_{n-1}^k(-1)^k(1-t)^k =$$

$$=\sum_{k=0}^{\infty}(-1)^k\left(\sum_{n=k+1}^{\infty}\beta_n C_{n-1}^k\right)(1-t)^k=\sum_{k=0}^{\infty}\alpha_{k+1}(1-t)^k \ .$$

Applying term by term integration over the contours, laying inside the disc D and connecting the points 1 and z ($|z|<r-1$), we obtain

$$\int_1^z\frac{\varphi(t)-1}{t}dt=\sum_{k=0}^{\infty}\frac{\alpha_{k+1}}{k+1}(1-z)^{k+1}=\sum_{k=1}^{\infty}\frac{\alpha_k}{k}(1-z)^k \ .$$

Thus, if $|z|<r-1$, then

$$E_\alpha(1-z)=z\exp\left\{\sum_{k=1}^{\infty}\frac{\alpha_k}{k}(1-z)^k\right\}=z\exp\left\{\int_1^z\frac{\varphi(t)-1}{t}dt\right\}=\exp\left\{\int_1^z\frac{\varphi(t)}{t}dt\right\}. \quad (10)$$

Now, the statement of Theorem 2 follows from (8), (10) and the uniqueness theorem of analytical functions.

Example 1. Let p be natural number and $\varphi(t)=(1-t)^p$. Then $\varphi(t)=1+\sum_{n=1}^{p}(-1)^n C_p^n t^n$, $\beta_n=(-1)^n C_p^n$ when $n\le p$, and $\beta_n=0$ when $n>p$, $\alpha_{k+1}=(-1)^k\sum_{n=k+1}^{p}\beta_n C_{n-1}^k$.

On the other hand, $\dfrac{\varphi(t)-1}{t}=\dfrac{(1-t)^p-1}{t}=-((1-t)^{p-1}+(1-t)^{p-2}+\ldots+1)$, i.e. $\alpha_{k+1}=-1$ when $k=0,1,\ldots,p-1$, and $\alpha_{k+1}=0$ when $k=p,p+1,\ldots$ Thus, we get $b_\varphi^{(l)}(z,\varsigma)=E(1-b_0^{(l)}(z,\varsigma),p-1)$, $l=1,2$. This representation is proved by M.M. Djrbashyan [3] (see also [7]).

4. Let $\varphi\in\Phi$ and $\varphi(t)=1+\sum_{n=1}^{\infty}\beta_n t^n$, $|t|<1$. From (8) it follows that for functions $b_\varphi^{(l)}=b_\varphi^{(l)}(z,\varsigma)$, $l=1,2$, representations are true

$$b_\varphi^{(l)}(z,\varsigma)=b_0^{(l)}(z,\varsigma)\exp\left\{\beta_0+\sum_{n=1}^{\infty}\frac{\beta_n}{n}\left(b_0^{(l)}(z,\varsigma)^n\right)\right\}, \quad (11)$$

where the limit $\beta_0=-\lim\limits_{\substack{t\to1\\|t|<1}}\sum_{n=1}^{\infty}\frac{\beta_n}{n}t^n$ exists due to the definition of class Φ.

Example 2. Let $\alpha>-1$ be any real number and $\varphi(t)=(1-t)^\alpha$. Then $(1-t)^\alpha=1+\sum_{n=1}^{\infty}\beta_n^{(1)}(\alpha)t^n$, where $\beta_n^{(1)}(\alpha)=(-1)^n\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$ $(n\ge1)$.

Note that $\beta_n^{(1)}(\alpha)\ge0$ for $-1<\alpha\le0$, and $\beta_n^{(1)}(\alpha)\le0$ for $0<\alpha\le1$.

Example 3. Let $\alpha>-1$ be any real number and $\varphi(t)=\left(\dfrac{1-t}{1+t}\right)^\alpha$. Then $\left(\dfrac{1-t}{1+t}\right)^\alpha=1+\sum_{n=1}^{\infty}\beta_n^{(2)}(\alpha)t^n$, where $\beta_1^{(2)}(\alpha)=-2\alpha$,

$$\beta_n^{(2)}(\alpha) = \frac{(-1)^n}{n}\Big(\sum_{k=1}^{n-1} C_n^k \alpha(\alpha-1)\cdots(\alpha-k+1)\,\alpha(\alpha+1)\cdots(\alpha+n-k-1)+$$
$$+\alpha(\alpha-1)\cdots(\alpha-n+1)+\alpha(\alpha+1)\cdots(\alpha+n-1)\Big),\ n\ge 2.$$

There holds also the recurrent formula

$$\beta_n^{(2)}(\alpha) = ((1-)^n - 1)\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} - \sum_{k=1}^{n-1}\frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!}\beta_{n-k}^{(2)} \quad (n\ge 2).$$

Note that $\beta_n^{(2)}(\alpha)\ge 0$ for $-1<\alpha\le 0$, and $\operatorname{sgn}\beta_n^{(2)}(\alpha) = (-1)^n$ for $0<\alpha\le 1$, $n\ge 1$.

From the following obvious inequalities

$$\sum_{n=1}^{\infty}\frac{|\beta_n|}{n}\operatorname{Re}(b_0^{(l)}(z,\varsigma))^n \le \sum_{n=1}^{\infty}\frac{|\beta_n|}{n}\left|b_0^{(l)}(z,\varsigma)\right|^n \le \lim_{t\to 1-0}\sum_{n=1}^{\infty}\frac{|\beta_n|}{n}t^n$$

and formulas (7) we obtain $\left|b_\varphi^{(l)}(z,\varsigma)\right| \le \left|b_0^{(l)}(z,\varsigma)\right|$ when $\beta_n \ge 0\ (n\ge 1)$,

$$\left|b_\varphi^{(l)}(z,\varsigma)\right| \ge \left|b_0^{(l)}(z,\varsigma)\right| \text{ when } \beta_n \le 0\ (n\ge 1).$$

Particularly, due to (1), (2), (3) the following inequalities hold:

$$\left|b_\alpha^{(l)}(z,\varsigma)\right| \le \left|b_0^{(l)}(z,\varsigma)\right| \text{ when } -1<\alpha\le 0\ (l=1,2,3), \tag{12}$$
$$\left|b_\varphi^{(l)}(z,\varsigma)\right| \ge \left|b_0^{(l)}(z,\varsigma)\right| \text{ when } 0<\alpha\le 1\ (l=1,3).$$

The inequality (12) in case $l=2$ is established in [5] with the help of other method.

If $\left|1-b_0^{(l)}(z,\varsigma)\right|<1$, then writing the functions $b_0^{(l)} = b_\varphi^{(l)}(z,\varsigma)$ in the form $b_\varphi^{(l)}(z,\varsigma) = \exp\left\{\int_0^{\left|1-b_0^{(l)}\right|}\frac{\varphi(1+re^{i\vartheta})}{1+re^{i\vartheta}}e^{i\vartheta}dr\right\}$, where $\vartheta = \arg(b_0^{(l)}-1)$, we obtain the inequalities

$$\left|\log b_\varphi^{(l)}(z,\varsigma)\right| \le \frac{1}{1-\left|1-b_0^{(l)}\right|}\int_0^{\left|1-b_0^{(l)}\right|}\left|\varphi(1+re^{i\vartheta})\right|dr \quad (l=1,2). \tag{13}$$

Assume that $|\varphi(t)| = O(|1-t|^\rho)$ with $\rho>0$ holds for $t\to 1$, $|t|<1$. Then, due to (13) the inequality holds

$$\left|\log b_\varphi^{(l)}(z,\varsigma)\right| \le O(1)\frac{\left|1-b_0^{(l)}(z,\varsigma)\right|^{1+\rho}}{1-\left|1-b_0^{(l)}\right|}. \tag{14}$$

For $\varsigma\in D$, $|z|\le R<1$ we have

$$\left|1-b_0^{(1)}(z,\varsigma)\right| = \frac{1-|\varsigma|^2}{|1-\bar{\varsigma}z|} \le \frac{2}{1-R}(1-|\varsigma|); \tag{15}$$

for $\varsigma\in G$, $\operatorname{Im} z\le -\rho<0$ we have

$$\left|1-b_0^{(2)}(z,\varsigma)\right| = \frac{2|\operatorname{Im}\varsigma|}{|z-\varsigma|} \le \frac{2}{\rho}|\operatorname{Im}\varsigma|. \tag{16}$$

From inequalities (14), (15), (16) the convergence conditions (4), (6) of infinite products (5), (7) follow immediately.

Example 4. The case of analytic function φ with singularity in a point $z=1$ is of interest. Let $\varphi(t)=e^{\frac{t}{t-1}}=e\cdot e^{\frac{1}{t-1}}$, $t\in D$. From (13), (16) in the case $z,\varsigma\in G$, $|\operatorname{Im} z|\ge\rho>0$, $|\operatorname{Im}\varsigma|\le\frac{\rho}{3}$ we get the following estimation:

$$\left|\log b_{\varphi}^{(2)}(z,\varsigma)\right|\le 3e\int_{0}^{\left|1-b_0^{(2)}\right|}e^{\frac{\cos\vartheta}{r}}dr\le 3\left|b_0^{(2)}(z,\varsigma)-1\right|\exp\left\{1+\frac{\operatorname{Re}(b_0^{(2)}-1)}{\left|b_0^{(2)}(z,\varsigma)-1\right|^2}\right\}\le$$

$$\le\frac{6\sqrt{e}}{\rho}\left|\operatorname{Im}\varsigma\right|\exp\left\{-\frac{1}{2}\frac{|\operatorname{Im} z|}{|\operatorname{Im}\varsigma|}\right\}.$$

Thus, if for the sequence $\{z_n\}_1^{\infty}\subset G$ $\left(\lim\limits_{n\to\infty}\operatorname{Im} z_n=0\right)$ for any $\rho>0$ holds

$$\sum_{n=1}^{\infty}e^{-\frac{\rho}{|\operatorname{Im} z_n|}}<+\infty, \tag{17}$$

then the infinite product $B_{\varphi}^{(2)}(z,\{z_n\})=\prod\limits_{n=1}^{\infty}\exp\left\{\int_{1}^{b_0^{(2)}(z,z_n)}e^{\frac{t}{t-1}}\frac{dt}{t}\right\}$ converges absolutely and uniformly in any half-plane $\{z:\operatorname{Im} z<-\rho\}$, and represents functions analytic in half-planes G with zeros $\{z_n\}_1^{\infty}$. For example, if $|\operatorname{Im} z_n|=\frac{1}{\log^2 n}$, then the condition (17) holds, but none of conditions (6) is satisfied.

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Mathematics

ON FACTORIZATION OF A CLASS OF SECOND ORDER MATRIX-FUNCTIONS

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The paper suggests a factorization construction method for a class of second order matrix-functions. The left lower element of these matrix-functions may be represented as combinations of other three elements of matrix-function and two functions, meromorphic respectively in interior and exterior regions of the unit disk.

***Keywords*:** factorization, matrix-function, partial indices.

1°. We denote by W the Wiener algebra of continuous functions defined on the unit circle Γ and representable as absolutely convergent Fourier series. By $W^{n\times m}$ we denote the set of all those matrix-functions (MF) of order $n\times m$, the components of which belong to W. For short we use standard notations W^n and W instead of $W^{n\times 1}$ and $W^{1\times 1}$.

Let $\langle f\rangle_k = \dfrac{1}{2\pi i}\int\limits_{\Gamma} f(z)\, z^{-k-1}\, dz$ be the Fourier coefficients of $f\in W^{n\times m}$. We define by equalities

$$(\mathrm{P}_j^- f)(t)=\sum_{k=-\infty}^{j-1}\langle f\rangle_k\, t^k\ ,\ \ (\mathrm{P}_j^+ f)(t)=\sum_{k=j}^{+\infty}\langle f\rangle_k\, t^k$$

projections $\mathrm{P}_j^{\pm}$ ($j\in Z$) acting in $W^{n\times m}$.

Consider the following spaces: $W_+^{n\times m}=\operatorname{Im}\mathrm{P}_0^+$, $W_-^{n\times m}=\operatorname{Im}\mathrm{P}_1^-$, $\overset{\circ}{W}{}_-^{n\times m}=\operatorname{Im}\mathrm{P}_0^-$. Under factorization of a MF $G\in W^{2\times 2}$ in W we understand its representation in the form $G=G_-\Lambda G_+$, where G_-, $G_-^{-1}\in W_-^{2\times 2}$, G_+, $G_+^{-1}\in W_+^{2\times 2}$, $\Lambda(t)=\operatorname{diag}\left[t^{\chi_1},t^{\chi_2}\right]$, $t\in\Gamma$, χ_1, $\chi_2\in Z$ and $\chi_1\le\chi_2$. The numbers χ_1, χ_2 are called partial indices of the MF G. Constructive methods for building factorizations are known only for narrow classes of matrix-functions (see, e.g., [1–6]).

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Let $G_{11}, G_{12}, G_{22}, h_1, h_2 \in W$. We assume that there exist some polynomials q_- and q_+ ($\deg q_- = \nu_+$, $\deg q_+ = n_-$), whose roots lie correspondingly in Γ_+ and Γ_-, and there exists a non-negative integer N ($N \geq n_-$), such that $p_- = \frac{q_+ h_1}{\tau^N} \in W_-$, $p_+ = q_- h_2 \in W_+$, where $\tau(t) = t$ $(t \in \Gamma)$.

A class of second order matrix-functions, defined on the unit circle $\Gamma = \{z \in \mathbb{C}\,;\ |z| = 1\}$ of a form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ h_1 G_{11} + h_2 G_{22} - h_1 h_2 G_{12} & G_{22} \end{pmatrix} \tag{1}$$

under assumption that h_1, h_2 are meromorphic functions correspondingly in domains $\Gamma_- = \{z \in \mathbb{C}\,;\ |z| > 1\}$ and $\Gamma_+ = \{z \in \mathbb{C}\,;\ |z| < 1\}$, is considered in [7]. In [7] we obtained some explicit formulas for the partial indices of MF of the form (1). In the present paper for this class of matrix-functions we suggest a constructive method of recovering factors $G_\pm$. The matrix-function factorization problem (1) is reduced to a successive factorization of two scalar functions and to the solution of explicit finite systems of linear equations.

2°. We assume below that

$$q_-(0) \neq 0 \,. \tag{2}$$

This assumption is not essential by the following reason. If $q_-(0) = 0$ and $\alpha \in \Gamma_+$ are such that $q_-(\alpha) \neq 0$, then it is not difficult to verify, that MF $G_\zeta = G \circ \zeta$, where $\zeta(z) = (z + \alpha)/(1 + \overline{\alpha} z)$, also has structure of (1), if one replaces q_- with $\hat{q}_-(t) = (1 + \overline{\alpha} t)^{\nu_+} q_-(\zeta(t))$. One can easily see, that $q_-(0) = q_-(\alpha) \neq 0$. Besides, since ζ maps Γ_+ into Γ_+, then factorization of MF G in a simple way is restored by factorization of G_ζ.

3°. We consider functions

$$V_1 = \tau^N (q_- G_{11} - p_+ G_{12})/(q_+ q_-), \quad V_2 = \tau^N (\tau^{-N} q_+ G_{22} - p_- G_{12})/(q_+ q_-) \,. \tag{3}$$

Further, we assume that the following conditions are satisfied

$$V_i(t) \neq 0 \quad (t \in \Gamma). \tag{4}$$

Factorization of MF G exists, if and only if conditions (4) are satisfied (see [7]).

Assume that $\chi_i = \operatorname{ind} V_i = \frac{1}{2\pi} \operatorname{var} \arg V_i(t) \Big|_{t \in \Gamma}$, $i = 1, 2$, $\chi_0 = \max\{\chi_1, \chi_2\}$, $\nu_- = N_+(\chi_1 - \chi_2)$,

$$V_i^\pm = \exp \mathrm{P}_0^\pm (\ln(t^{-\chi_i} V_i)), \quad i = 1, 2, \quad V_{12}^\pm = V_2^\pm \exp \mathrm{P}_0^\pm \left(\frac{\tau^{N - \chi_1} G_{12}}{V_1^- V_2^+ q_+ q_-} \right), \tag{5}$$

$$V_\pm = \begin{pmatrix} V_1^\pm & V_{12}^\pm \\ 0 & V_2^\pm \end{pmatrix}. \tag{6}$$

The MF $G_0=\tau^{-\chi_0}G$ allows representation $G_0=AB$ (see [7]), where

$$A=\begin{pmatrix}\frac{q_+}{t^N} & 0\\ p_- & 1\end{pmatrix}\begin{pmatrix}t^{\chi_1-\chi_0} & 0\\ 0 & t^{\chi_2-\chi_0}\end{pmatrix}V_-,\quad B=V_+\begin{pmatrix}1 & 0\\ p_+ & q_-\end{pmatrix}.$$

For $j\in Z$ we denote $\hat{j}=\frac{1}{2}(j+|j|)$, $\check{j}=\frac{1}{2}(j-|j|)$. We define also a MF $U=\tau^{-\nu_-}P^-_{\nu_-}(B^{-1})P^+_{\nu_-}(A^{-1})$ and B_j, A_j ($j>-\nu_-$, $\hat{j}+\nu_->1$), U_j, K_j ($j>-\nu_-$) by following formulas:

$$\mathrm{B}_j=\begin{pmatrix}\langle B^{-1}\rangle_{-1-\check{j}} & \langle B^{-1}\rangle_{-2-\check{j}} & \cdots & \langle B^{-1}\rangle_{-\nu_--j+1}\\ \langle B^{-1}\rangle_{-2-\check{j}} & \langle B^{-1}\rangle_{-3-\check{j}} & \cdots & \langle B^{-1}\rangle_{-\nu_--j}\\ \cdots & \cdots & \cdots & \cdots\\ \langle B^{-1}\rangle_{-\nu_+} & \langle B^{-1}\rangle_{-\nu_+-1} & \cdots & \langle B^{-1}\rangle_{-(\nu_++\nu_-)+2-\hat{j}}\end{pmatrix},\tag{7}$$

$$\mathrm{A}_j=\begin{pmatrix}0 & \langle A^{-1}\rangle_{\nu_--1} & \langle A^{-1}\rangle_{\nu_--2} & \cdots & \langle A^{-1}\rangle_{1-\hat{j}}\\ 0 & 0 & \langle A^{-1}\rangle_{\nu_--1} & \cdots & \langle A^{-1}\rangle_{2-\hat{j}}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & \langle A^{-1}\rangle_{\nu_--1}\end{pmatrix},\tag{8}$$

$$\mathrm{U}_j=\begin{pmatrix}\langle U\rangle_{-1-\check{j}} & \langle U\rangle_{-2-\check{j}} & \cdots & \langle U\rangle_{-\nu_--j}\\ \langle U\rangle_{-2-\check{j}} & \langle U\rangle_{-3-\check{j}} & \cdots & \langle U\rangle_{-\nu_--j-1}\\ \cdots & \cdots & \cdots & \cdots\\ \langle U\rangle_{-\nu_+} & \langle U\rangle_{-\nu_+-1} & \cdots & \langle U\rangle_{-(\nu_++\nu_-)+1-\hat{j}}\end{pmatrix},\tag{9}$$

$$\mathrm{K}_j=\mathrm{U}_j+\mathrm{B}_j\mathrm{A}_j \text{ for } \nu_-+\hat{j}>1,\ \mathrm{K}_j=\mathrm{U}_j \text{ for } \nu_-+\hat{j}=1.\tag{10}$$

Results of [7] imply, that partial indices of MF G_0 are equal to $-\nu_-+\nu_0$ and $\nu_+-\nu_0$, where $\nu_0=\mathrm{card}\{j\,;\ r_j=2\theta_j+r_{j-1},\ j=-\nu_-+1,\dots,\nu_+\}$. Here $r_{-\nu_-}=\nu_-$, $r_j=\mathrm{rang}\mathrm{K}_j$ for $j>-\nu_-$ (particularly, $r_{\nu_+}=\nu_+$), $\theta_j=1$ when $j>0$, and $\theta_j=0$ when $j<0$, $j\in Z$. We denote $\xi_1=-\nu_-+\nu_0+1$, $\xi_2=\nu_+-\nu_0+1$.

4°. We denote by L_j ($j<0$) the space of vector-polynomials of z^{-1} of the form $\sum\limits_{k=j}^{-1}\varphi_k z^k$, where $\varphi_k\in\mathrm{C}^2$. For $j=0$ we assume that $\mathrm{L}_0=\{0\}$. Let $\mathrm{C}^{2n}=\mathrm{C}^2\times\mathrm{C}^2\times\cdots\times\mathrm{C}^2$. We define operators $\Psi_j:\mathrm{C}^{2(\nu_-+\hat{j})}\to\mathrm{L}_{-(\nu_-+\hat{j})}$, $\Psi_j q=\sum\limits_{k=-(\nu_-+\hat{j})}^{-1}q_k t^k$, where $q=\left[q_{-(\nu_-+\hat{j})}\cdots q_{-1}\right]$, $q_k\in\mathrm{C}^2$ ($k=-(\nu_-+\hat{j}),\dots,-1$).

We define the families of Hankel operators $H_j^-: \overset{\circ}{W}{}_-^2 \to W_+^2$, $H_j^+: W_+^2 \to \overset{\circ}{W}{}_-^2$ ($j \in Z$) and Toeplitz operators $T_j: W_+^2 \to W_+^2$ ($j \in Z$) by formula

$$H_j^- \varphi = \mathrm{P}_0^+\left(\tau^{\hat{j}}(A^{-1}\varphi)\right),\ H_j^+ \varphi = \mathrm{P}_0^-\left(\tau^{\breve{j}}(B^{-1}\varphi)\right),\ T_j \varphi = \mathrm{P}_0^+\left(\tau^{-j} G_0 \varphi\right).$$

Reasoning by analogy to the proof of Proposition 3 in [7], one can verify that for $j > -\nu_-$ the kernels of finite-dimensional operators $\mathrm{K}_j = H_j^+ H_j^-\big|_{\mathrm{L}_{-(\nu_- + \hat{j})}}$ are connected with kernels K_j by relations: $\ker \mathrm{K}_j = \Psi_j \ker \mathrm{K}_j$. The Lemma in [7] implies

$$N_j = \ker T_j = \left\{\tau^{\breve{j}} B^{-1} \mathrm{P}_0^+\left(\tau^{\hat{j}} A^{-1} \Psi_j q\right);\, q \in \ker \mathrm{K}_j\right\},\quad j > -\nu_-. \qquad (11)$$

The Toeplitz operators T_j may be defined (by the same formula) on wider spaces $L_2^+ = \mathrm{P}_0^+(L_2(\Gamma))$. Note that in this case kernels $N_j = \ker T_j$ are not extendable. Indeed, since $G_0 \in W^{2\times 2}$, then its any factorization in L_2 is a factorization in W (see [2]). And since any solution $T_j \varphi = 0$ in L_2^+ has the form $G_+^{-1} p$, where p is a polynomial, then φ belongs also to W_+^2.

Along with the spaces N_j ($j \in Z$) we consider also hereditary spaces $N_j = N_j + \tau N_j = \left\{\varphi + \tau\psi;\ \varphi, \psi \in N_j\right\}$. Taking into account the abovementioned remark, it is easy to see that $N_j \subset N_{j+1}$, $j \in Z$. The direct complement M_j of the space N_j in N_{j+1} is called $(2, j)_+$-index subspace of MF G_0. It is known (see [8]) that $N_j = \{0\}$ for $j \le \xi_1 - 1$, and $N_j = N_{j-1}$ for all $j \in Z \setminus \{\xi_1, \xi_2\}$. Besides, the following statement is true.

Proposition 1. If $\xi_1 \ne \xi_2$, then spaces $M_{\xi_1 - 1}$, $M_{\xi_2 - 1}$ are one-dimensional, and if $\xi_1 = \xi_2$, then the unique nonzero $(2, j)_+$-index subspace $M_{\xi_1 - 1}$ is two-dimensional.

Proof. Since MF G_0 allows factorization in the sense of the Section 1°, then it allows factorization in the space L_2 (see [5]), and, therefore, it has a finite $(r_+, 2)$-indexation (see [9], Theorem 4). Besides, the $(r_+, 2)$-partial indices coincide with $\xi_1 - 1$, $\xi_2 - 1$. Consequently, due to Theorem 5 in [8], $\dim M_{\xi_1 - 1} + \dim M_{\xi_2 - 1} = 2$, if $\xi_1 \ne \xi_2$, and $\dim M_{\xi_1 - 1} = 2$, if $\xi_1 = \xi_2$. Taking into account that $\dim M_{\xi_i - 1}$ ($i = 1, 2$) are nonzero spaces (see [9]), we complete the proof of Proposition 1.

We choose vector-functions (VF) φ_1 and φ_2 as follows. Assume, that for $\xi_1 \ne \xi_2$ φ_1 and φ_2 are bases for index subspaces $M_{\xi_1 - 1}$ and $M_{\xi_2 - 1}$ respectively, and VFs φ_1 and φ_2 are a basis of subspace $M_{\xi_1 - 1}$ when $\xi_1 = \xi_2$.

We define $G_+^{-1}=[\varphi_1,\varphi_2]$ (i.e. columns of MF G_+^{-1} are formed with VFs φ_1 and φ_2), $\Lambda_0=\mathrm{diag}\left[t^{\xi_1-1},t^{\xi_2-1}\right]$ and $G_-=G_0G_+^{-1}\Lambda_0^{-1}$, then the following statement is true.

Proposition 2. The representation $G_0=G_-\Lambda_0G_+$ is a factorization of MF G_0.

Proof. G_0 has a finite $(r_+,2)$-indexation with $(r_+,2)$-partial indices ξ_1-1, ξ_2-1, and, therefore, due to Theorem 2 from [9], the representation $G_0=G_-\Lambda_0G_+$ is a $(r_+,2)$-index factorization. It follows from Theorem 4 [9], that G_0 is a factorization in L_2 as well. Since $G\in W^{2\times 2}$, then a factorization in L_2 coincides with factorization in W, i.e. with factorization in the sense of the Section 1° (see [2]). The proof is thus completed.

5°. We introduce the spaces $N_j(0)=\{\varphi(0)\,;\varphi\in N_j\}$, $\mathcal{N}_j(0)=\{\varphi(0);\varphi\in \mathcal{N}_j\}$ and $M_j(0)=\left\{\varphi(0);\ \varphi\in M_j\right\}$. Obviously, $\mathcal{N}_j(0)=N_j(0)$, $N_j(0)=\{0\}$ when $j\le\xi_1-1$, and $M_j(0)=\{0\}$ when $j\notin\left\{\xi_1-1,\xi_2-1\right\}$. Let $\xi_1\neq\xi_2$. Since $\det G_+^{-1}(0)\neq 0$, then Propositions 1 and 2 imply $\dim M_{\xi_1-1}(0)=\dim M_{\xi_2-1}(0)=1$ and $M_{\xi_1-1}(0)\dotplus M_{\xi_2-1}(0)=\mathrm{C}^2$. Hence, taking into account the equalities $N_j(0)\cap M_j(0)=\{0\}$, $N_{j+1}(0)=N_j(0)\dotplus M_j(0)$, $j\in Z$ (see [8]), we get $N_{\xi_1}(0)=\cdots=N_{\xi_2-1}(0)=M_{\xi_1-1}(0)$ and $N_j=\mathrm{C}^2$ for $j\ge\xi_2$. For $\xi_1=\xi_2$ we have $\dim M_{\xi_1-1}(0)=2$, and for $j\ge\xi_2$ we get $N_j=\mathrm{C}^2$.

6°. For $j\in Z$, satisfying the inequality $\nu_-+\hat{j}\ge 1$, we define matrices K_j' by formula $\mathrm{K}_j'=\left[\left\langle A^{-1}\right\rangle_{\nu_--\hat{j}}\cdots\left\langle A^{-1}\right\rangle_{1-j}\right]$. Vectors $\overline{q}_1\in\mathrm{C}^{2\left(\nu_-+\hat{\xi}_1\right)}$, $\overline{q}_2\in\mathrm{C}^{2\left(\nu_-+\hat{\xi}_2\right)}$ are called a factorization pair for the MF G, if $\overline{q}_i\in\ker\mathrm{K}_{\xi_i}$ ($i=1,2$), and vectors $\mathrm{K}_{\xi_1}'\overline{q}_1$, $\mathrm{K}_{\xi_2}'\overline{q}_2$ are nonzero and linearly independent. Obviously, the existence of factorization pair $\overline{q}_1$, $\overline{q}_2$ is equivalent to the existence of nonzero linearly independent vectors $y_1,y_2\in\mathrm{C}^2$, such that

$$\begin{pmatrix}\mathrm{K}_{\xi_i} & 0\\ \mathrm{K}_{\xi_i}' & -E_2\end{pmatrix}\begin{pmatrix}\overline{q}_i\\ \overline{y}_i\end{pmatrix}=0,\qquad i=1,2, \tag{12}$$

where E_2 is a unity matrix.

Proposition 3. MF G possesses a factorization pair.

Proof. Assume that $\nu_-+\hat{j}>0$ and $\varphi\in N_j$. Due to (11) there exists $\overline{q}=\left[q_{-(\nu_-+\hat{j})}\cdots q_{-1}\right]\in\ker\mathrm{K}_j$ ($q_s\in\mathrm{C}^2$, $s=-\left(\nu_-+\hat{j}\right),\dots,j-1$), such that $\varphi(t)=t^{\check{j}}\,B^{-1}(t)\mathrm{P}_0^+\left(t^{\hat{j}}A^{-1}\Psi_jq\right)$. Using equality $(\Psi_jq)(t)=\sum\limits_{k=-(\nu_-+\hat{j})}^{-1}q_kt^k$, one can verify that

$$\varphi(t)=B^{-1}(t)\sum_{m=0}^{\infty}\left(\sum_{k=-(\nu_-+\hat{j})}^{-1}\left\langle A^{-1}\right\rangle_{m-k-\hat{j}}q_k\right)t^{m+\breve{j}}. \tag{13}$$

$B(0)$ is an invertible matrix, and hence $N_j(0)=\{B^{-1}(0)\mathrm{K}'_j\overline{q}\ ;\ \overline{q}\in\ker\mathrm{K}_j\}$. Now the existence of the factorization pair follows from the properties of spaces $N_j(0)$ (see Section 5°). Proposition is thus proved. Note also that in the neighborhood of $z=0$ the function $B(z)\varphi(z)$ is analytical, and, therefore,

$$\sum_{k=-(\nu_-+\hat{j})}^{-1}\left\langle A^{-1}\right\rangle_{m-k-\hat{j}}q_k=0,\qquad m=0,1,\ldots,\breve{j}-1. \tag{14}$$

7°. The main result of the present paper is the following

Theorem. Let $\overline{q}_1$, $\overline{q}_2$ be a factorization pair for G, $\chi_i=\xi_i+\chi_0$,

$$\varphi_i=t^{\breve{\xi}_i}B^{-1}(t)H^-_{\xi_i}\Psi_{\xi_i}\overline{q}_i,\quad i=1,2, \tag{15}$$

$$G_+^{-1}=[\varphi_1,\varphi_2],\ \Lambda(t)=\operatorname{diag}[t^{\chi_1},t^{\chi_2}],\ G_-=GG_+^{-1}\Lambda^{-1}.$$

Then the representation $G=G_-\Lambda G_+$ is a factorization of MF G.

Proof. It follows from the definition of factorization pair that $\mathrm{K}_{\xi_i}q_i=0$ ($i=1,2$). Due to (11), VFs φ_j, defined by the equality (15), belong to spaces N_{ξ_i}.

We consider first the case $\xi_1=\xi_2$. Since $\mathrm{K}'_{\xi_1}\overline{q}_1$ and $\mathrm{K}'_{\xi_2}\overline{q}_2$ are linearly independent and $\det B(0)\neq 0$, then $\varphi_1(0)$ and $\varphi_2(0)$ are linearly independent as well. Therefore, VFs φ_1, φ_2 are also linearly independent. We have $N_{\xi_1}=M_{\xi_1-1}$, and hence φ_1, φ_2 form a basis of M_{ξ_1-1}, and the proof of Theorem follows from Proposition 2.

Assume now $\xi_1\neq\xi_2$. $\varphi_1(0)\neq 0$, i.e. $\varphi_1\neq 0$, since $\det B(0)\neq 0$ and $\mathrm{K}'_{\xi_1}\overline{q}_1\neq 0$. Further, $M_{\xi_1-1}=N_{\xi_1}$ is an one-dimensional space, and, therefore, φ_1 is the basis of this space. Since $\mathrm{K}'_{\xi_1}\overline{q}_1$ and $\mathrm{K}'_{\xi_2}\overline{q}_2$ are linearly independent, then vectors $\varphi_1(0)$, $\varphi_2(0)$ are also linearly independent. Taking into account that $N_{\xi_1-1}(0)=N_{\xi_1}(0)=\operatorname{span}\{\varphi_1(0)\}$, we obtain that $\varphi_2(0)\notin N_{\xi_2-1}(0)=N_{\xi_2}(0)$. Thus, $\varphi_2\in N_{\xi_2}$, but $\varphi_2\notin N_{\xi_2-1}$, i.e. φ_2 belongs to some subspace M_{ξ_2-1}. It remains to apply the Proposition 2.

Theorem is thus proved.

Basing on the Theorem proved above we suggest the following scheme of constructing a factorization of the MF G:

1. Constructing MF $V_\pm$ according to (5), (6).
2. Constructing matrices K_j ($j=-\nu_-+1,\ldots,\nu_+$) according to (7)–(10).
3. Determining numbers $r_j=\operatorname{rang}\mathrm{K}_j$ ($j=-\nu_-+1,\ldots,\nu_+$), $r_{-\nu_-}=\nu_-$, ν_0, ξ_i, χ_i ($i=1,2$).
4. Constructing a factorization pair according to (12).
5. Recovering the factorization of the MF G by means of formulas (15).

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Mathematics

BOUNDARY VALUE PROBLEM FOR THE PSEUDOPARABOLIC EQUATIONS

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In the present paper the boundary value problem for the Sobolev type equation

$$\begin{cases} \dfrac{\partial}{\partial t} L\left(u\left(t,x\right)\right)+M\left(u\left(t,x\right)\right)=f\left(t,x\right), \quad t>0,\ x=(x_1,\dots,x_n)\in\Omega\subset\mathrm{R}^n, \\ u\big|_{\partial\Omega}=0, \\ \left(Lu\right)\left(0,x\right)=g\left(x\right), \quad x\in\Omega, \end{cases}$$

is considered, where L and M are second-order differential operators. It is proved that under some conditions this problem in the corresponding space has the unique solution.

***Keywords*:** Sobolev type equations, pseudoparabolic equations, monotone and radial operators.

1. Let $\Omega\subset\mathrm{R}^n$ be a bounded domain with the smooth boundary Γ. We consider the following boundary value problem:

$$\begin{cases} \dfrac{\partial}{\partial t} L(u(t,x))+M(u(t,x))=f(t,x), \quad t>0,\ x=(x_1,\dots,x_n)\in\Omega\subset\mathrm{R}^n, & (1) \\ u\big|_{\partial\Omega}=0, & (2) \\ (Lu)(0,x)=g(x), \quad x\in\Omega, & (3) \end{cases}$$

where $L\left(u\right)=-\sum\limits_{i,j=1}^{n-1}\dfrac{\partial}{\partial x_i}\left(b_{ij}\left(t,x\right)\dfrac{\partial u}{\partial x_j}\right)$, $M\left(u\right)=-\sum\limits_{i,j=1}^{n}\dfrac{\partial}{\partial x_i}\left(a_{ij}\left(x\right)\dfrac{\partial}{\partial x_j}\right)$,

$$f\left(t,x\right)\in L_2\left(\left(0,T\right);W_2^{-1}(\Omega)\right),\ g\left(x\right)\in W_2^{-1}(\Omega).$$

We suppose that the functions $b_{ij}\left(t,x\right)$ and $a_{ij}\left(t,x\right)$ $(i,j=1,2,\dots,n)$ are defined in $\left[0,T\right]\times\overline{\Omega}$, $b_{ij}\left(t,x\right)=b_{ji}\left(t,x\right)$, $a_{ij}\left(t,x\right)=a_{ji}\left(t,x\right)$ $(i,j=1,2,\dots,n)$ and

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for every $t\in[0,T]$ and $x\in\bar{\Omega}$ the following quadratic form is positively defined:

$$\sum_{i,j=1}^{n} b_{ij}(t,x)\xi_i\xi_j \ge c_0|\xi|^2, \tag{4}$$

where $\xi=(\xi_1,\ldots,\xi_n)$, $c_0=const>0$.

The case of the problem (1)–(3) with $u|_{t=0}=g(x)$, instead of (3) (first boundary value problem), has been considered by R.A. Aleksandrian [1], G.S. Hakobyan, R.L. Shakhbaghyan [2], Kh. Gaevskii, K. Greger, K. Zakharis [3], R.E. Showalter [4], H.A.Mamikonyan [5] etc.

In this paper we study a new boundary value problem.

For the fixed $t\in[0,T]$ we define mappings $L(t)$ and $M(t)$ from $\overset{\circ}{W}{}_2^1(\Omega)$ to $W_2^{-1}(\Omega)$ by formulas

$$\langle L(t)v,w\rangle=\sum_{i,j=1}^{n}\int_{\Omega} b_{ij}(t,x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx, \tag{5}$$

$$\langle M(t)v,w\rangle=\sum_{i,j=1}^{n}\int_{\Omega} a_{ij}(t,x)\frac{\partial v}{\partial x_i}\cdot\frac{\partial w}{\partial x_j}dx, \tag{6}$$

where $v\in\overset{\circ}{W}{}_2^1(\Omega)$ and $w\in\overset{\circ}{W}{}_2^1(\Omega)$. It is easy to see that for $\forall v\in\overset{\circ}{W}{}_2^1(\Omega)$ formulas (5) and (6) define linear bounded functionals $L(t)v$ and $M(t)v$, which belong to $W_2^{-1}(\Omega)$. At the same time differential expressions $L(u)$ and $M(u)$ generate operators $(Lu)(t)=L(t)u(t,x)$ and $(Mu)(t)=M(t)u(t,x)$ that map $L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ into $L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

Let's give some definitions (see [3, 4]). Let X be a real, reflexive Banach space.

Definition 1. The operator $A:X\to X^*$ is called

- radially continuous, if for $\forall x,y\in X$ the function $\varphi(s)=\langle A(x+sy),y\rangle$ is continuous in $[0,1]$;
- Lipschitz-continuous, if there exists a positive constant M such that
$$\|Ax-Ay\|_*\le M\|x-y\| \quad \text{for } \forall x,y\in X;$$
- monotone, if $\langle Ax-Ay,x-y\rangle\ge 0$ for $\forall x,y\in X$;
- strictly monotone, if there exists a positive constant m such that
$$\langle Ax-Ay,x-y\rangle\ge m\|x-y\|^2 \quad \text{for } \forall x,y\in X.$$

Lemma 1. The operators $L(t)$, $M(t)$: $\overset{\circ}{W}{}_2^1(\Omega)\to W_2^{-1}(\Omega)=\left(\overset{\circ}{W}{}_2^1(\Omega)\right)^*$ are radially continuous and uniformly bounded with respect to t.

Proof. Indeed, for every functions $v(x), w(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ we have

$$\varphi(s) = \langle L(t)(v + sw), w \rangle = \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x) \frac{\partial(v + sw)}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} dx =$$

$$= \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x) \frac{\partial v}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} dx + s \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x) \frac{\partial w}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} dx = \langle L(t)\, v, w \rangle + s \langle L(t)\, w, w \rangle.$$

Similarly, we get $\varphi(s) = \langle M(t)(v + sw), w \rangle = \langle M(t)v, w \rangle + s \langle M(t) w, w \rangle$, hence the functionals $\varphi(s)$ and $\psi(s)$ are linear. From the formulas (5) and (6) it follows that

$$\left| \langle L(t) v, w \rangle \right| = \left| \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x) \frac{\partial v}{\partial x_i} \cdot \frac{\partial w}{\partial x_j} dx \right| \le$$

$$\le \sum_{i,j=1}^{n} \int_{\Omega} \left| b_{ij}(t,x) \right| \left| \frac{\partial v}{\partial x_i} \right| \left| \frac{\partial w}{\partial x_j} \right| dx \le c_1 \|v\|_{\overset{\circ}{W}{}_2^1(\Omega)} \cdot \|w\|_{\overset{\circ}{W}{}_2^1(\Omega)},$$

hence we have $\|L(t)v\|_* \le c_1 \|v\|_{\overset{\circ}{W}{}_2^1}$, $\|M(t)v\|_* \le c_1 \|v\|_{\overset{\circ}{W}{}_2^1}$. Now the Lipschitz continuity of the operators $L(t)$ and $M(t)$ follows from their linearity.

Lemma 2. The operators $L(t)$ are uniformly strictly monotone with respect to t.

Proof. From condition (4) it follows that for every $v(x), w(x) \in \overset{\circ}{W}{}_2^1(\Omega)$ we have

$$\left| \langle L(t)v - L(t)w, v - w \rangle \right| = \langle L(t)(v - w), v - w \rangle = \sum_{i,j=1}^{n} \int_{\Omega} b_{ij}(t,x) \frac{\partial(v - w)}{\partial x_i} \cdot \frac{\partial(v - w)}{\partial x_j} dx \ge$$

$$\ge c_0 \int_{\Omega} \sum_{i,j=1}^{n} \left| \frac{\partial(v - w)}{\partial x_i} \right|^2 dx = c_0 \|v - w\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}.$$

Definition 2. Let X and Y be linear spaces and $s = [0,T]$. A mapping $G: L_2(0,T;X) \to L_2(0,T;Y)$ is called Volterra-type, if from the condition $u(s) = v(s)$ for almost all $s \in [0,t]$, $t \in S$, it follows that $(Gu)(s) = (Gv)(s)$ for almost all $s \in [0,t]$. It is evident that the operator M is of Volterra-type.

From the Lemma 1, Lemma 2 and Lemma 2.2 (see [1]) we get

Theorem 1. The operator $L: L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right) \to L_2\left(0,T;W_2^{-1}(\Omega)\right)$ is radially continuous, strictly monotone, and there exists the inverse operator L^{-1}, which is Lipschitz continuous and

$$\left(L^{-1} f\right)(t) = L^{-1}(t) f(t) \quad \forall t \in [0,T], \quad \forall f \in L_2\left(0,T;W_2^{-1}(\Omega)\right).$$

Together with the problem (1)–(3), let's consider the following one

$$\begin{cases} v' + \mathrm{A}v = f, \\ v(0) = g, \end{cases} \tag{7}$$

where $\mathrm{A} = ML^{-1} : L_2\left(0,T;W_2^{-1}(\Omega)\right) \to L_2\left(0,T;W_2^{-1}(\Omega)\right)$. Since the operator A satisfies the conditions of Theorem 1.3 (see [3]), we conclude that the problem (7) has the unique solution. Denote it by v_*. Then the function $v_* = L^{-1}v$ is the solution of the problem (1)–(3). Thus, we can formulate the following (see Theorem 2.4, [3])

Theorem 2. Let the functions $b_{ij}(t,x) = b_{ji}(t,x)$, $a_{ij}(t,x) = a_{ji}(t,x)$ ($i,j = 1,2,\ldots,n$) be continuous in the domain $[0,T]\times\Omega$, and condition (4) holds for any $t \in [0,T]$ and any $x \in \bar{\Omega}$. Then the problem (1)–(3) has a unique solution and $L(u) \in C\left(0,T;W_2^{-1}(\Omega)\right)$, $\frac{\partial}{\partial t}\left(L(u)\right) \in L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

2. Now we consider the problem (1)–(3) with the assumption that the operators L and M are second order nonlinear differential operators:

$$L(u) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(b_i(t,x,\nabla u)\right), \qquad M(u) = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(a_i(t,x,\nabla u)\right),$$

where the functions $b_i(t,x,\xi_1,\ldots,\xi_n)$, $a_i(t,x,\xi_1,\ldots,\xi_n)$ are defined and continuous in $[0,T]\times\bar{\Omega}\times R^n$, and have continuous derivatives with respect to ξ_j ($j = 1,2,\ldots,n$).

We suppose that the functions $b_i(t,x,\xi)$ and $a_i(t,x,\xi)$ ($\xi = (\xi_1,\ldots,\xi_n)$, $i = 1,2,\ldots,n$) satisfy the conditions:

1) $\left|b_i(t,x,\xi)\right| \le c_1\left(|\xi|+1\right)$, $c_1 = const > 0$, $i = 1,2,\ldots,n$,

2) $\left|b_{ij}(t,x,\xi)\right| = \left|\frac{\partial b_1}{\partial \xi_j}\right| \le c_2$, $c_2 = const > 0$, $i,j = 1,2,\ldots,n$,

3) $\sum\limits_{i,j=1}^{n} b_{ij}(t,x,\xi)\eta_1\eta_2 \ge c_3|\eta|^2$ $\forall t \in [0,T]$, $\forall x \in \bar{\Omega}$ and $\forall \eta = (\eta_1,\ldots,\eta_n) \in R^n$,

4) $\left|a_i(t,x,\xi)\right| \le c_4\left(|\xi|+1\right)$, $\left|\frac{\partial a_i(t,x,\xi)}{\partial \xi_j}\right| \le c_5$, $i,j = 1,2,\ldots,n$.

For fixed $t \in [0,T]$ define the operators $L(t)$ and $M(t)$ from $\overset{\circ}{W}{}_2^1(\Omega)$ to $W_2^{-1}(\Omega)$ by formulas

$$\left\langle L(t)v,w\right\rangle = \sum_{i=1}^{n}\int_{\Omega} b_i(t,x,\nabla u)\frac{\partial w}{\partial x_i}dx, \tag{7}$$

$$\left\langle M(t)v,w\right\rangle = \sum_{i=1}^{n}\int_{\Omega} a_i(t,x,\nabla u)\frac{\partial w}{\partial x_i}dx. \tag{8}$$

Operators $L(t)$ and $M(t)$ $(t\in[0,T])$ generate mappings L and M from $L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)$ to $L_2\left(0,T;W_2^{-1}(\Omega)\right)$ by formulas

$$(Lu)(t)=L(t)(u(t,x)), \tag{9}$$

$$(Mu)(t)=M(t)(u(t,x)). \tag{10}$$

Lemma 3 ([5]). Let the conditions 1)–3) hold. Then the operator $L(t)$ is radially continuous and strictly monotone.

Proof. For every $s_1, s_2\in[0,1]$ we have

$$|\varphi(s_1)-\varphi(s_2)|=|\langle L(t)(u_1+s_1v),v\rangle-\langle L(t)(u_2+s_2v),v\rangle|=$$

$$=|\langle L(t)(u+s_1v)-L(t)(u+s_2v),v\rangle|=$$

$$=\left|\sum_{i=1}^{n}\int_{\Omega}b_i(t,x,\nabla u+s_1\nabla v)-b_i(t,x,\nabla u+s_2\nabla v)\frac{\partial v}{\partial x_i}dx\right|=$$

$$=\left|\sum_{i=1}^{n}\int_{\Omega}\int_0^1\left(\sum_{j=1}^{n}\frac{\partial b_i(t,x,\nabla u+s_1\nabla v+\tau(s_2-s_1)\nabla v)}{\partial\xi_j}(s_2-s_1)\frac{\partial v}{\partial x_i}\cdot\frac{\partial v}{\partial x_j}\right)dt\,dx\right|\le C|s_1-s_2|\|v\|^2_{\overset{\circ}{W}{}_2^1(\Omega)},$$

thus, the operator $L(t)$ is radially continuous.

Now we prove that the operator $L(t)$ is strictly monotone. Indeed, from the condition 3) we get

$$\sum_{i=1}^{n}\int_{\Omega}\left[b_i(t,x,\nabla u)-b_i(t,x,\nabla v)\right]\frac{\partial(u-v)}{\partial x_i}dx=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\int_{\Omega}\int_0^1 b_{ij}(t,x,\nabla v+\tau(\nabla u-\nabla v))\frac{\partial(u-v)}{\partial x_i}\cdot\frac{\partial(u-v)}{\partial x_j}dt\,dx\ge c_3\|u-v\|^2_{\overset{\circ}{W}{}_2^1(\Omega)}.$$

The proof of Lemma 3 is complete.

It is easy to verify that the operator

$$M:L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)\to L_2\left(0,T;W_2^{-1}(\Omega)\right)$$

is Lipschitz continuous and of Voltera type. From Lemma 2 and Lemma 2.2 (see [3]) it immediately follows

Lemma 4. The operator $L:L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)\to L_2\left(0,T;W_2^{-1}(\Omega)\right)$ is radially continuous, strictly monotone, and there exists the inverse operator

$$L^{-1}:L_2\left(0,T;\overset{\circ}{W}{}_2^1(\Omega)\right)\to L_2\left(0,T;W_2^{-1}(\Omega)\right),$$

whereas $(L^{-1}f)(t)=L^{-1}(t)f(t)$ for $\forall t\in[0,T]$ and $\forall f\in L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

From Lemma 4 and Theorem 2.4 (see [3]) it immediately follows

Theorem 3. Let the functions $b_i(t,x,\xi)$ and $a_i(t,x,\xi)$ $(i=1,2,\dots,n)$ satisfy the conditions 1)–4). Then the problem (1)–(3), where the operators L and M are defined by formulas (9) and (10), has a unique solution, and $L(u)\in C\left(0,T;W_2^{-1}(\Omega)\right)$, $\frac{\partial}{\partial t}L(u)\in L_2\left(0,T;W_2^{-1}(\Omega)\right)$.

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Mathematics

NEUMANN PROBLEM FOR FOURTH ORDER DEGENERATE ORDINARY DIFFERENTIAL EQUATIONS

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In the present paper the Neumann problem for the equation $Lu \equiv \left(t^{\alpha}u''\right)'' + au = f$, where $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$, is considered. Firstly, the weighted Sobolev space W_{α}^{2} and generalized solution for the above-mentioned equation are defined. Then, the existence and uniqueness of the generalized solution is studied, as well as the spectrum and the domain of corresponding operator are described.

***Keywords*:** Neumann problem, weighted Sobolev spaces, generalized solution, spectrum of linear operators.

The Problem Formulation. Consider the Neumann problem for the following ordinary differential equation of the fourth order

$$Lu \equiv \left(t^{\alpha}u''\right)'' + au = f, \tag{1}$$

where $0 \le \alpha \le 4$, $t \in [0,b]$, $f \in L_2(0,b)$, $a = const$.

Define the weighted Sobolev space W_{α}^{2} and consider behavior of functions from W_{α}^{2} in neighborhood of $t = 0$. Then we define the generalized solution to the Neumann problem for the equation (1). Under some conditions on factor a we have to prove the existence and uniqueness of the generalized solution. Moreover, we have to give a description of the spectrum of operator L and the operator domain $D(L)$. Note that the Dirichlet problem for degenerate ordinary differential equations of second and fourth orders have been considered in [1, 2] and for higher orders – in [3].

The Space W_{α}^{2}. Let $\alpha \ge 0$ and t belong to the finite interval $(0,b)$. Consider the set W_{α}^{2} of the functions $u(t)$, which have generalized derivative of

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the second order, such that the following semi-norm $\|u\|_1 = \int_0^b t^\alpha |u''(t)|^2 dt$ is finite. First note, that for the functions $u \in W_\alpha^2$ for every $t_0 \in (0,b]$ there exist finite values $u(t_0)$ and $u'(t_0)$ (see [3]). Below we study the behavior of functions $u(t)$ and $u'(t)$ in neighborhood of $t = 0$.

Proposition 1. For $u \in W_\alpha^2$ the following inequalities hold:

$$|u(t)|^2 \le (c_1 + c_2 t^{3-\alpha}) \|u\|_1^2, \ \alpha \ne 1, \ \alpha \ne 3, \tag{2}$$

$$|u'(t)|^2 \le (c_1 + c_2 t^{1-\alpha}) \|u\|_1^2, \ \alpha \ne 1. \tag{3}$$

In (2) $t^{3-\alpha}$ is replaced with $|\ln t|$ for $\alpha = 3$, $t^{3-\alpha}$ – with $t^2 |\ln t|$ for $\alpha = 1$. In (3) $t^{1-\alpha}$ is replaced with $|\ln t|$.

The proof is carried out analogously to those in [2] and [4]. Namely we use the representation $u(t) = u(t_0) + \int_{t_0}^t u'(\tau) d\tau$ and apply the Cauchy inequalities.

From Proposition 1 it follows that for $0 \le \alpha < 1$ (weak degeneration) the values $u(0)$ and $u'(0)$ are finite, for $1 \le \alpha < 3$ only $u(0)$ is finite, while for $\alpha \ge 3$ both $u(0)$ and $u'(0)$ may turn to infinity. From the inequality (2) for $0 \le \alpha < 4$ we get the inequality

$$\|u\|_{L_2(0,b)} \le c \|u\|_1, \tag{4}$$

i.e. we have the following inclusion

$$W_\alpha^2 \subset L_2(0,b). \tag{5}$$

Inclusion (5) is true also for $\alpha = 4$. Indeed, using Hardy's inequality (see [5]), we get

$$\int_0^b |u(t)|^2 dt = \int_0^b \left| u(b) + \int_t^b u'(\tau) d\tau \right|^2 dt \le c \int_0^b t^2 |u'(t)|^2 dt,$$

$$\int_a^b t^2 |u'(t)|^2 dt = \int_0^b t^2 \left| u'(b) + \int_t^b u''(\tau) d\tau \right|^2 dt \le (c_1 + c_2) \int_0^b t^4 |u''(t)|^2 dt.$$

Inclusion (5) for $\alpha > 4$ doesn't hold. Indeed, the function $u(t) = t^{-\frac{1}{2}}$ belongs to W_α^2 for $\alpha > 4$, but $u \notin L_2(0,b)$. Therefore, to remain in frames of $L_2(0,b)$, we further assume that $0 \le \alpha \le 4$. Now we can define the following norm in W_α^2:

$$\|u\|_{W_\alpha^2}^2 = \int_0^b \left(t^\alpha |u''(t)|^2 + |u(t)|^2 \right) dt. \tag{6}$$

The space W_α^2 is a Hilbert space with scalar product $(u,v)_\alpha = (t^\alpha u'', v'') + (u,v)$, where $(\cdot,\cdot)$ stands for the scalar product in $L_2(0,b)$. Obviously, for $0 \le \alpha \le 4$ we have the following inequality

$$\|u\|_{L_2(0,b)} \le c\|u\|_{W_\alpha^2}. \tag{7}$$

Proposition 2. The inclusion (4) for $0 \le \alpha < 4$ is compact.

Indeed, using inequality (3) we get

$$\|u(t+h)-u(t)\|^2_{L_2(0,b)} = \int_0^b |u(t+h)-u(t)|^2\,dt = \int_0^b \left|\int_t^{t+h} u'(\tau)d\tau\right|^2 dt \le$$

$$\le \int_0^b \left|\int_t^{t+h}\left(\sqrt{c_1}+c_3 t^{\frac{1-\alpha}{2}}\right)d\tau\right|^2 dt\cdot\|u\|_1^2 = \int_0^b\left(\sqrt{c_1}h + c_4\left((t+h)^{\frac{3-\alpha}{2}} - t^{\frac{3-\alpha}{2}}\right)\right)^2 dt\cdot\|u\|_1^2 \le$$

$$\le\left(c_5|h| + 2c_4\int_0^b\left|(t+h)^{\frac{3-\alpha}{2}} - t^{\frac{3-\alpha}{2}}\right|^2 dt\right)\cdot\|u\|_1^2 \le c|h|^\beta\|u\|_1^2,$$

i.e. we have the following inequality: $\|u(t+h)-u(t)\|_{L_2(0,b)} \le c\|u\|_{W_\alpha^2}$.

The result now follows from the pre-compactness criterion in $L_2(0,b)$. Note also that for $\alpha = 4$ the continuous inclusion (5) is not compact (see [1]).

The Neumann Problem.

Definition 1. The function $u \in W_\alpha^2$ is called the generalized solution to Neumann problem for equation (1), if for every $v \in W_\alpha^2$ we have the equality

$$(t^\alpha u'', v'') + a(u,v) = (f,v). \tag{8}$$

Note that, if the generalized solution $u \in W_\alpha^2$ is classical, then for $\alpha = 0$ we get the following conditions (see [6]): $u''(0) = u'''(0) = u''(b) = u'''(b) = 0$. Consider the particular case of the equation (1) when $a = 1$.

$$Bu \equiv \left(t^\alpha u''\right)'' + u = f. \tag{9}$$

Proposition 3. For every $f \in L_2(0,b)$ the generalized solution of Neumann problem for equation (9) exists and is unique.

Proof. Uniqueness of the generalized solution of equation (9) immediately follows from equality (8) (with $a=1$), if we put $f=0$ and $v=u$. To prove the existence define the functional $l_f(v) = (f,v)$, $f \in L_2(0,b)$, over the space W_α^2. Using the inequality (7) we get

$$|l_f(v)|^2 = \left|\int_0^b f(t)\overline{v(t)}dt\right|^2 \le \|f\|^2_{L_2(0,b)}\|v\|^2_{L_2(0,b)} \le c\|f\|^2_{L_2(0,b)}\|v\|_{W_\alpha^2},$$

i.e. $l_f(v)$ is a linear continuous functional over the space W_α^2. Using Riesz lemma on representation we get $l_f(v) = (u_0, v)_\alpha$, $u_0 \in W_\alpha^2$. Therefore, the function u_0 is the generalized solution to equation (9) (see [1]).

Define the operator $B: L_2(0,b) \to L_2(0,b)$ corresponding to Definition 1.

Definition 2. We say that the function $u \in W_\alpha^2$ belongs to the domain $D(B)$ of operator B, if there exists $f \in L_2(0,b)$ such that the equality (7) is valid. In this case we write $Bu = f$.

Theorem 1. The operator $B: L_2(0,b) \to L_2(0,b)$ is positive and self-adjoint. The bounded operator $B^{-1}: L_2(0,b) \to L_2(0,b)$ for $0 \le \alpha < 4$ is compact.

Proof. The symmetry and positivity of operator B is a direct consequence of Definition 2. The coincidence of $D(B)$ and $D(B^*)$ (B^* is the adjoint to operator B) follows from the existence of a generalized solution of (9) for every $f \in L_2(0,b)$ (see Proposition 3). Note that Definition 2 implies the inequality $\|u\|_{W_\alpha^2} \le c\|Bu\|_{L_2(0,b)}$. Compactness of the operator B^{-1} for $0 \le \alpha < 4$ now follows from the Proposition 2.

Corollary. For $0 \le \alpha < 4$ the operator B has discrete spectrum, and its eigenfunction system is complete in $f \in L_2(0,b)$ (see [7]).

Note that now we can rewrite the equation (1) in the form $Bu = (1-a)u + f$, i.e. we can refer the number $1-a$ as a spectral parameter.

Theorem 2. The domain of operator L consists of functions $u(t,)$ for which $u(0)$ is finite when $0 \le \alpha < 7/2$ and $u'(0)$ is finite for $0 \le \alpha < 2$. The values $u(0)$ and $u'(0)$ can not be specified arbitrarily, but are determined by the right-hand side of (1).

Proof. Since $D(L) = D(L - aI)$, it is sufficient to study the properties of Neumann problem for equation

$$\left(t^\alpha u''\right)'' = f. \tag{10}$$

Let $1 \le \alpha < 2$. The derivative of the general solution to equation (10) has the following form:

$$u'(t) = c_1 + c_2 t^{2-\alpha} + \int_0^t \tau^{-\alpha} \int_0^\tau (\tau - \eta) f(\eta) d\eta d\tau.$$

We have $\left|\int_0^t \tau^{-\alpha} \int_0^\tau (\tau-\eta) f(\eta) d\eta d\tau\right| \le c t^{\frac{5}{2}-\alpha} \|f\|_{L_2(0,b)}$.

For $\alpha \ge 2$ the value $u'(0)$, generally speaking, can be infinite. Now assume $3 \le \alpha < 7/2$. Then we can write the solution in the following form:

$$u(t) = c_3 + c_4 t - \int_0^t \int_\tau^b \eta^{-\alpha} \int_0^\eta (\eta - \xi) f(\xi) d\xi d\eta d\tau.$$

Then we have

$$\left|\int_0^t \int_\tau^b \eta^{-\alpha} \int_0^\eta (\eta - \xi) f(\xi) d\xi d\eta d\tau\right| \le c\left|t^{\frac{7}{2}-\alpha} - \left(\frac{7}{2} - \alpha\right) b^{\frac{5}{2}-\alpha} t\right| \|f\|_{L_2(0,b)},$$

which completes the proof.

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Mathematics

REGRESSION MODELS GENERATED BY DISTRIBUTIONS OF MODERATE GROWTH

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Regression model generated by two-parametric distribution of moderate growth and arising in bioinformatics is considered. Consistency (in a weak sense) of least square estimates for parameters is proved. Distributions of least square estimates for parameters and of Gaussian noise variation estimate are obtained. Results may be used for statistical hypothesis testing with regard to parameters of model.

***Keywords*:** consistent least square estimate, regression model, distribution of moderate growth.

§ 1. Introduction. Two-parametric distribution of *moderate growth* is introduced in [1]. It takes the following form

$$\begin{cases} p_n(\alpha) = p_0(\alpha)\dfrac{\theta^n}{\psi_n}\displaystyle\prod_{m=0}^{n-1}\left(1+\frac{c-1}{\psi_m}\right), \quad n=1,2,\dots, \\ p_0(\alpha) = \left\{1+\displaystyle\sum_{n\ge1}\frac{\theta^n}{\psi_n}\prod_{m=0}^{n-1}\left(1+\frac{c-1}{\psi_m}\right)\right\}^{-1} \end{cases} \tag{1.1}$$

with parametric set $A=\{\alpha=(c,\theta):\ 0<c<+\infty,\ 0<\theta\le1\}$. The moderate growth of distribution $\{p_n(\alpha)\}_0^\infty$. (1.1) is defined by conditions on sequence $\{\psi_n\}_1^\infty$:

$$\psi_0=1,\ \{\psi_n\}_1^\infty \text{ is non-decreasing, } \lim_{n\to+\infty}\psi_n=+\infty,\ \lim_{n\to+\infty}\left(\psi_n/\psi_{n-1}\right)=1$$

under the constraint

$$S_\psi=\sum_{n\ge1}\left(1/\psi_n\right)<+\infty. \tag{1.2}$$

Below $N(a,\sigma^2)$ denotes the Gaussian distribution function with *mean a* and *variance* σ^2.

In [2] the following regression model generated by model (1.1)–(1.2) is considered. We take logarithm from both sides of (1.1) and replace $\ln\left(1+\dfrac{c-1}{\psi_m}\right)$ by $(c-1)/\psi_m$ (so called *linearization*). Then, denoting $f_\alpha(n)=\eta\cdot n+(c-1)S_\psi(n)$,

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$\eta = \ln\theta$, $S_\psi(n) = \sum_{m-0}^{n-1} 1/\psi_m$ and $y_n = \ln\big(p_n(\alpha)/p_0(\alpha)\big) + \ln\psi_n$, we build the *regression model* of the form

$$y_n = f_\alpha(n) + \varepsilon_n, \quad n = 1,2,\dots,N. \tag{1.3}$$

Given the integer $N \ge 1$ and the *observed* sequence $\{y_n\}_1^N$ one needs to find estimates for *unknown* parameters c and η under, for instance, the assumption: $\varepsilon_n \sim N(0,\sigma^2)$, $n = \overline{1,N}$, *are independent Gaussian noises with unknown variation* σ^2. So, the estimate of σ^2 is also needed.

Denote $r_N = c_N \cdot S_N - \left(\sum n \cdot S_\psi(n)\right)^2$, $c_N = \sum n^2$, $S_N = \sum \left(S_\psi(n)\right)^2$.

Here and everywhere below in sums limits on n (from 1 to N) are omitted for simplicity. In [2] for *unbiased* least square estimates (LSEs) $\hat{c}_N$ and $\hat{\eta}_N$ of parameters c and $\eta (= \ln\theta)$ in the model (1.3) the following formulas are found:

$$\begin{cases} \hat{c}_N - 1 = r_N^{-1}\left\{c_N\left(\sum y_n \cdot S_\psi(n)\right) - \left(\sum n \cdot y_n\right)\left(\sum n \cdot S_\psi(n)\right)\right\}, \\ \hat{\eta}_N = r_N^{-1}\left\{S_N\left(\sum n \cdot y_n\right) - \left(\sum n \cdot S_\psi(n)\right)\left(\sum y_n \cdot S_\psi(n)\right)\right\}. \end{cases} \tag{1.4}$$

In this paper, first of all, based on (1.4) the *normality* and the *consistency* in a weak sense of LSEs $\hat{c}_N$ and $\hat{\eta}_N$ are established. Next, we consider the estimate $\overset{2}{\sigma}_N = \dfrac{1}{N-2}\sum e_n^2$ for variation σ^2, where $e_n = y_n - f_\alpha(n)$ are so called *residuals* (*remainders*) of regression (1.3), and $f_\alpha(n) = \hat{\eta}_N n + (\hat{c}_N - 1)S_\psi(n)$, $n = 1,2,\dots,N$, obviously, are LSEs for $f_\alpha(n)$ (the predicted value of $f_\alpha(n)$).

We prove some "good" properties of LSEs $\overset{2}{\sigma}_N, \hat{c}_N$ and $\hat{\eta}_N$.

§ 2. Normality and Consistency.

Theorem 1. The LSEs $\hat{\eta}_N$ and $\hat{c}_N$ for η and c have distribution functions $N\left(\eta, \dfrac{\sigma^2}{r_N} S_N\right)$ and $N\left(c, \dfrac{\sigma^2}{r_N} c_N\right)$ respectively. They are consistent in a weak sense, i.e. $\hat{\eta}_N \xrightarrow{p} \eta$, $\hat{c}_N \xrightarrow{p} \eta$ as $N \to +\infty$. Here the sign „ $\xrightarrow{p}$ '' denotes the convergence in probability.

Proof. Denote $y = (y_1,\dots,y_N)^`$, $\varepsilon = (\varepsilon_1,\dots,\varepsilon_N)^`$, $\beta = (\eta, c-1)^`$, where «`» is the symbol of allocation, and present the *regression model* (1.3) in the following form $y = X\beta + \varepsilon$,

$$X = \begin{pmatrix} 1 & S_\psi(1) \\ \vdots & \vdots \\ n & S_\psi(n) \end{pmatrix}. \tag{2.1}$$

It is well-known (see, for instance [3]), that LSE β, minimizing the regression remainders squares sum for (2.1), due to $e^`e = \sum e_n^2$, $e = y - X\beta$, takes the form

$$\beta = (X`X)^{-1} X`y`. \tag{2.2}$$

Easily seen that the matrix $X`X$ takes the form

$$X`X = \begin{pmatrix} c_N & x`S_\psi \\ \vdots & \vdots \\ x`S_\psi & S_N \end{pmatrix}$$

with $x = (1,2,\dots,N)$, $S_\psi = (S_\psi(1), S_\psi(2),\dots,S_\psi(N))$, where we assume that $\det(X`X) = r_N \neq 0$. Then, evaluations lead to the form (2.2) of estimate β.

From Gauss–Markov theorem [3] it follows that the estimate β is *optimal* (in the sense of minimums of variations $D\hat{\eta}_N$ and $D\hat{c}_N$) in the class of linear with respect to y, unbiased estimates for parameter β. Taking into account that y_n, $n = \overline{1,N}$, has distribution function $N(f_\alpha(n), \sigma^2)$, the estimate (2,2) is linear with respect to y, and X is not random, we conclude that LSEs $\hat{c}_N$ and $\hat{\eta}_N$ are Gaussian. Further, from the representation (2.2)

$$\beta = (X`X)^{-1} X`(X\beta + \varepsilon) = \beta + (X`X)^{-1} X`$$

it follows $E\beta = \beta + (X`X)^{-1} X`E\varepsilon = \beta,$ where E denotes the sign of mathematical expectation.

So, β is the *unbiased* estimate for parameter β. Let us evaluate the variances $D\hat{\eta}_N$ and $D\hat{c}_N$. For this purpose we need in covariance matrix $V\beta$ of estimate β: $V\beta = E(\beta - E\beta)(\beta - E\beta)` = EA\varepsilon(A\varepsilon)` = A(E\varepsilon\varepsilon`)A` = \sigma^2 AA`$, where $A = (X`X)^{-1} X`$. Since $AA` = (X`X)^{-1}$, therefore,

$$V\beta = \sigma^2 (X`X)^{-1}. \tag{2.3}$$

On the other hand, because of the form

$$(X`X)^{-1} = r_N^{-1} \begin{pmatrix} S_N & -(x`S_\psi) \\ \vdots & \vdots \\ -(x`S_\psi) & c_N \end{pmatrix},$$

due to (2.3), we obtain $D\hat{\eta}_N = \dfrac{\sigma^2}{r_N} S_N, \quad D\hat{c}_N = \dfrac{\sigma^2}{r_N} c_N.$

Let us pass to the proof of consistency of estimates $\hat{c}_N$ and $\hat{\eta}_N$. It is enough to show that $D\hat{\eta}_N \to 0$ and $D\hat{c}_N \to 0$ as $N \to +\infty$ (see [4]).

Due to Couchy–Shwartz inequality, we have $0 \le 1 - \dfrac{\left(\sum n \cdot S_\psi(n)\right)^2}{\sum n^2 \sum \left(S_\psi(n)\right)^2} < 1$.

That is why

$$\lim_{n\to+\infty} \frac{r_N}{\sum \left(S_\psi(n)\right)^2} = +\infty, \tag{2.4}$$

which follows from the representation

$$\frac{r_N}{S_N}=\sum n^2-\frac{\sum n\cdot S_\psi(n)}{\sum\left(S_\psi(n)\right)^2}=\sum n^2\left(1-\frac{\left(\sum n\cdot S_\psi(n)\right)^2}{\left(\sum n^2\right)\left(\sum\left(S_\psi(n)^2\right)^2\right)}\right).$$

The limit relation (2.4) says that $D\hat{\eta}_N\to 0$ as $N\to+\infty$.

Similarly one may prove that $D\hat{c}_N\to 0$ as $N\to+\infty$. Theorem 1 is proved.

§ 3. Properties of LSE $\overset{2}{\sigma}_N$. Let the constraint (1.2) holds.

Theorem 2. The statistics (1.5) is unbiased estimate for σ^2, and the statistics $\chi^2_{N-2}=(N-2)\dfrac{\overset{2}{\sigma}_N}{\sigma^2}$ has χ^2-distribution $H^2(N-2)$ with $(N-2)$ degrees of freedom.

Proof. Let us present the predicted (by regression) value $\hat{y}=X\beta$ in the form $\hat{y}=(X`X)^{-1}X`y=My$, and the vector of regression remainders $e=y-\hat{y}$ in the form $e=y-X\beta=(I_N-M)y=By=B(X\beta+\varepsilon)=B\varepsilon$, because $Bx=0$. Here I_N is a usual unit matrix of order N.

The matrixes H and B satisfy conditions: $H`=H, H^2=H$, $B`=B, B^2=B$.

Write down the following chain of equalities:

$E(e`e)=E(\sum e_n^2)=\mathrm{Etr}(ee`)=\mathrm{Etr}(B\varepsilon\varepsilon`B)=\mathrm{tr}(BE(\varepsilon\varepsilon`)B`)=\sigma^2\cdot\mathrm{tr}(BB`)=\sigma^2\cdot\mathrm{tr}B,$

where "tr" denotes the *trace* of matrix B. On the other hand, $\mathrm{tr}B=\mathrm{tr}(I_N)-\mathrm{tr}(M)$ and $\mathrm{tr}(M)=\mathrm{tr}(X(X`X)^{-1}X`)=\mathrm{tr}(X`X(X`X)^{-1})=\mathrm{tr}I_2=2$. That is why, finally, we obtain $E(ee`)=\sigma^2(N-2)$, i.e. $E\overset{2}{\sigma}_n=\sigma^2$.

Next we have $\chi^2_{N-2}=\dfrac{1}{\sigma^2}(ee`)B(\varepsilon/\sigma)$, where $(\varepsilon/\sigma)$ is an $N$-dimensional standard Gaussian vector (with zero means and unit variations). Since $B`=B$, $B^2=B$, therefore, (see [3]) the statistics χ^2_{N-2} has χ^2-distribution with $k=\mathrm{rank}(B)$ degrees of freedom. But in this case of B we have $\mathrm{rank}(B)=\mathrm{tr}B=N-2$. Theorem 2 is proved.

Remark. It is of interest that for a given N:

The estimates $\overset{2}{\sigma}_N$, $(\hat{c}_N,\hat{\eta}_N)$ are independent. (3.1)

Indeed, taking into account that $\overset{2}{\sigma}_n=\dfrac{1}{N-2}(ee`)$, it is enough to prove that LSEs $\beta=\beta_N=(\hat{\eta}_N,\hat{c}_N)$ and remainders of regression vector E are non-correlated, because they have Gaussian distribution.

Since $Ee=0$, therefore, $\mathrm{cov}(\beta_N,e)=E(\beta_N-\beta)e`=EA\varepsilon(\varepsilon B)`=AE(\varepsilon\varepsilon`)B`=$ $=\sigma^2(AB)=0$, where the equalities $AB=(X`X)^{-1}X`(I_N-X(X`X)^{-1}X`)=0$ were used.

§ 4. Properties of LSEs $\hat{c}_N$ and $\hat{\eta}_N$. Let the constraint (1.2) holds.

Theorem 3. The statistics

$$t_{N-2}^{(1)} = \frac{(\hat{\eta}_N - \eta)}{\sigma_N}\left(\frac{r_N}{S_N}\right)^{1/2} \quad \text{and} \quad t_{N-2}^{(2)} = \frac{(\hat{c}_N - c)}{\sigma_N}\cdot\left(\frac{r_N}{S_N}\right)^{1/2}$$

have Student's distribution $T(N-2)$ with $N-2$ degrees of freedom.

Proof. Due to Theorem 1, $\hat{\eta}_N - \eta$ has Gaussian distribution $N\left(0, \sigma^2_{\hat{\eta}_N}\right)$, where $\sigma^2_{\hat{\eta}_N} = D\hat{\eta}_N = \frac{\sigma^2}{r_N} S_N$.

Let us take as an estimate for $\sigma^2_{\hat{\eta}_N}$ the statistics $\frac{\sigma_N^2}{r_N} S_N$, i.e. $S^2_{\hat{\eta}_N} = \hat{\sigma}^2_{\eta_N} = (\sigma_N^2 S_N)/r_N$.

Due to Theorem 2, the statistics $\chi^2_{N-2} = (N-2)\frac{\sigma_N^2}{\sigma^2} = \frac{1}{\sigma^2}\sum e_n^2$ has distribution $H^2(N-2)$. Now let us consider the following statistics:

$$t_{N-2}^{(1)} = \frac{(\hat{\eta}_N - \eta)/\sigma_{\hat{\eta}_N}}{S_{\hat{\eta}_N}/\sigma_{\hat{\eta}_N}} = \frac{\xi_0}{\sqrt{\frac{1}{N-2}\chi^2_{N-2}}},$$

where $\xi_0 = (\hat{\eta}_N - \eta)/\sigma_{\hat{\eta}_N}$ has distribution $N(0,1)$ and

$$\frac{S_{\hat{\eta}_N}}{\sigma_{\hat{\eta}_N}} = \frac{\sigma_N (S_N/r_n)^{1/2}}{\sigma (S_N/r_n)^{1/2}} = \frac{\sigma_N}{\sigma} = \sqrt{\frac{1}{N-2}\chi^2_{N-2}}.$$

According to the Remark, $\hat{\eta}_N$ and e are independent. That is why random variables ξ_0 and χ^2_{N-2} are independent too. It implies, due to definition of random variable, which has Student's distribution, that the statistics $t_{N-2}^{(1)}$ has Student's distribution $T(N-2)$ with $N-2$ degrees of freedom.

Theorem 3 is proved.

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Mechanics

BUCKLING OF ISOTROPIC PLATES WITH TWO OPPOSITE SIMPLY SUPPORTED EDGES AND THE OTHER TWO EDGES ROTATIONALLY RESTRAINED UNLOADED

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The paper presents buckling loads of isotropic rectangular plates with two simply supported opposite edges and the other two edges elastically supported against rotation. An analytical method that uses the Lévy solution method is employed to determine the buckling loads of the mentioned rectangular plates. The convergence and comparison of the results with those available in the literature indicate the accuracy and the validity of the proposed technique. Effects of the elastic restraint parameters on the mode shapes are illustrated in graphic forms.

Keywords: buckling, elastic restraint, isotropic plates, rectangular plates.

Introduction. Plates of different shapes with different boundary conditions having various applied in-plane force distributions, as well as different buckling factors are considered and documented in [1–4]. Buckling of isotropic plate is discussed in numerous classical monographs. Solution procedures, development of characteristic equations and graphical presentation of buckling curves in terms of dimensionless buckling coefficient and edge restraint coefficient, are well known. Effect of the rotational restraint on the buckling load is thoroughly studied in literature as well. This paper follows the previous papers and presents in standard form parametric information on buckling.

Analysis. Consider the equation governing buckling deflection w (stability equation) for a rectangular isotropic plate, subjected to distributed compressive load P along the x-axis. The rectangular plate simply supported along the edges $x=0$ and $x=a$, and elastically supported against rotation along the other edges is shown in Fig. 1.

$$D\Delta^2 w + P\frac{\partial^2 w}{\partial x^2} = 0, \tag{1}$$

$$w = 0\ , \quad \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, a . \tag{2}$$

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The boundary conditions on the rotationally restrained edges at $y = \pm 0,5b$ are

$$\left.\begin{aligned} & w_{yy} + \frac{K_R}{D} w_y = 0, \\ & w = 0 \end{aligned}\right\} \quad \text{at } y = \pm 0,5b.$$

Here D is a flexural stiffness, defined as $D = \dfrac{Eh^3}{12(1-\nu^2)}$, where E is the Young's modulus, ν is the Poisson's ratio, w is the deflection, Δ is the Laplace operator, and K_R is the restraining moment along the rotationally restrained edge per unit length and per unit rotation [5]. The dimensionless coefficient of rotational restraint R (alternatively identified as ε in the literature) can be defined as $R = \dfrac{K_R b}{D}$, where b is the plate's width. The boundary condition on the restrained edges can then be written as

$$\left.\begin{aligned} & w_{yy} + \frac{R}{b} w_y = 0, \\ & w = 0 \end{aligned}\right\} \quad \text{at } y = \pm 0,5b. \tag{3}$$

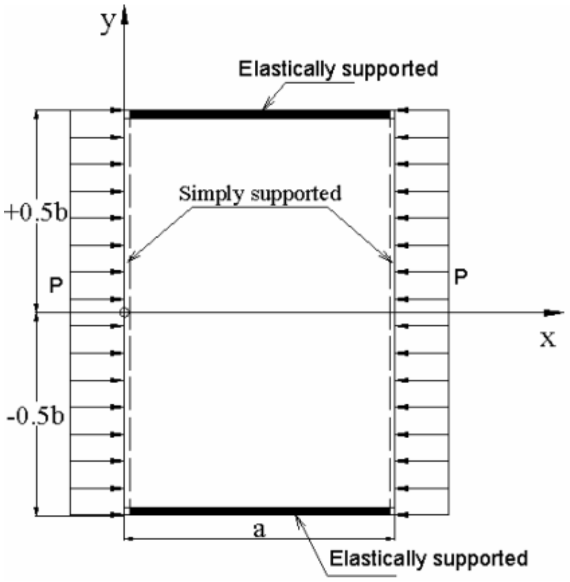

Fig. 1. Uniformly compressed rectangular plate simply supported along the edges x=0 and x=a, and elastically supported against rotation along the other edges (SSEE plate).

The solution to (1) is chosen in a form that satisfies (2)

$$w(x,y) = \sum_{m=1}^{\infty} f_m(y) \sin(\frac{m\pi}{a} x), \quad \mu_m = \frac{m\pi}{a}. \tag{4}$$

The function $f_m(y)$ is the eigenfunction, corresponding to the least eigenvalue (which is to be determined), represents the buckling shape along the y-axis. m represents the number of half-waves in the direction x. Substituting (4) into (1), we find for $f_m(y)$ the following linear ordinary differential equation:

$$\frac{d^4 f_m}{dy^4} - 2\mu_m^{\ 2}\frac{d^2 f_m}{dy^2} + \left[\mu_m^{\ 4} - \frac{P}{D}\mu_m^{\ 2}\right] f_m = 0. \tag{5}$$

Substituting (4) into (3), we obtain

$$\left.\begin{array}{l} \dfrac{d^2 f_m}{dy^2} + \dfrac{R}{b}\cdot\dfrac{df_m}{dy} = 0, \\ f = 0 \end{array}\right\} \quad \text{at} \quad y = \pm 0{,}5b. \tag{6}$$

The form of the solution to (5) depends on the nature of the roots λ of the equation $\lambda^4 - 2\mu_m^{\ 2}\lambda^2 + \left[\mu_m^{\ 4} - \frac{P}{D}\mu_m^{\ 2}\right] = 0$.

Assuming that $\frac{P}{D} > \mu_m^{\ 2}$, the general solution is

$$f_m(y) = C_1\cosh\lambda_1 y + C_2\sinh\lambda_1 y + C_3\cos\lambda_2 y + C_4\sin\lambda_2 y, \tag{7}$$

where

$$(\lambda_1)^2 = \sqrt{\mu_m^{\ 2}\frac{P}{D}} + \mu_m^{\ 2}, \quad (\lambda_2)^2 = \sqrt{\mu_m^{\ 2}\frac{P}{D}} - \mu_m^{\ 2}. \tag{8}$$

Since the deflection w at the buckling load must be a symmetric function of y, in the right hand side of (7) there remain only the first and the third terms. Thus,

$$f_m(y) = C_1\cosh\lambda_1 y + C_3\cos\lambda_2 y, \tag{9}$$

and the general mode shape is given by $w(x,y) = \sum_{m=1}^{\infty}(C_1\cosh\lambda_1 y + C_3\cos\lambda_2 y)\sin(\mu_m x)$.

The unknown constants C_1 and C_3 are determined from the edge conditions at $y = \pm 0{,}5b$. Substituting (9) into (6) and considering $w_{xx} = 0$ at $y = \pm 0{,}5b$, a set of simultaneous equations with regard to C_1 and C_3 is obtained:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{Bmatrix} C_1 \\ C_3 \end{Bmatrix} = 0.$$

Here $A_{11} = \lambda_1^{\ 2}\cosh(\lambda_1 b/2) + (R/b)\lambda_1\sinh(\lambda_1 b/2)$, $A_{12} = -\lambda_2^{\ 2}\cos(\lambda_2 b/2) - (R/b)\lambda_2\sin(\lambda_2 b/2)$, $A_{21} = \cosh(\lambda_1 b/2)$, $A_{22} = \cos(\lambda_2 b/2)$.

The condition $\Delta = 0$ yields the following characteristic buckling equation:

$$\Delta = A_{11}A_{22} - A_{12}A_{21} = 0,$$

$$\begin{aligned} &\lambda_1^{\ 2}\cos(\lambda_2 b/2)\cosh(\lambda_1 b/2) + (R/b)\lambda_1\cos(\lambda_2 b/2)\sinh(\lambda_1 b/2) + \\ &+\lambda_2^{\ 2}\cosh(\lambda_1 b/2)\cos(\lambda_2 b/2) + (R/b)\lambda_2\cosh(\lambda_1 b/2)\sin(\lambda_2 b/2) = 0. \end{aligned} \tag{10}$$

Since λ_1 and λ_2 contain P (8), (10) can be solved, using an iterative scheme, for the smallest P, denoted by P_{cr}, once the geometric and material parameters of the plate are known. The critical buckling may be written as $P_{cr} = k\frac{\pi^2 D}{b^2}$, where k is a numerical factor (buckling coefficient), depending on the plate aspect ratio and material properties.

The transcendental equation is solved to determine the buckling coefficient, k, as a function of plate properties, the value of the edge restraint coefficient R, the plate aspect ratio a/b and the mode number m. The buckling mode is given by the mode number, for which the smallest k is obtained for a given set of parameters.

Results and Discussion. Matlab program was written to perform the parametric studies reported below. Parametric studies were performed to investigate the influence of material properties, rotational restraint and mode number on the buckling characteristics of this plate. Buckling load $\overline{P}$ was obtained for different coefficients of rotational restraint values as a function of the plate aspect ratio a/b. Buckling curves are shown in Fig. 2. Due to the fact that the rotational restraint is a variable parameter $R = 0,\ 4,\ 10,\ 30,\ 10000$, $R = 0$ corresponds to a simple support, and $R = \infty$ corresponds to a clamped support. Intermediate values of R imply partial rotational restraint. In the numerical studies $R = 10000$ was taken to represent $R = \infty$ [4, 5].

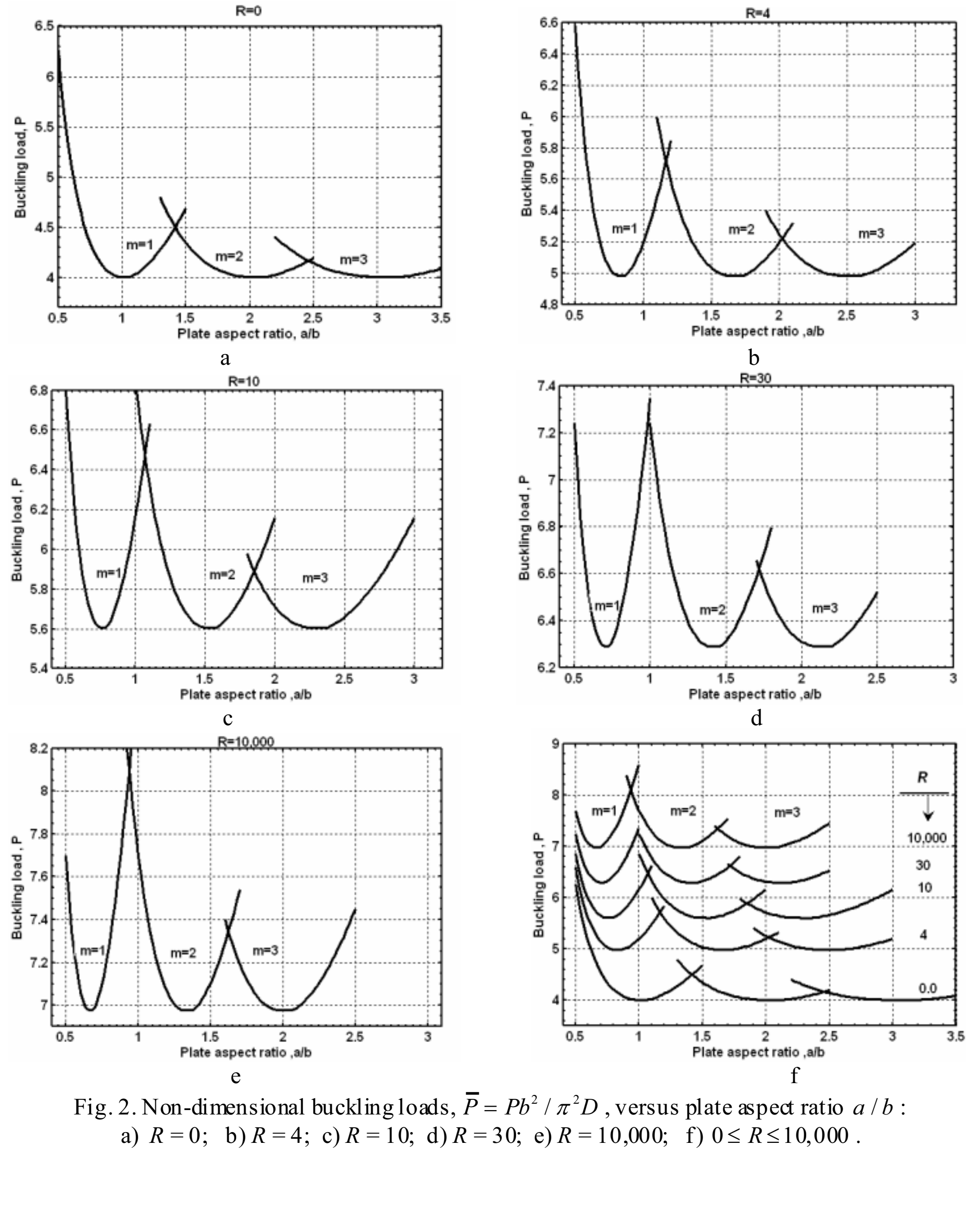

Fig. 2. Non-dimensional buckling loads, $\overline{P} = Pb^2/\pi^2 D$, versus plate aspect ratio a/b:
a) $R = 0$; b) $R = 4$; c) $R = 10$; d) $R = 30$; e) $R = 10{,}000$; f) $0 \le R \le 10{,}000$.

Conclusions. Buckling of an isotropic plate, free supported on its loaded edges, and rotationally restrained on its unloaded edges has been considered in this paper. The correct form of the characteristic transcendental equation for this buckling problem has been provided. Parametric studies have been conducted and buckling curves have been presented.

- $R = 10000 \equiv \infty$. The minimum value of the buckling load $\bar{P} = 6{,}976$ occurs at $a/b = 0{,}665$. There is a mode change at $a/b = 0{,}936$ from $m = 1$ to $m = 2$, the buckling load for this aspect ratio is $\bar{P} = 8{,}095$, and the minimum buckling load $\bar{P} = 6{,}976$ for mode $m = 2$ occurs at $a/b = 1{,}329$. These results corresponds to a clamped support [4].
- $R = 0$. For this case the minimum value of the buckling load $\bar{P} = 4{,}008$ occurs at $a/b = 1$. The result corresponds to exact a simply support [4].
- $0 < R < 10000$. The other buckling curves are between two sets of the mentioned buckling load.

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Informatics

ON THE TYPE CORRECTNESS OF POLYMORPHIC λ-TERMS. 2

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In this paper the polymorphic lambda terms are considered, where no type information is provided for the variables. The aim of this work is to prove that presented typification algorithm [1] typifies such terms in most common way.

Keywords: type, term, constraint, skeleton, expansion, principal typing.

1. Introduction. Types are used in programming languages to analyze programs without executing them, for purposes such as detecting programming errors earlier, for doing optimizations etc. In some programming languages no explicit type information is provided by the programmer, hence some system of type inference is required to recover the lost information and do compile time type checking. One of such type inference systems is the well known Hindley/Milner system [2], used in languages such as Haskell, SML, OCaml etc. An important property of the type systems is the property of *principal typings* [3, 4], which allows the compiler to do *compositional analysis*, i.e. analysis of modules in absence of information about other modules [3, 4]. Unfortunately the Hindley/Milner system doesn't support the property of *principal typings* [3]. This paper is the continuation of [1], in which we consider the extension of the type inference system called *System E*. In section 2 we prove that the type inference algorithm returns the principal typing of a term.

2. Principal Typing of a Term.

2.1. Preliminary Definitions and Facts. Before proving that the type inference algorithm returns the principal typing of a term let us present some definitions and facts.

Definition 2.1. Let $Q_1, Q_2 \in Skeleton$. Q_1 and Q_2 are equivalent, written $Q_1 \approx Q_2$, iff $term(Q_1) = term(Q_2)$, $typing(Q_1) = typing(Q_2)$, $constraint(Q_1) = constraint(Q_1)$. In other words, $Q_1 \approx Q_2$, iff the judgements $(M \triangleright Q_1) : (A ђ \tau) / \Delta$ and $(M \triangleright Q_2) : (A ђ \tau) / \Delta$ are both inferable or not inferable.

Lemma 2.1. The following skeletons are equivalent:

1. $(Q_1 \cap (Q_2 \cap Q_3)) \approx ((Q_1 \cap Q_2) \cap Q_3)$; 2. $(Q_1 \cap Q_2) \approx (Q_2 \cap Q_1)$;

3. $(\omega^M \cap Q) \approx Q$; 4. $e(Q_1 \cap Q_2) \approx (eQ_1 \cap eQ_2)$; 5. $e\omega^M \approx \omega^M$,

where $Q_1, Q_2, Q_3 \in Skeleton$ and $M \in Term$ and $e \in ExpansionVariable$.

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Let us consider the judgement $(M \rhd Q):(A \vdash \tau)/\Delta$ that is inferable. In many cases we will consider the maximal subtrees of the inference tree of that judgement that have root node corresponding to one of the following type inference rules: [VAR], [CONST], [OMEGA], [ABS] and [APP].

Lemma 2.2. Let $(M \rhd Q):(A \vdash \tau)/\Delta$ be an inferable judgement. Then there exist E-paths $e'_1,\dots,e'_n$, environments $A_1,\dots,A_n$, $Q_1,\dots,Q_n \in Skeleton$, $\tau_1,\dots,\tau_n \in Type$ and $\Delta_1,\dots,\Delta_n \in Constraint$, $n \ge 1$, such that $Q \approx e'_1Q_1 \cap \dots \cap e'_nQ_n$, $A{=}e'_1A_1 \cap \dots \cap e'_nA_n$, $\tau{=}e'_1\tau_1 \cap \dots \cap e'_n\tau_n$, $\Delta{=}e'_1\Delta_1 \cap \dots \cap e'_n\Delta_n$ and judgements $(M \rhd Q_i):(A_i \vdash \tau_i)/\Delta_i$, $i{=}1,\dots,n$, are inferable, and in the last step of inference of that judgements one of the following rules is used: [VAR], [CONST], [OMEGA], [ABS] and [APP].

A free occurrence of the subskeleton $x^{:\tau}$ in skeleton Q is defined in a conventional way, i.e. the occurrence of the subskeleton $x^{:\tau}$ in skeleton Q is called free, if it doesn't fall within the scope of a lambda that uses variable x, otherwise, the occurrence is called bounded. It is easy to see, that if the skeleton Q' is obtained from the skeleton Q by renaming some term variables, then $Q \approx Q'$. Let us introduce the following notations:

1. We denote by $Q\langle Q_1,\dots,Q_n\rangle$, $n \ge 0$, the skeleton Q, in which mutually different subskeletons $Q_1,\dots,Q_n$ are considered.

2. We denote by $Q\langle Q_1 := Q'_1,\dots,Q_n := Q'_n\rangle$, $n \ge 0$, those skeletons that are obtained from the skeleton $Q\langle Q_1,\dots,Q_n\rangle$ through substituting the subskeletons $Q_1,\dots,Q_n$ by $Q'_1,\dots,Q'_n$ respectively. The substitution mentioned above is called canonical, iff all free occurrences of subskeletons in Q_i are also free in $Q\langle Q_1,\dots,Q_n\rangle$, and all free occurrences of subskeletons in Q'_i are also free in $Q\langle Q_1 := Q'_1,\dots,Q_n := Q'_n\rangle$, $i{=}1,\dots,n$. Henceforth only canonical substitutions of skeletons will be considered.

Definition 2.2. Let $Q\langle Q'\rangle \in Skeleton$. Then the *E-Path* of the skeleton Q' in Q, written as $E\text{-}Path(Q\langle Q'\rangle)$, is calculated as follows:

1. If $Q{=}Q'$, then $E\text{-}Path(Q\langle Q'\rangle) = \varepsilon$;

2. If $Q{=}eQ_1$, and Q' is a subskeleton of Q_1, then $E\text{-}Path(Q\langle Q'\rangle) =$ $= eE\text{-}Path(Q_1\langle Q'\rangle)$, where $Q_1 \in Skeleton$ and $e \in ExpansionVariable$;

3. If $Q{=}(\lambda x.Q_1)$, and Q' is a subskeleton of Q_1, then $E\text{-}Path(Q\langle Q'\rangle) =$ $= E\text{-}Path(Q_1\langle Q'\rangle)$, where $Q_1 \in Skeleton$ and $x \in TermVariable$;

4. If $Q{=}(Q_1 \cap Q_2)$, and Q' is a subskeleton of Q_1, then $E\text{-}Path(Q\langle Q'\rangle) =$ $= E\text{-}Path(Q_1\langle Q'\rangle)$, where $Q_1, Q_2 \in Skeleton$;

5. If $Q{=}(Q_1 \cap Q_2)$, and Q' is a subskeleton of Q_2, then $E\text{-}Path(Q\langle Q'\rangle) =$

$=E\text{-}Path\left(Q_2\left\langle Q'\right\rangle\right)$, where $Q_1, Q_2 \in Skeleton$;

6. If $Q=(Q_1Q_2)^{:\tau}$, and Q' is a subskeleton of Q_1, then $E\text{-}Path\left(Q\left\langle Q'\right\rangle\right)=$ $=E\text{-}Path\left(Q_1\left\langle Q'\right\rangle\right)$, where $Q_1, Q_2 \in Skeleton$ and $\tau \in Type$;

7. If $Q=(Q_1Q_2)^{:\tau}$, and Q' is a subskeleton of Q_2, then $E\text{-}Path\left(Q\left\langle Q'\right\rangle\right)=$ $=E\text{-}Path\left(Q_2\left\langle Q'\right\rangle\right)$, where $Q_1, Q_2 \in Skeleton$ and $\tau \in Type$.

Let us present some simple propositions without proof.

Proposition 2.1. Let $Q \in Skeleton$ and $type(Q) = \vec{e}_1\tau_1 \cap \ldots \cap \vec{e}_n\tau_n$, $n \geq 1$, where $\tau_1,\ldots,\tau_n \in Type$ and $\vec{e}_1,\ldots,\vec{e}_n$ are *E-Paths*. Then $\exists Q_1,\ldots,Q_n \in Skeleton$ such that $Q \approx \vec{e}_1 Q_1 \cap \ldots \cap \vec{e}_n Q_n$ and $type(Q_i) = \tau_i$, $i=1,\ldots,n$.

Proposition 2.2. Let $Q\left\langle Q_1,\ldots,Q_n\right\rangle \in Skeleton$, $n \geq 0$, and $Q'_1,\ldots,Q'_n \in Skeleton$. Then if $type(Q_i)=type(Q'_i)$ $\forall i=1,\ldots,n$, then $type\left(Q\left\langle Q_1,\ldots,Q_n\right\rangle\right)=type\left(Q\left\langle Q_1 := Q'_1,\ldots,Q_n := Q'_n\right\rangle\right)$.

Proposition 2.3. Let $Q\left\langle x^{:\tau_1},\ldots,x^{:\tau_n}\right\rangle \in Skeleton$ and only subskeletons $x^{:\tau_1},\ldots,x^{:\tau_n}$ have free occurrence in Q and $Q_1,\ldots,Q_n \in Skeleton$, where $x \in TermVariable$ and $\tau_1,\ldots,\tau_n \in Type$, $n \geq 0$. Then if $term\left(Q\left\langle x^{:\tau_1},\ldots,x^{:\tau_n}\right\rangle\right) = M_1$ and $term(Q_i) = M_2$ $\forall i=1,\ldots,n$, then $term\left(Q\left\langle x^{:\tau_1} := Q_1,\ldots,x^{:\tau_n} := Q_n\right\rangle\right)=$ $= M_1[x := M_2]$.

Proposition 2.4. Let $Q\left\langle x^{:\tau}\right\rangle \in Skeleton$ and $Q' \in Skeleton$ and $type(Q')=\tau$, where $\tau \in Type$ and $x \in TermVariable$. Then $constraint\left(Q\left\langle x^{:\tau} := Q'\right\rangle\right)=$ $=constraint\left(Q\left\langle x^{:\tau}\right\rangle\right) \cap E\text{-}Path\left(Q\left\langle x^{:\tau}\right\rangle\right) constraint\,(Q')$.

Let us consider the term $M \notin \beta\text{-}NF$ and one step of β-reduction: $M \rightarrow_\beta M'$. Now we are going to show that if $(A ђ\, \tau)$ is a typing of term M, then it is also a typing of term M'.

Lemma 2.3. Let $Q\left\langle x^{:\tau_1},\ldots,x^{:\tau_n}\right\rangle \in Skeleton$ and only subskeletons $x^{:\tau_1},\ldots,x^{:\tau_n}$ have free occurrence in Q and $\vec{e}_i = E\text{-}Path\left(Q\left\langle x^{:\tau_i}\right\rangle\right)$, $i=1,\ldots,n$, where $\tau_1,\ldots,\tau_n \in Type$ and $x \in TermVariable$, $n \geq 0$. Then $env(Q)(x) = \vec{e}_1\tau_1 \cap \ldots \cap \vec{e}_n\tau_n$.

Proof. By induction on form of skeleton Q.

1. Let $Q=\omega^M$, where $M \in Term$. We must show that $env(Q)(x) = \omega$. By the rule [OMEGA], $env(Q) = env_\omega \Rightarrow env(Q)(x) = \omega$.

2. Let $Q=c^{:\tau}$, where $c \in Constant$ and $\tau \in Type$. We must show that $env(Q)(x) = \omega$. By the rule [CONST], $env(Q) = env_\omega \Rightarrow env(Q)(x) = \omega$.

3. Let $Q{=}y^{:\tau}$, where $x \neq y \in TermVariable$ and $\tau \in Type$. We must show that $env(Q)(x) = \omega$. By the rule [VAR], $env(Q) = env_{\omega}[y \to \tau] \Rightarrow env(Q)(x) = \omega$.

4. Let $Q{=}x^{:\tau}$, where $\tau \in Type$. We must show that $env(Q)(x) = \tau$. By the rule [VAR], $env(Q) = env_{\omega}[x \to \tau] \Rightarrow env(Q)(x) = \tau$.

5. Let $Q{=}eQ'$, where $e \in ExpansionVariable$ and $Q' \in Skeleton$. Assume that only subskeletons $x^{:\tau_1},\dots,x^{:\tau_n}$ have free occurrence in Q' and $\vec{e}'_i{=}E\text{-}Path\left(Q'\left\langle x^{:\tau_i}\right\rangle\right)$, where $i{=}1,\dots,n$, $n \geq 0$. We must show that $env(Q)(x){=}$ $=\vec{e}_1\tau_1 \cap \dots \cap \vec{e}_n\tau_n$. By induction hypothesis, $env(Q')(x){=}\vec{e}'_1\tau_1 \cap \dots \cap \vec{e}'_n\tau_n$. By the rule [E-VAR], $env(Q){=}e\,env(Q') \Rightarrow env(Q)(x){=}e\,env(Q')(x){=}e(\vec{e}'_1\tau_1 \cap$ $\dots \cap \vec{e}'_n\tau_n){=}\vec{e}_1\tau_1 \cap \dots \cap \vec{e}_n\tau_n$.

6. Let $Q{=}(\lambda y.Q')$, where $y \in TermVariable$ and $Q' \in Skeleton$. Assume that only subskeletons $x^{:\tau_1},\dots,x^{:\tau_n}$ have free occurrence in Q' and $\vec{e}'_i{=}E\text{-}Path\left(Q'\left\langle x^{:\tau_i}\right\rangle\right)$, where $i{=}1,\dots,n$, $n \geq 0$. We must show that $env(Q)(x){=}$ $=\vec{e}_1\tau_1 \cap \dots \cap \vec{e}_n\tau_n$. By induction hypothesis, $env(Q')(x){=}\vec{e}'_1\tau_1 \cap \dots \cap \vec{e}'_n\tau_n$. By the rule [ABS], $env(Q) = env(Q')[y \to \omega] \Rightarrow env(Q)(x) = env(Q')(x) =$ $= \vec{e}'_1\tau_1 \cap \dots \cap \vec{e}'_n\tau_n {=} \vec{e}_1\tau_1 \cap \dots \cap \vec{e}_n\tau_n$.

7. Let $Q{=}(\lambda y.Q')$, where $Q' \in Skeleton$. We must show that $env(Q)(x) = \omega$. By the rule [ABS], $env(Q) = env(Q')[x \to \omega] \Rightarrow env(Q)(x) = \omega$.

8. Let $Q{=}(Q_1 \cap Q_2)$, where $Q_1, Q_2 \in Skeleton$. Assume that only subskeletons $x^{:\tau'_1},\dots,x^{:\tau'_m}$ have free occurrence in Q_1 and only subskeletons $x^{:\tau''_1},\dots,x^{:\tau''_k}$ have free occurrence in Q_2, and $\vec{e}'_i{=}E\text{-}Path\left(Q_1\left\langle x^{:\tau'_i}\right\rangle\right)$, $\vec{e}''_j{=}$ $E\text{-}Path\left(Q_2\left\langle x^{:\tau''_j}\right\rangle\right)$, where $i{=}1,\dots,m$, $j{=}1,\dots,k$ and $m,k \geq 0$. We must show that $env(Q)(x) = \vec{e}'_1\tau'_1 \cap$ $\dots \cap \vec{e}'_m\tau'_1 \cap \vec{e}''_1\tau''_1 \cap \dots \cap \vec{e}''_k\tau''_k$. By induction hypothesis, $env(Q_1)(x) = \vec{e}'_1\,\tau'_1 \cap$ $\dots \cap \vec{e}'_m\,\tau'_1$ and $env(Q_2)(x) = \vec{e}''_1\tau''_1 \cap \dots \cap \vec{e}''_k\tau''_k$. By the rule [INT], $env(Q) = env(Q_1)$ $\cap env(Q_2) \Rightarrow env(Q)(x) = env(Q_1)(x) \cap env(Q_2)(x) = \vec{e}'_1\tau'_1 \cap \dots \cap \vec{e}'_m\tau'_1 \cap \vec{e}''_1\tau''_1 \cap \dots \cap \vec{e}''_k\tau''_k$.

9. Let $Q{=}(Q_1Q_2)^{:\tau}$, where $Q_1, Q_2 \in Skeleton$ and $\tau \in Type$. Assume that only subskeletons $x^{:\tau'_1},\dots,x^{:\tau'_m}$ have free occurrence in Q_1, only subskeletons $x^{:\tau''_1},\dots,x^{:\tau''_k}$ have free occurrence in Q_2, $\vec{e}'_i{=}E\text{-}Path\left(Q_1\left\langle x^{:\tau'_i}\right\rangle\right)$ and $\vec{e}''_j{=}$ $E\text{-}Path\left(Q_2\left\langle x^{:\tau''_j}\right\rangle\right)$, where $i{=}1,\dots,m$, $j{=}1,\dots,k$ and $m,k \geq 0$. We must show that $env(Q)(x) = \vec{e}'_1\tau'_1 \cap$ $\dots \cap \vec{e}'_m\tau'_1 \cap \vec{e}''_1\tau''_1 \cap \dots \cap \vec{e}''_k\tau''_k$. By induction hypothesis, $env(Q_1)(x) = \vec{e}'_1\tau'_1 \cap \dots \cap \vec{e}'_m\tau'_1$ and $env(Q_2)(x) = \vec{e}''_1\tau''_1 \cap \dots \cap \vec{e}''_k\tau''_k$. By the rule [APP], $env(Q) = env(Q_1) \cap$ $env(Q_2) \Rightarrow env(Q)(x) = env(Q_1)(x) \cap env(Q_2)(x) = \vec{e}'_1\tau'_1 \cap \dots \cap \vec{e}'_m\tau'_1 \cap \vec{e}''_1\tau''_1 \cap \dots \cap \vec{e}''_k\tau''_k$.

Lemma 2.4. Let $M_1, M_2 \in Term$ and $x \in TermVariable$. If $(A \vdash \tau)$ is a typing of term $((\lambda x.M_1)M_2)$, then it is also a typing of term $M_1[x := M_2]$.

Proof. Because $(A \vdash \tau)$ is a typing of term $((\lambda x.M_1)M_2)$, there exist $Q \in Skeleton$ and $\Delta \in Constraint$ such that the judgement $(((\lambda x.M_1)M_2) \rhd Q):(A \vdash \tau)/\Delta$ is inferable, and Δ is solved. There are three cases to consider:

1. In the last step of inference of the judgement $(((\lambda x.M_1)M_2) \rhd Q):(A \vdash \tau)/\Delta$ the rule [OMEGA] is used. Then $Q{=}\omega^{((\lambda x.M_1)M_2)}$, $(A \vdash \tau) = (env_\omega \vdash \omega)$ and $\Delta{=}\omega$. Because $(env_\omega \vdash \omega)$ is a typing of any term, it is also a typing of term $M_1[x := M_2]$.

2. In the last step of inference of the judgement $(((\lambda x.M_1)M_2) \rhd Q):(A \vdash \tau)/\Delta$ rule [APP] is used. Then $Q{=}(Q_1Q_2)^{:\tau}$ and the judgements $((\lambda x.M_1) \rhd Q_1):(A_1 \vdash \tau_1)/\Delta_1$ and $(M_2 \rhd Q_2):(A_2 \vdash \tau_2)/\Delta_2$ are inferable, and $A{=}A_1 \cap A_2$ and $\tau{=}\tau_1 \cap \tau_2$, and $\Delta{=}\Delta_1 \cap \Delta_2 \cap (\tau_1 \doteq (\tau_2 \to \tau))$ (1). Because Δ is solved, (1) $\Rightarrow \tau_1{=}(\tau_2 \to \tau)$ (2), and constraints Δ_1, Δ_2 are solved. (2) $\Rightarrow$ in the last step of inference of the judgement $((\lambda x.M_1) \rhd Q_1):(A_1 \vdash \tau_1)/\Delta_1$ the rule [ABS] is used. Hence, $Q_1{=}(\lambda x.Q')$ and the judgement $(M_1 \rhd Q'):(A' \vdash \tau')/\Delta'$ is inferable, and $A_1{=}A'[x \to \omega]$ and $\tau_1{=}(\tau_2 \to \tau) = (A'(x) \to \tau')$, and $\Delta_1{=}\Delta'$ (3). (3)$\Rightarrow \tau_2{=}A'(x)$ and $\tau{=}\tau'$, and constraint Δ' is solved (4). Assume that only subskeletons $x^{:\tau'_1},\ldots,x^{:\tau'_n}$, $n \geq 0$, have free occurrence in Q'. By Lemma 2.3, $A'(x) = \vec{e}_1\tau'_1 \cap \ldots \cap \vec{e}_n\tau'_n$ (5), where $\vec{e}_i{=}E\text{-}Path\left(Q'\left\langle x^{:\tau'_i}\right\rangle\right)$, $i{=}1,\ldots,n$. (4),(5)$\Rightarrow$ $\Rightarrow \tau_2{=}\vec{e}_1\tau'_1 \cap \ \ldots \cap \vec{e}_n\tau'_n$ (6). By Proposition 2.1 and (6), $Q_2 \approx \vec{e}_1Q'_1 \cap \ldots \cap \vec{e}_nQ'_n$ and $type(Q'_i) = \tau'_i$, $i{=}1,\ldots,n$ (7). Let us consider the following skeleton: $Q'' = Q'\left\langle x^{:\tau'_1} := Q'_1,\ldots,x^{:\tau'_n} := Q'_n\right\rangle$. Now we will calculate $term(Q''), env(Q''),$ $type(Q'')$ and $constraint(Q'')$.

2a. (1), (3)$\Rightarrow term(Q') = term\left(Q'\left\langle x^{:\tau'_1},\ldots,x^{:\tau'_n}\right\rangle\right)$ and $term(Q_2) = M_2$. (7)$\Rightarrow term(Q_2) = term(\vec{e}_1Q'_1 \cap \ \ldots \cap \vec{e}_nQ'_n) = term(Q') = \ldots = term(Q'_n) = M_2$. Hence by Proposition 2.3, $term(Q'') = M_1[x := M_2]$ (8).

2b. Let us show that $env(Q'')(y) = A(y)$ $\forall y \in TermVariable$ such that $y \neq x$. By Lemma 2.3, $A(y)$ depends on subskeletons of the form $y^{:\tau''}$ that have free occurrence in skeleton $Q{=}((\lambda x.Q')Q_2)$, and their *E-Paths* in that skeleton and $env(Q'')(y)$ depend on subskeletons of the form $y^{:\tau''}$ that have free occurrences in skeleton $Q'' = Q'\left\langle x^{:\tau'_1} := Q'_1,\ldots,x^{:\tau'_n} := Q'_n\right\rangle$ and their *E-Paths* in that skeleton. Due to (7), $Q \approx ((\lambda x.Q')(\vec{e}_1Q'_1 \cap \ldots \cap \vec{e}_nQ'_n))$. Hence, it is easy to see that the both skeletons Q and Q'' have the same free occurrences of subskeletons of the form $y^{:\tau''}$ with the same *E-Path*$\Rightarrow env(Q'')(y) = A(y)$ (9). Now let us show that $env(Q'')(x) =$ $= A(x)$. Assume that there is no subskeleton of form $x^{:\tau''}$ that have free occurrence in skeleton Q_2, otherwise, we would rename the bound variable x in skeleton

$(\lambda x.Q')$. Hence, it is easy to see that both skeletons Q and Q'' have no free occurrences of subskeletons of form $x^{:\tau''} \Rightarrow env(Q'')(x)=A(x)=\omega$ (10). (9), (10) $\Rightarrow env(Q'') = A$ (11).

2c. (3)$\Rightarrow type(Q') = type\left(Q'\left\langle x^{:\tau'_1},\dots,x^{:\tau'_n}\right\rangle\right) = \tau'$. By proposition 2.2, (4) and (7), $type(Q'') = type(Q') = \tau' = \tau \Rightarrow type(Q'') = \tau$ (12).

2d. (1),(3)$\Rightarrow constraint(Q_2)=\Delta_2$ and $constraint(Q')=\Delta'=\Delta_1$. (7)$\Rightarrow$ $\Rightarrow constraint(Q_2)=\vec{e}_1 constraint\ (Q'_1) \cap \dots \cap \vec{e}_n constraint(Q'_n)$ (13). By Proposition 2.4, $constraint(Q'')=constraint(Q') \cap$ $E\text{-}Path\left(Q'\left\langle x^{:\tau'_1}\right\rangle\right) constraint(Q'_1) \cap \dots$ $\cap E\text{-}Path\left(Q'\left\langle x^{:\tau'_n}\right\rangle\right) constraint(Q'_n) = constraint(Q') \cap \vec{e}_1 constraint(Q'_1) \cap \dots \cap \vec{e}_n constraint(Q'_n)$ (14). (13), (14)$\Rightarrow constraint(Q'') = \Delta_1 \cap \Delta_2$ (15). (8), (11), (12), (15) $\Rightarrow$ $\Rightarrow \left(M_1[x := M_2] \triangleright Q'\left\langle x^{:\tau'_1} := Q'_1,\dots,x^{:\tau'_n} := Q'_n\right\rangle\right) : (A \,ђ\, \tau) / \Delta_1 \cap \Delta_2$ is inferable, and constraints Δ_1, Δ_2 are solved. Hence, $(A \,ђ\, \tau)$ is typing of term $M_1[x := M_2]$.

3. In the last step of inference of the judgement $(((\lambda x.M_1)M_2) \triangleright Q) : (A \,ђ\, \tau) / \Delta$ the rules [OMEGA] and [APP] is not used. By Lemma 2.2, $Q \approx \vec{e}_1 Q_1 \cap \dots \cap \vec{e}_n Q_n$, $A=\vec{e}_1 A_1 \cap \dots \cap \vec{e}_n A_n$, $\tau=\vec{e}_1 \tau_1 \cap \dots \cap \vec{e}_n \tau_n$ and $\Delta=\vec{e}_1 \Delta_1 \cap \dots \cap \vec{e}_n \Delta_n$, and the following judgements are inferable: $(((\lambda x.M_1)M_2) \triangleright Q_i) : (A_i \,ђ\, \tau_i) / \Delta$, $i=1,\dots,n$, $n \geq 1$. In our case in the last step of inference of the judgements $(((\lambda x.M_1)M_2) \triangleright Q_i) : (A_i \,ђ\, \tau_i) / \Delta$, $i=1,\dots,n$, the rule [OMEGA] or rule [APP] is used $\Rightarrow$ by 1st and 2nd points of our proof, $(A_i \,ђ\, \tau_i)$ is typing of term $M_1[x := M_2] \Rightarrow$by the rules [INT], [E-VAR], $(\vec{e}_1 A_1 \cap \dots \cap \vec{e}_n A_n \,ђ\, \vec{e}_1 \tau_1 \cap \dots \cap \vec{e}_n \tau_n) = (A \,ђ\, \tau)$ is the typing of term $M_1[x := M_2]$.

Lemma 2.5. Let $M, M' \in Term$ and $M \to_\beta M'$. If $(A \,ђ\, \tau)$ is a typing of term M, then it is also a typing of term M'.

Proof. Let us denote by M_β the β-redex corresponding to the one step of beta reduction $M \to_\beta M'$. We will prove Lemma by induction on the form of term M.

1. Let $M=M_\beta$. By Lemma 2.4, $(A \,ђ\, \tau)$ is typing of term M'.

2. Let $M=(\lambda x.M_1)$, where $M_1 \in Term$ and M_β is subterm of M_1. $(A \,ђ\, \tau)$ is typing of term $(\lambda x.M_1) \Rightarrow \exists Q \in Skeleton$, s.t. the judgement $((\lambda x.M_1) \triangleright Q) : (A \,ђ\, \tau) / \Delta$ is inferable and Δ is solved. By Lemma 2.2, $Q \approx \vec{e}_1 Q_1 \cap \dots \cap \vec{e}_n Q_n$, $A=\vec{e}_1 A_1 \cap \dots \cap \vec{e}_n A_n$, $\tau=\vec{e}_1 \tau_1 \cap \dots \cap \vec{e}_n \tau_n$ and $\Delta=\vec{e}_1 \Delta_1 \cap \dots \cap \vec{e}_n \Delta_n$, and the following judgements are inferable: $((\lambda x.M_1) \triangleright Q_i) : (A_i \,ђ\, \tau_i) / \Delta_i$, $i=1,\dots,n$, $n \geq 1$. In our case in the last step of inference of the judgements $((\lambda x.M_1) \triangleright Q_i) : (A_i \,ђ\, \tau_i) / \Delta_i$, $i=1,\dots,n$, the rule [OMEGA] or rule [ABS] is used. Let us show that $(A_i \,ђ\, \tau_i)$ is a typing of term M'. In that case $(A \,ђ\, \tau) = (\vec{e}_1 A_1 \cap \dots \cap \vec{e}_n A_n \,ђ\, \vec{e}_1 \tau_1 \cap \dots \cap \vec{e}_n \tau_n)$ will also be typing of term M' (using the rules [E-VAR] and [INT]), which we need to prove.

$M \to_\beta M' \Rightarrow \exists M_1' \in Term$ such that $M' = (\lambda x.M_1')$ and $M_1 \to_\beta M_1'$. There are two cases to consider:

2a. In the last step of inference of the judgement $((\lambda x.M_1) \rhd Q_i) : (A_i \text{ ђ } \tau_i)/\Delta_i$ the rule [OMEGA] is used. Then $(A_i \text{ ђ } \tau_i) = = (env_\omega \text{ ђ } \omega)$. Because $(env_\omega \text{ ђ } \omega)$ is a typing of any term, it is also a typing of term M'.

2b. In the last step of inference of the judgement $((\lambda x.M_1) \rhd Q_i) : (A_i \text{ ђ } \tau_i)/\Delta_i$ the rule [ABS] is used. Then $\exists \tau_i' \in Type$ and environment A_i' such that $(A_i' \text{ ђ } \tau_i')$ is a typing of term M_1, $A_i = A_i'[x \to \omega]$ and $\tau_i = (A_i'(x) \to \tau_i')$. By induction hypothesis, $(A_i' \text{ ђ } \tau_i')$ is a typing of term $M_1' \Rightarrow (A_i \text{ ђ } \tau_i) = (A_i'[x \to \omega] \text{ ђ } (A_i'(x) \to \tau_i'))$ is a typing of term M' (using the rule [ABS]).

3. Let $M = (M_1 M_2)$, where $M_1, M_2 \in Term$ and M_β is a subterm of M_1. $(A \text{ ђ } \tau)$ is typing of term $(M_1 M_2) \Rightarrow \exists Q \in Skeleton$, s.t. the judgement $((M_1 M_2) \rhd Q) : (A \text{ ђ } \tau)/\Delta$ is inferable and Δ is solved. By Lemma 2.2, $Q \approx \vec{e}_1 Q_1 \cap \ldots \cap \vec{e}_n Q_n$, $A = \vec{e}_1 A_1 \cap \ldots \cap \vec{e}_n A_n$, $\tau = \vec{e}_1 \tau_1 \cap \ldots \cap \vec{e}_n \tau_n$ and $\Delta = \vec{e}_1 \Delta_1 \cap \ldots \cap \vec{e}_n \Delta_n$, and the following judgements are inferable: $((M_1 M_2) \rhd Q_i) : (A_i \text{ ђ } \tau_i)/\Delta_i$, $i = 1, \ldots, n$, $n \geq 1$. In our case in the last step of inference of the judgements $((M_1 M_2) \rhd Q_i) : (A_i \text{ ђ } \tau_i)/\Delta_i$, $i = 1, \ldots, n$, the rule [OMEGA] or rule [APP] is used. Let us show that $(A_i \text{ ђ } \tau_i)$ is a typing of term M'. In that case $(A \text{ ђ } \tau) = (\vec{e}_1 A_1 \cap \ldots \cap \vec{e}_n A_n \text{ ђ } \vec{e}_1 \tau_1 \cap \ldots \cap \vec{e}_n \tau_n)$ will also be typing of term M' (using the rules [E-VAR] and [INT]), which we need to prove. $M \to_\beta M' \Rightarrow \exists M_1' \in Term$ such that $M' = (M_1' M_2)$ and $M_1 \to_\beta M_1'$. There are two cases to consider:

3a. In the last step of inference of the judgement $((M_1 M_2) \rhd Q_i) : (A_i \text{ ђ } \tau_i)/\Delta_i$ the rule [OMEGA] is used. Then $(A_i \text{ ђ } \tau_i) = = (env_\omega \text{ ђ } \omega)$. Because $(env_\omega \text{ ђ } \omega)$ is a typing of any term, it is also a typing of term M'.

3b. In the last step of inference of the judgement $((M_1 M_2) \rhd Q_i) : (A_i \text{ ђ } \tau_i)/\Delta_i$ the rule [APP] is used. Then $\exists \tau_i^1, \tau_i^2 \in Type$ and environments A_i^1, A_i^2 such that $(A_i^1 \text{ ђ } \tau_i^1)$ is a typing of term M_1, and $(A_i^2 \text{ ђ } \tau_i^2)$ is a typing of term M_2 and $A_i = A_i^1 \cap A_i^2$, $\tau_i^1 = (\tau_i^2 \to \tau_i)$. By induction hypothesis, $(A_i^1 \text{ ђ } \tau_i^1)$ is typing of term $M_1' \Rightarrow (A_i \text{ ђ } \tau_i) = (A_i^1 \cap A_i^2 \text{ ђ } \tau_i)$, is a typing of term $M' = (M_1' M_2)$ (using the rule [APP]).

4. Let $M = (M_1 M_2)$, where $M_1, M_2 \in Term$ and M_β is a subterm of M_2. The proof is similar to the proof of 3rd point.

2.2 Type Inference Algorithm and Principal Typing of Term. In this subsection we will prove that in case of success the type inference algorithm returns the principal typing of term. First of all let us consider terms that are in β-normal form.

Lemma 2.6. Let $M \in Term$ and $M \in \beta\text{-}NF$. Then if the judgement

$(M \rhd Q') : (A' \vdash \tau') / \Delta'$ is inferable, constraint Δ' is solved, and the rule [OMEGA] is not used during the inference of that judgement, then:

1. $(A \vdash \tau) = Typify(M)$, i.e. the type inference algorithm succeeds for input M.

2. $\exists E \in Expansion$, s.t. $A' = [E]A$ and $\tau' = [E]\tau$.

3. If in the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ one of the rules [VAR], [CONST], [ABS] or [APP] is used, then the expansion E is a subtitution of the following form: $E = \{a_0 := \tau_{a_0}\}$ in case of [VAR]; $E = \varepsilon$ in case of [CONST]; $E = \{e_0 := E_{e_0}\}$ in case of [ABS]; $E = \{a_0 := \tau_{a_0}, e_1 := E_{e_1}, e_2 := E_{e_2}\}$ in case of [APP].

Proof. By induction on the form of term M.

1. Let $M = x$, where $x \in TermVariable$. Then by the type inference algorithm definition, $A = env_\omega[x \to a_0]$ and $\tau = a_0$, which is the proof of first part of Lemma's statement. There are two cases to consider:

1a. In the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ the rule [VAR] is used. Then $Q' = x^{:\tau'}$, $A' = env_\omega[x \to \tau']$ and $\Delta' = \omega$. Let $E = \{a_0 := \tau'\} \Rightarrow A = [E]A$ and $\tau' = [E]\tau$, which we need to prove.

1b. In the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ the rule [E-VAR] or rule [INT] is used. By Lemma 2.2, $Q' \approx \vec{e}_1 Q'_1 \cap \ldots \cap \vec{e}_n Q'_n$, $A' = \vec{e}_1 A'_1 \cap \ldots \cap \vec{e}_n A'_n$, $\tau' = \vec{e}_1 \tau'_1 \cap \ldots \cap \vec{e}_n \tau'_n$ and $\Delta' = \vec{e}_1 \Delta'_1 \cap \ldots \cap \vec{e}_n \Delta'_n$, and the following judgements are inferable: $(M \rhd Q'_i) : (A'_i \vdash \tau'_i) / \Delta'_i$, $i = 1, \ldots, n$, $n \geq 1$. It is easy to see that the condition of Lemma holds also for the judgements $(M \rhd Q'_i) : (A'_i \vdash \tau'_i) / \Delta'_i$, $i = 1, \ldots, n$. In our case in the last step of inference of the judgements $(M \rhd Q'_i) : (A'_i \vdash \tau'_i) / \Delta'_i$, $i = 1, \ldots, n$, the rule [VAR] is used. Hence, by point 1a, $\exists E_1, \ldots, E_n \in Expansion$, s.t. $A'_i = [E_i]A_i$, $\tau'_i = [E_i]\tau_i$, $i = 1, \ldots, n$ (1).

Let $E = \vec{e}_1 E_1 \cap \ldots \cap \vec{e}_n E_n$. By (1), $[E]A = \vec{e}_1[E_1]A \cap \ldots \cap \vec{e}_n[E_n]A = \vec{e}_1 A'_1 \cap \ldots \cap \vec{e}_n A'_n = A'$ and $[E]\tau = \vec{e}_1[E_1]\tau \cap \ldots \cap \vec{e}_n[E_n]\tau = \vec{e}_1 \tau'_1 \cap \ldots \cap \vec{e}_n \tau'_n = \tau'$, which we need to prove.

2. Let $M = c$, where $c \in Constant$. Then by the type inference algorithm definition [1], $A = env_\omega$ and $\tau = \Sigma(c)$, which is the proof of first part of Lemma's statement. There are two cases to consider:

2a. In the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ the rule [CONST] is used. Then $Q' = x^{:\Sigma(c)}$, $A' = env_\omega$, $\tau' = \Sigma(c)$ and $\Delta' = \omega$. Let $E = \varepsilon \Rightarrow A' = [E]A$ and $\tau' = [E]\tau$, which is to be proved.

2b. In the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ the rule [E-VAR] or the rule [INT] is used. The proof is similar to the proof of point 1b.

3. Let $M = (\lambda x.M_1)$, where $x \in TermVariable$ and $M_1 \in Term$. Let $P_1 = initial(M_1)$ and $P = initial(M) = (\lambda x.e_0 P_1) \Rightarrow constraint(P) = e_0 constraint(P_1)$. Hence, by definition of the unification algorithm [1] and unification rules $unify_\beta$, $unify_x$, $unify_c$ [1], $\sigma = Unify(constraint(P)) \Leftrightarrow \sigma_1 = Unify\ (constraint(P_1))$, where

$\sigma = e_0 / \sigma_1$ (2). Due to (2), due to the definition of the type inference algorithm [1] and definitions of algorithms *env* and *type* [1], $(A \vdash \tau) = Typify(M) \Leftrightarrow (A_1 \vdash \tau_1) = Typify(M_1)$, where $A = e_0 A_1[x \to \omega]$ and $\tau = (e_0 A_1(x) \to e_0 \tau_1)$ (3). There are two cases to consider:

3a. In the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ the rule [ABS] is used. Then $Q' = (\lambda x.Q_1')$, $A' = A_1'[x \to \omega]$, $\tau' = (A_1'(x) \to \tau_1')$ and the judgement $(M_1 \rhd Q_1') : (A_1' \vdash \tau_1') / \Delta_1'$ is inferable (4). It is easy to see that the condition of Lemma holds also for the judgement $(M_1 \rhd Q_1') : (A_1' \vdash \tau_1') / \Delta_1'$. Hence, by the induction hypothesis, $Typify(M_1)$ succeeds and $\exists E_1 \in Expansion$, s.t. $A_1' = [E_1] A_1$ and $\tau_1' = [E_1] \tau_1$ (5). By (3) and (5), the first part of Lemma's statement is proved. Let $E = \{e_0 := E_1\}$. By (3), (4) and (5), $[E]A = [\{e_0 := E_1\}] e_0 A_1 [x \to \omega] = [E_1]$, $A_1[x \to \omega] = A_1' [x \to \omega] = A'$ and $[E]\tau = [\{e_0 := E_1\}](e_0 A_1(x) \to e_0 \tau_1) = ([E_1] A_1(x) \to [E_1]\tau_1) = (A_1'(x) \to \tau_1') = \tau'$, which is to be proved.

3b. In the last step of the inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ the rule [E-VAR] or the rule [INT] is used. The proof is similiar to the proof of point 1b.

4. Let $M = (M_1 M_2)$. We will not present the proof of this case.

Lemma 2.7. Let $M \in Term$ and $M \in \beta$*-NF*. Then if $(A' \vdash \tau')$ is a typing of term M and $(A \vdash \tau) = Typify(M)$, then:

1. $\exists E \in Expansion$, s.t. $A' = [E]A$ and $\tau' = [E]\tau$.

2. If in the last step of inference of the judgement $(M \rhd Q') : (A' \vdash \tau') / \Delta'$ one of the rules [VAR], [CONST], [ABS] or [APP] is used, then $E = \omega$ or E is a subtitution of the following form: $E = \{a_0 := \tau_{a_0}\}$ in case of [VAR]; $E = \varepsilon$ in case of [CONST]; $E = \{e_0 := E_{e_0}\}$ in case of [ABS]; $E = \{a_0 := \tau_{a_0}, e_1 := E_{e_1}, e_2 := E_{e_2}\}$ in case of [APP].

Proof. The proof is very similiar to the proof of Lemma 2.6.

Now let us present the main theorem on the principal typing of a term.

Theorem 2.1. Let $M \in Term$ and $M \in Term$, s.t. $M \twoheadrightarrow_\beta M'$ and $M' \in \beta$*-NF*.

1. If there exists a typing of term M' such that during the inference of the corresponding judgement the rule [OMEGA] is not used, then $Typify(M)$ succeeds.

2. If $(A \vdash \tau) = Typify(M)$, then $(A \vdash \tau)$ is the principal typing of term M.

Proof. Let $\Delta = constraint(initial(M))$ and $\Delta' = constraint(initial(M'))$. By Lemma 2.12 of [1], $Unify(\Delta) = [Unify(\Delta')]\sigma$, where $\sigma = [\sigma_m] \ldots [\sigma_2]\sigma_1$ (1), and substitutions $\sigma_1, \ldots, \sigma_m$, $m \ge 0$, are created by the rule $unify_\beta$ during the work of the unification algorithm for input Δ. Hence, by definition of the type inference algorithm, both $Typify(M)$ and $Typify(M')$ are simultaneously executed or fail.

1. Because there exists a typing of term M' such that during the inference of the corresponding judgement the rule [OMEGA] is not used, then

due to Lemma 2.6, $Typify(M')$ succeeds $\Rightarrow Typify(M)$ succeeds as well, which is proves the first part of the Theorem.

2. We have that $(A \vdash \tau) = Typify(M)$. By Lemma 2.6 of [1], each application of the rule $unify_\beta$ corresponds to one step of β-reduction. Hence, $\exists M_1,\ldots,M_m \in Term$, such that $M = M_0 \to_\beta M_1 \to_\beta \ldots \to_\beta M_m = M'$, $A_i = [\sigma_i] A_{i-1}$, $\tau_i = [\sigma_i] \tau_{i-1}$, where $A_j = env(initial(M_j))$ and $\tau_j = type(initial\ (M_j))$, $i = 1,\ldots,m$, $j = 0,\ldots,m$ (2). (1),(2) $\Rightarrow$ $A_m = env(initial(M')) = [\sigma] A_0 = [\sigma] A$ and $\tau_m = type(initial\ (M')) = [\sigma]\tau_0 = [\sigma]\tau$ (3). By (1) and (3), $(A \vdash \tau) = Typify(M) = ([[Unify(\Delta')]\sigma] A_0 \vdash [[Unify(\Delta')]\sigma]\tau_0) = ([Unfiy(\Delta')][\sigma]A_0 \vdash [Unfiy(\Delta')][\sigma]\tau_0) = ([Unify(\Delta)]A_0 \vdash [Unify(\Delta)]\tau_0) = ([Unfiy(\Delta')]A_m \vdash [Unfiy(\Delta')]\tau_m) = Typify(M')$ (4). Let $(A' \vdash \tau')$ is a typing of term M. By Lemma 2.5, $(A' \vdash \tau')$ is also a typing of term $M' \in \beta\text{-}NF$. Hence by (4) and Lemma 2.7, $\exists E \in Expansion$, s.t. $A' = [E]A$ and $\tau' = [E]\tau$, which means that $Typify(M)$ is the principal typing of term M.

Remark 2.1. The type inference algorithm returns the principal typing of a term that has a β-normal form, except for the situations, when it is impossible to type a β-normal form of the given term without using the rule [OMEGA]. For terms that do not have a β-normal form the type inference algorithm never returns.

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Physics

VACUUM FLUCTUATIONS IN COSMOLOGICAL MODELS WITH COMPACTIFIED DIMENSIONS

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We study quantum effects of scalar fields in cosmological models of Friedman–Robertson–Walker with a power-low scale factor and spatial topology $R^p \times (S^1)^q$. Recurrent formulae are obtained for positive-frequency Wightman function, vacuum expectation values of the field squared and energy density.

Keywords**:** cosmology, vacuum fluctuations, Kaluza–Klein theories.

1. Introduction. It is expected that in string theory the most natural topology for the universe is that of a flat compact three-manifold [1]. In inflationary scenario universes with compact spatial dimensions, under certain conditions, should be considered a rule rather than an exception [2]. The models of a compact universe with nontrivial topology may play an important role by providing proper initial conditions for inflation (on the cosmological consequences of the non-trivial topology and observational bounds on the size of compactified dimensions see, for example, [3]). The quantum creation of the universe with toroidal spatial topology is discussed in [4–8] within the framework of various supergravity theories. Vacuum expectation values of the field squared has been considered in the previous work [9].

The compactification of spatial dimensions leads to the modification of the spectrum of vacuum fluctuations and, as a result, to Casimir-type contributions to the vacuum expectation values of physical observables (on the topological Casimir effect and its role in cosmology see [10] and references therein). The effects of the toroidal compactification of spatial dimensions in dS space-time on the properties of quantum vacuum for a scalar field with general curvature coupling parameter are investigated in [11]. The one-loop quantum effects for a fermionic field on background of dS space-time with spatial topology $R^p \times (S^1)^q$ are studied in [12]. In the present paper we investigate the effect of the compactification of one of spatial dimensions in the Friedmann–Robertson–Walker (FRW) cosmological models with power-law scale factor. For a scalar field with general curvature coupling parameter we evaluate the vacuum energy density.

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In this paper we consider the Wightman function for the background FRW space-time with topology $R^p \times (S^1)^q$. We decompose this function in two parts: the first one is the corresponding function for the uncompactified FRW space-time, and the second one is induced by the compactness of the spatial dimensions. We use the Wightman function for the evaluation of the vacuum energy density. As the part corresponding to the uncompactified FRW space-time is well-investigated in literature, we are mainly concerned with the topological part. The asymptotic behavior of the latter is investigated in detail in early and late stages of the cosmological evolution.

2. Wightman Function in FRW Space-time With Compact Spatial Dimensions. We consider a quantum scalar field with curvature coupling parameter ξ on background of the $(D+1)$-dimensional FRW space-time. The field equation has the form

$$\left(\nabla_l \nabla^l + m^2 + \xi R\right)\varphi = 0\,, \tag{1}$$

where ∇_l is the covariant derivative operator. The values of the curvature coupling parameter $\xi = 0$ and $\xi = \xi_D \equiv (D-1)/4D$ correspond to the most important special cases of minimally and conformally coupled fields. We will write the corresponding line element in the form most appropriate for cosmological applications:

$$ds^2 = dt^2 - a^2(t)\sum_{i=1}^{D}(dz^i)^2, \quad a(t) = \alpha t^c\,. \tag{2}$$

For the further discussion, in addition to the synchronous time coordinate t it is convenient to introduce the conformal time τ in accordance with

$$t = \left[\alpha(1-c)\tau\right]^{1/(1-c)}, \quad a(t) = \alpha\left[\alpha(1-c)\tau\right]^{c/(1-c)}. \tag{3}$$

Note that $0 \le \tau < \infty$ for $0 < c < 1$, and $-\infty < \tau \le 0$ for $c > 1$. In terms of this coordinate the line element takes conformally flat form:

$$ds^2 = \Omega^2(\tau)\left[d\tau^2 - \sum_{i=1}^{D}(dz^i)^2\right], \quad \Omega^2(\tau) = \alpha^2\left[\alpha(1-c)\tau\right]^{2c/(1-c)}, \tag{4}$$

and the corresponding Ricci scalar has the form

$$R = \frac{Dc\left[(D+1)c-2\right]}{\left[\alpha(1-c)\tau\right]^{2/(1-c)}}. \tag{5}$$

We will assume that the spatial coordinates z^l, $l = D_1+1,\ldots,D$, are compactified to S^1: $0 \le z^l \le L_l$, and for the other coordinates we have $-\infty < z^l < +\infty$, $l = 1,\ldots,D_1$. Hence, we consider the spatial topology $R^{D_1} \times (S^1)^{D_2}$. For $D_1 = 0$ as a special case we obtain the toroidally compactified FRW space-time. The results obtained here can be used to describe two types of models. For the first one $D = 4$, and it corresponds to the universe with Kaluza–Klein type single extra dimension. For the second model $D = 3$, and the results given below describe how the properties of the universe are changed by one-loop quantum effects, induced by the compactness of a single spatial dimension.

For a scalar field with periodic boundary condition one has $\varphi(\eta, \mathbf{z}_{D_1}, \mathbf{z}_{D_2} + \mathbf{L}_{D_2}) = \varphi(\eta, \mathbf{z}_{D_1}, \mathbf{z}_{D_2})$, where $\eta = |\tau|$, $\mathbf{z}_{D_1} = (z^1,\ldots,z^{D_1})$, $\mathbf{z}_{D_2} = (z^{D_1+1},\ldots,z^D)$,

$\boldsymbol{L}_{D_2}=(L_{D_1+1},\ldots,L_D)$. In this paper we are interested in the effects of non-trivial topology on the vacuum expectation value (VEV) of the energy density. This VEV is obtained from the corresponding Wightman function in the coincidence limit of the arguments.

To evaluate the Wightman function we employ the mode-sum formula

$$G^{+}_{p,q}(x,x')=\langle 0|\varphi(x)\varphi(x')|0\rangle=\sum_{\sigma}\varphi_\sigma(x)\varphi^{*}_\sigma(x'), \tag{6}$$

where $\{\varphi_\sigma(x),\varphi^{*}_\sigma(x)\}$ is a complete set of positive and negative frequency solutions to the classical field equation and satisfying the periodicity condition along the compactified dimension. The collective index σ specifies these solutions. For the problem under consideration and in the case of a massless field the eigenfunctions have the form [9]

$$\varphi_\sigma(x)=C_\sigma\eta^{b}H^{(2)}_{\nu}(k\eta)e^{i\boldsymbol{k}_p\cdot\boldsymbol{z}_p+i\boldsymbol{k}_q\cdot\boldsymbol{z}_q}, \tag{7}$$

with the notations

$$\boldsymbol{k}_{D_1}=(k_1,\ldots,k_{D_1}),\quad \boldsymbol{k}_{D_2}=(k_{D_1+1},\ldots,k_D),\quad k=\sqrt{\boldsymbol{k}^2_{D_1}+\boldsymbol{k}^2_{D_2}},$$
$$k_l=2\pi n_l/L_l,\quad n_l=0,\pm1,\pm2,\ldots,\quad l=D_1+1,\ldots,D,$$
$$b=\frac{1}{2}\cdot\frac{cD-1}{c-1}, \tag{8}$$

and the order of the Hankel function $H^{(2)}_\nu(x)$ is defined by the relation

$$\nu=\frac{1}{2|1-c|}\left\{(cD-1)^2-4\xi Dc\left[(D+1)c-2\right]\right\}^{1/2}. \tag{9}$$

Note that for a conformally coupled field $\nu=1/2$.

The coefficient C_σ with $\sigma=(\boldsymbol{k}_{D_1},n_{D_1+1},\ldots,n_D)$ is found from the orthonormalization condition:

$$-i\int\sqrt{|g|}g^{00}\left[\varphi_\sigma(x)\partial_\tau\varphi^{*}_{\sigma'}(x)-\varphi^{*}_{\sigma'}(x)\partial_\tau\varphi_\sigma(x)\right]d^Dx=\delta_{\sigma\sigma'}, \tag{10}$$

where the integration goes over the spatial hypersurface $\tau=const$, and $\delta_{\sigma\sigma'}$ is understood as the Kronecker delta for discrete indices and as the Dirac delta-function for continuous ones. This leads to the result

$$|C_\sigma|^2=\frac{\left[\alpha|1-c|\right]^{(D-1)c/(c-1)}e^{-(\nu-\nu^{*})\pi i/2}}{2^{p+2}\pi^{p-1}\alpha^{D-1}V_q},\quad V_q=L_{p+1}\cdots L_D. \tag{11}$$

Substituting the eigenfunctions (7) with the normalization coefficient (11) into the mode-sum formula for the Wightman function, one finds

$$G^{+}_{p,q}(x,x')=\frac{A(\eta\eta')^{b}}{2^{p}\pi^{p+1}V_q}\times$$
$$\times\int e^{i\boldsymbol{k}_p\cdot\Delta\boldsymbol{z}_q}\sum_{\boldsymbol{n}_q=-\infty}^{\infty}e^{i\boldsymbol{k}_q\cdot\Delta\boldsymbol{z}_q}K_\nu(k\eta e^{\mathrm{sign}(\tau)\pi i/2})K_\nu(k\eta' e^{-\mathrm{sign}(\tau)\pi i/2})\,d\boldsymbol{k}_p, \tag{12}$$

where $\Delta z^l=z^l-z'^l$, $k=\sqrt{\boldsymbol{k}^2_p+\boldsymbol{k}^2_{\boldsymbol{n}_q-1}+k^2_{p+1}}$ and

$$A=\alpha^{1-D}\left[\alpha|1-c|\right]^{(D-1)c/(c-1)}. \tag{13}$$

In (12) we wrote the Hankel function in terms of the MacDonald function $K_\nu(z)$. It can be seen that after the application of the Abel–Plana summation formula [10, 13] to the series over n_{D_1+1}, the following recurrence formula is obtained:

$$G^{+}_{p,q}(x,x') = G^{+}_{p+1,q-1}(x,x') + \Delta_{p+1}G^{+}_{p,q}(x,x'), \tag{14}$$

where the first term on the right is the Wightman functions in the FRW space-time with $p+1$ uncompactified and $q-1$ toroidally compactified dimensions, and the second term is induced by the compactness of the z^{p+1}-direction and is given by the formula

$$\Delta_{p+1}G^{+}_{p,q}(x,x') = \frac{A(\eta\eta')^{b}}{(2\pi)^{p+1}V_{q-1}}\int e^{i\boldsymbol{k}_p\cdot\Delta\boldsymbol{z}_p}\sum_{\boldsymbol{n}_{q-1}=-\infty}^{\infty} e^{i\boldsymbol{k}_{q-1}\cdot\Delta\boldsymbol{z}_{q-1}}d\boldsymbol{k}_p \times$$
$$\times\int_0^\infty \frac{y\cosh\left(\Delta z^{p+1}\sqrt{y^2+\boldsymbol{k}_p^2+\boldsymbol{k}^2_{\boldsymbol{n}_q-1}}\right)}{\sqrt{y^2+\boldsymbol{k}_p^2+\boldsymbol{k}^2_{\boldsymbol{n}_q-1}}\left(e^{L_{p+1}\sqrt{y^2+\boldsymbol{k}_p^2+\boldsymbol{k}^2_{\boldsymbol{n}_q-1}}}-1\right)}\times \tag{15}$$
$$\times\left\{K_\nu(\eta y)\left[I_{-\nu}(\eta' y)+I_\nu(\eta' y)\right]+\left[I_\nu(\eta y)+I_{-\nu}(\eta y)\right]K_\nu(\eta' y)\right\}dy,$$

where $V_{q-1} = L_{p+2}\cdots L_D$ and the notation

$$\boldsymbol{k}^2_{\boldsymbol{n}_q-1} = \sum_{l=p+2}^{D}\left(2\pi n_l/L_l\right)^2 \tag{16}$$

is introduced. Note that in this formula the integration with respect to the angular part of $\boldsymbol{k}_p$ can be done explicitly.

3. Vacuum Energy Density. Now we turn to the investigation of the VEV for the vacuum energy density. Using the Wightman function we can evaluate this VEV by making use of the formula [14]

$$\langle 0|T_{00}|0\rangle = \lim_{x'\to x}\partial_0\partial'_0 G^{+}(x,x') + \left[\left(\xi-\frac{1}{4}\right)g_{00}\nabla_l\nabla^l - \xi\nabla_0\nabla_0 - \xi R_{00}\right]\langle 0|\varphi^2|0\rangle, \tag{17}$$

where R_{ik} is the Ricci tensor for the FRW space-time with the 00-component

$$R_0^0 = \frac{Dc\Omega^{-2}}{(1-c)\tau^2},$$

As in the case of the Wightman function, the renormalized VEV of the energy density is presented as the sum

$$\langle T_0^0\rangle_{p,q} = \langle T_0^0\rangle_{p+1,q-1} + \Delta_{p+1}\langle T_0^0\rangle_{p,q}, \tag{18}$$

where the part due to the compactness of the z^{p+1}-direction is given by the expressions

$$\Delta_{p+1}\langle T_0^0\rangle_{p,q} = \frac{2^{(1-p)/2}A\Omega^{-2}}{\pi^{(p+3)/2}V_{q-1}}\sum_{\boldsymbol{n}_{q-1}=-\infty}^{\infty}\int_0^\infty y^{3-2b}\sum_{n=1}^{\infty}\frac{f_{(p-1)/2}\left(nL_{p+1}\sqrt{y^2+\boldsymbol{k}^2_{\boldsymbol{n}_q-1}}\right)}{\left(nL_{p+1}\right)^{p-1}}F^{(0)}(\eta y)\,dy, \tag{19}$$

with the notation

$$F^{(0)}(z)=\frac{1}{2}\tilde{I}'_{\nu}(z)\tilde{K}'_{\nu}(z)+\frac{D\xi c}{z(1-c)}\left(\tilde{I}(z)\tilde{K}(z)\right)'-\frac{1}{2}\left[1+D\xi c\frac{4-(D+3)c}{z^2(1-c)^2}\right]\tilde{I}_{\nu}(z)\tilde{K}_{\nu}(z). \quad (20)$$

In (20) we have defined the functions

$$f_{\nu}(x)=x^{\nu}K_{\nu}(x), \quad \tilde{K}_{\nu}(z)=z^bK_{\nu}(z), \quad \tilde{I}_{\nu}(z)=z^b\left[I_{\nu}(z)+I_{-\nu}(z)\right]. \quad (21)$$

After the recurring application of formula (18), the vacuum energy density in FRW model with spatial topology $R^p\times\left(S^1\right)^q$ is presented in the form

$$\left\langle T_0^0\right\rangle_{p,q}=\left\langle T_0^0\right\rangle_{FRW}+\left\langle T_0^0\right\rangle_c ,$$

where $\left\langle T_0^0\right\rangle_{FRW}$ is the corresponding quantity for uncompactified FRW space-time and the part

$$\left\langle T_0^0\right\rangle_c=\sum_{l=1}^{q}\varDelta_{D-l+1}\left\langle T_0^0\right\rangle_{D-l,l} \quad (22)$$

is induced by the toroidal compactification of the q-dimensional subspace.

For a conformally coupled massless scalar field one has $\nu=1/2$ and $\left[I_{-\nu}(x)+I_{\nu}(x)\right]K_{\nu}(x)=1/x$. For the function $F^{(0)}(z)$ we have:

$$F^{(0)}(z)=z^{2b-3}\left[\frac{c(1-D)}{2(1-c)}-z^2\right]. \quad (23)$$

Using the formula

$$\int_0^{\infty}f_{(p-1)/2}\left(a\sqrt{z^2+b^2}\right)dz=\sqrt{\frac{\pi}{2}}\cdot\frac{f_{p/2}(ab)}{a}, \quad (24)$$

for the case of a conformally coupled field we find

$$\Delta_{p+1}\left\langle T_0^0\right\rangle_{p,q}=-\frac{2\Omega^{-D-1}}{(2\pi)^{p/2+1}V_{q-1}}\sum_{\mathbf{n}_{q-1}=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{f_{p/2+1}\left(nL_{p+1}k_{\mathbf{n}_{q-1}}\right)}{\left(nL_{p+1}\right)^{p+2}}. \quad (25)$$

Formula (25) could also be obtained from the corresponding result in $(D+1)$-dimensional Minkowski space-time with spatial topology $R^p\times\left(S^1\right)^q$, taking into account that two problems are conformally related: $\Delta_{p+1}\left\langle T_0^0\right\rangle_{p,q}=\Omega^{-D-1}\Delta_{p+1}\left\langle T_0^0\right\rangle_{p,q}^{(M)}$. A similar formula takes place for the total topological part.

The general formulas for the topological part in the VEV of the energy density are simplified in the asymptotic regions of the parameters. For small values of the ratio L_{p+1}/η we can see that to the leading order $\Delta_{p+1}\left\langle T_0^0\right\rangle_{p,q}$ coincides with the corresponding result for a conformally coupled massless field given by formula (25). Note that in terms of the synchronous time coordinate we have $L_{p+1}/\eta=\alpha|1-c|L_{p+1}t^{c-1}$ and, hence, $\Delta_{p+1}\left\langle T_0^0\right\rangle_{p,q}\propto t^{-c(D+1)}$. Hence, the limit under consideration corresponds to the early stages of the cosmological expansion ($t\to 0$) in the case $c>1$ and to the late stages ($t\to+\infty$) in the case $c<1$.

For large values of the ratio L_{p+1}/η and in the case of real ν, using the asymptotic formulae for the modified Bessel functions for small values of the argument, to the leading order one has

$$F^{(0)}(z) \approx \frac{2^{2\nu-1} D\Gamma(\nu)}{\Gamma(1-\nu) z^{2\nu}} \left[D/2 - \nu + 2\xi(2\nu - D - 1) \right]. \tag{26}$$

From formula (19) we find

$$\Delta_{p+1} \left\langle T_0^0 \right\rangle_{p,q} \approx \frac{2^{q+\nu-1} D\left[D/2 - \nu + 2\xi(2\nu - D - 1) \right]}{(2\pi)^{(p+3)/2} V_q L_{p+1}^{p=2\nu} \alpha^{D+1} \eta^{2\nu-D}} \Gamma(\nu) \sum_{n,\mathbf{n}_{q-1}} \frac{f_{(p+1)/2-\nu}\left(nL_{p+1} k_{\mathbf{n}_{q-1}} \right)}{n^{(p+1)/2-\nu}}. \tag{27}$$

In terms of the synchronous time coordinate one has $\Delta_{p+1} \left\langle T_0^0 \right\rangle_{p,q} \infty\, t^{(D-2\nu)(1-c)}$. For small values η/L_{p+1} and imaginary ν, in a similar way as in the previous case, in terms of the synchronous time coordinate we find

$$\Delta_{p+1} \left\langle T_0^0 \right\rangle_{p,q} \approx \frac{2^{q-1} D |1-c|^{-D} B_0}{(2\pi)^{(p+3)/2} V_q L_{p+1}^{p}} \frac{t^{D(1-c)}}{\alpha^{2D+1}} \sin\left[2|\nu| t/\alpha + |\nu| \ln\left(L_{p+1}/\alpha \right) + \varphi_0 \right], \tag{28}$$

where B_0 and φ_0 are defined by the relation

$$B_0 e^{i\varphi_0} = 2^{i|\nu|} \frac{|\nu|(1/2 - 2\xi) + i\left[D/4 - (D+1)\xi \right]}{\Gamma(1 - i|\nu|)} \sum_{\mathbf{n}_{q-1}=-\infty}^{\infty} \sum_{n=\infty}^{\infty} \frac{f_{(p+1)/2 - i|\nu|}\left(nL_{p+1} k_{\mathbf{n}_{q-1}} \right)}{n^{p+1-2i|\nu|}}. \tag{29}$$

This limit corresponds to the late stages of the cosmological expansion $(t \to +\infty)$ in the case $c > 1$ and to the early stages $(t \to 0)$ in the case $c < 1$.

4. Conclusion. Compactified spatial dimensions appear in various physical models, including Kaluza–Klein type theories, supergravity, string theory and cosmology. In this paper we investigate the quantum vacuum effects in FRW space-time induced by non-trivial topology of spatial dimensions. We consider a scalar field with general curvature coupling parameter, satisfying the periodic boundary condition along the compactified dimensions. Among the most important characteristics of the vacuum are the VEV of the energy density. Though the corresponding operator is local, due to the global nature of the vacuum this VEV carry an important information on the global structure of the background space-time.

In order to derive formula for the vacuum energy density, we first construct the Wightman function. Using of the Abel–Plana summation formula, we have extracted from this function the part, corresponding to the Wightman function for the uncompactified FRW space-time. As the topological part is finite in the coincidence limit, by this way the renormalization procedure is reduced to that for the standard FRW case. The latter was already realized in literature [4]. As a result the vacuum energy density is decomposed into FRW and topological parts. For general values of the curvature coupling parameter the corresponding formula is simplified in the asymptotic regions of small and large values of the ratio L_l/η. In the first case the leading term in the energy density is the same as that for a conformally coupled field, and the topological part behaves like $t^{-c(D+1)}$. This limit corresponds to the early stages of the cosmological expansion in the case $c > 1$ and to the late stages in the case $c<1$. For large values of the ratio L_l/η the behavior of the topological part is different for real and pure imaginary values of the parameter ν. In the first case this part behaves like $t^{(D-2\nu)(1-c)}$, whereas in the second case the decay has an oscillatory nature $t^{D(1-c)}\sin\left(2|\nu|t/\alpha+\varphi_1\right)$. This limit corresponds to the late stages of the cosmological expansion when $c>1$ and to the early stages when $c<1$.

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PROCEEDINGS OF THE YEREVAN STATE UNIVERSITY

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Physics

BANDWIDTH AND DURATION OF NONLINEAR-DISPERSIVE SIMILARITON

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Studying the spectral peculiarities of nonlinear-dispersive similariton, generated in single-mode optical fiber without gain (passive fiber), we reveal that the bandwidth of such a similariton is conditioned only by the initial pulse power. This property of nonlinear-dispersive similariton can be used for the pulse duration measurements at the femtosecond time scale, alternatively to the autocorrelation technique.

Keywords: fiber, femtosecond pulses, chirp, similariton.

Introduction. Similaritons, the pulses that maintain their temporal profiles during the propagation in fibers, is a modern topic in ultrafast optics. Initially the shaping of pulses with parabolic temporal, spectral and phase profiles for high-intensity pulses was predicted by Anderson et al [1]. Later on, parabolic pulses were generated in fibers with gain or distributed dispersion [2–4]. The parabolic pulses generated in both kinds of fibers with either gain or distributed dispersion maintain their temporal profiles during further propagation beginning from a certain distance. The majority of studies in this field is related to the parabolic similaritons of fibers with gain or distributed dispersion. Recently, a new type of similariton was generated in a passive fiber under the combined impacts of Kerr nonlinearity and dispersion [5]. This nonlinear-dispersive (NL-D) similariton has a parabolic phase (linear chirp) independently of the input pulse characteristics, and bell-shaped spectral and temporal profiles. The chirp factor practically matches with the one for the pure dispersive propagation. To describe the similariton completely the features of its bandwidth and duration are also necessary.

In this work we experimentally study the peculiarities of bandwidth of such similaritons formed in passive optical fibers under the combined impacts of nonlinearity and dispersion. Together with our previous research [5], this study gives the comprehensive description of NL-D similaritons.

Experiment and Results. The objective of our study is the revealing of the spectral broadening regulation for NL-D similariton. In our experiment we shape similaritons from transform-limited and chirped input pulses. We use Coherent Verdi V10–Mira 900F femtosecond laser system with $\tau_0 = 100$ *fs* pulse duration at

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a 76 *MHz* repetition rate, $\Delta\lambda_{in} = 10$ *nm* bandwidth, and central wavelength at $\lambda_0 = 800$ *nm*. The schematic of our experiment is shown in Fig. 1.

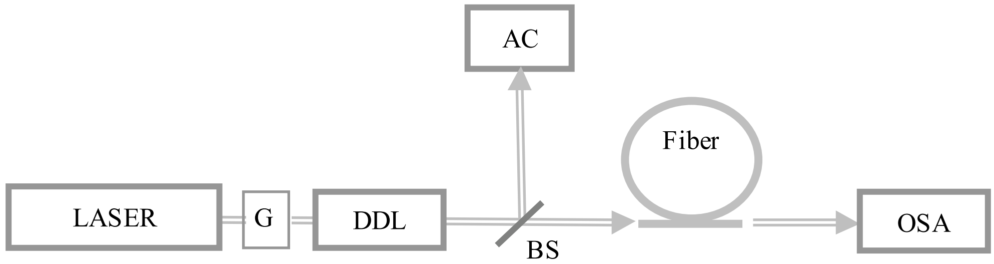
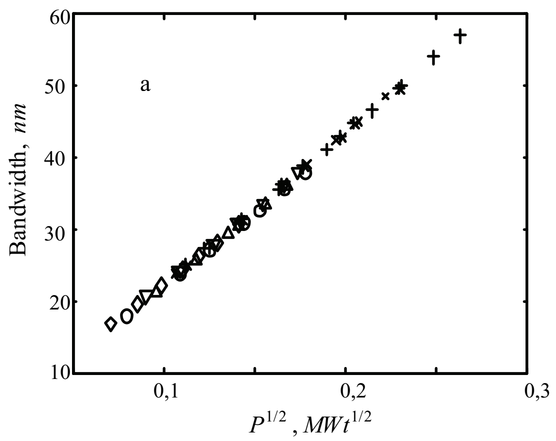


Fig. 1. Schematic of experimental setup.

We stretch the laser pulse, chirping it positively and negatively in SF11 glasses (G) with different thickness and dispersive delay line (DDL) consisting of two dispersion prisms. The beam splitter (BS) splits the laser radiation into the low- and high-power parts. We direct the low-power part to the autocorrelator APE PulseCheck (AC) to measure the input pulse autocorrelation duration, and the high-power part we inject into the fiber, where the similariton is shaped. Afterwards, we measure the bandwidth of similariton and the average power of radiation by means of optical spectrum analyzer (OSA Ando 6315). We also carry out the numerical modeling of the process under study, based on the solution of nonlinear Schrödinger equation with the terms of Kerr nonlinearity and group velocity dispersion (adequate to pulse durations of ≥ 50 *fs* [6]), using the split-step Fourier method.



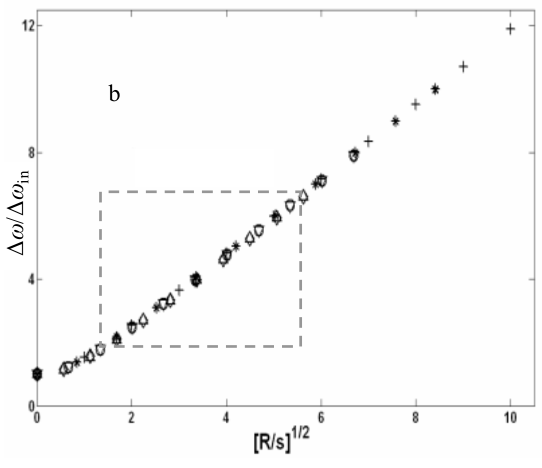


Fig. 2. Similariton's bandwidth versus the square root of its power: (a) – experiment; (b) – numerical simulation for an input Gaussian pulse (the bandwidth $\Delta\omega$ is normalized to the input pulse bandwidth $\Delta\omega_{in}$). + corresponds to the transform-limited pulse with the pulse stretching ratio $s = 1$, $\times - s = \sqrt{2}$, $\nabla - s = \sqrt{5}$, $\Diamond - s = \sqrt{10}$ (positive chirp coefficients); $* - s = \sqrt{2}$, $\circ - s = \sqrt{5}$, and $\Delta - s = \sqrt{10}$ (negative chirp coefficients); $\Delta\lambda_0 = 10$ *nm*. The selected section of (b) corresponds to the experimental range of (a).

Fig. 2 shows the experimental (a) and numerical (b) results in comparison with each other. Fig. 2 (a) shows the similariton's bandwidth $\Delta\lambda$ versus the square root of the pulse power $P = W/\tau_{in}$ (W is the pulse energy). Fig. 2 (b) shows an analogue numerical curve for an input Gaussian pulse. In numerical simulations the bandwidth $\Delta\omega$ is normalized to the input pulse bandwidth $\Delta\omega_{in}$. The experimental and numerical results are in good accordance with each other.

Discussion. NL-D similariton origins from the rectangular pulse shaped due to the pulse NL-D self-interaction in passive fiber [5, 6]. The pulse optimal compression ratio in the regime of rectangular pulses is $\tau_{in}/\tau_c \approx \sqrt{R}/1{,}8$ [6], where $R = WC^2/(\tau_{in}\Delta\omega_{in}^2)$, τ_{in} and τ_c are the input and compressed pulse durations, $\Delta\omega_{in}$ is the input pulse bandwidth, and $C = \sqrt{\frac{n_2 k_0}{k_2 S}}$ is a constant (n_2 is the fiber nonlinearity coefficient, S is the fiber mode area, $k_0 = 2\pi/\lambda_0$ is the wave number, and $k_2 \equiv \partial^2 k/\partial\omega^2\big|_{\omega_0}$). For a transform-limited input pulse $2\tau_{in}/\tau_c = \Delta\omega/\Delta\omega_{in}$, and thus, $\Delta\omega/\Delta\omega_{in} \approx \sqrt{R}$, where $\Delta\omega$ is the bandwidth of the rectangular pulse in fiber. Numerical studies show that NL-D similariton practically keeps the bandwidth of rectangular pulse, thus, the expression above can be used for it. Thus, for the bandwidth of NL-D similariton we have

$$\Delta\omega_{sim} = C\sqrt{P}\,, \tag{1}$$

where $P = W/\tau_{in}$ is the pulse power. Expression (1) motivates the obtained experimental and numerical results. Note for comparison that the bandwidth of parabolic similariton is $\Delta\omega(z) = [(gk_0 n_2 PW)/(2k_2^2 S)]^{1/3}\exp(gz/3)$, where g is the gain coefficient and z is the fiber length [3].

This peculiarity of NL-D similatiton's bandwidth can be used for pulse duration measurement at the femtosecond time scale. Since the value of C is given, we can have the pulse duration by measuring its bandwidth and energy.

Since the chirp of NL-D similariton is given by the fiber dispersion $\gamma \equiv \Delta\omega_{sim}/\Delta t = (k_2 z)^{-1}$ [5], for its duration we have

$$\Delta t = z\sqrt{k_0 k_2 n_2 P/S}\,. \tag{2}$$

Similariton's duration increases linearly during the propagation. Thus, we can control similariton's duration and bandwidth by changing the initial pulse power.

Conclusion. Our numerical and experimental studies show that the bandwidth $\Delta\omega$ of NL-D similariton generated in passive fiber depends linearly on $\sqrt{P}$ with the coefficient, given by the fiber parameters only.

The revealed property of NL-D similariton can be used for pulse duration measurements at the femtosecond time scale, alternatively to the autocorrelation technique.

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COMMUNICATIONS

Mathematics

EFFECTIVE ESTIMATES FOR MODEL GENERATED BY DISTRIBUTION OF MODERATE GROWTH

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In this note two-parametric model, generated by distribution of moderate growth, is considered. The parametric function of unknown parameters, for which there is unbiased, effective estimate, is obtained. The estimate itself is built too.

Keywords: distribution of moderate growth, parameters' estimation, effective estimate.

1^0. Introduction. In order to obtain an *effective* estimates for unknown parameter $\alpha=(\theta,c)$ with parametric set $A=\{\alpha=(\theta,c):0<\theta\le 1, c>0\}$ in two-parametric distribution of moderate growth (1)–(3) of paper [1], replacing $\ln\left(1+\dfrac{c-1}{\psi_k}\right)$ by $(c-1)/\psi_k$ we get the following distribution $\{p_\alpha(x)\}_0^\infty$:

$$\begin{cases} p_\alpha(x)=(g(\alpha))^{-1}\dfrac{\theta^x}{\psi_x}\exp\left\{(c-1)\displaystyle\sum_{m=0}^{x-1}\frac{1}{\psi_m}\right\}, \quad x=1,2,\dots, \\ p_\alpha(0)=(g(\alpha))^{-1}=\left(1+\displaystyle\sum_{x\ge 1}\frac{\theta^x}{\psi_x}\exp\left\{(c-1)\sum_{m=0}^{x-1}\frac{1}{\psi_m}\right\}\right)^{-1}. \end{cases} \tag{1.1}$$

The moderate growth is defined by conditions:

$$\psi_0=1,\ \{\psi_x\}_0^\infty \text{ increases, } \lim_{x\to+\infty}(\psi_k/\psi_{k-1})=1,\ S_\psi=\sum_{x\ge 1}(1/\psi_k)<+\infty. \tag{1.2}$$

We build the model (1.1)–(1.2), because the distribution $\{p_\alpha(x)\}_0^\infty$ of random variable (RV) $\xi\ge 0$ belongs to two-parametric *exponential* class, i.e. the representation holds (see [2])

$$p_\alpha(x)=h(x)\exp\left\{\sum_{i=1}^{2}A_i(\alpha)\cdot T_i(x)+B(\alpha)\right\}, \tag{1.3}$$

where all functions are finite and measurable with respect to corresponding variables. Namely, in our case

$h(x)=(\psi_x)^{-1}\exp\{-S_\psi(x)\},\ T_1(x)=x,\ T_2(x)=S_\psi(x),\ A_1(\alpha)=\ln\theta,\ A_2(\alpha)=c,$

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$B(\alpha)=-\ln g(\alpha)$. Here we denote $S_\psi(x)=\sum_{m=0}^{x-1}(1/\psi_m)$.

The belongings to exponential class allows, based on known facts of Mathematical Statistics, to find the *unique* parametric function $\tau(\alpha)=(\tau_1(\alpha),\tau_2(\alpha))$, which has *unbiased*, *effective* estimate $\tau^*=(\tau_1{}^*,\tau_2{}^*)$ for distribution (1.1)–(1.2).

Let $X^n=(X_1,X_2,...,X_n)$ be a sample of size $n>1$ of RV ξ, $\bar{X}^n=\frac{1}{n}\sum_{i=1}^n X_i$, $\bar{S}_\psi(X^n)=\frac{1}{n}\sum_{i=1}^n S_\psi(X_i)$. Note that the statistics $\left(\sum_{i=1}^n X_i,\sum_{i=1}^n S_\psi(X_i)\right)$ is *complete*, *sufficient* statistics for two-dimensional parameter $A(\alpha)=(\ln\theta,c)$. It follows from the form of likelihood function $p_\alpha(X_1)\cdots p_\alpha(X_n)$ of distribution (1.1)–(1.2) and from the Lehmann Theorem ([2], p.146). Then, due to Lehmann–Sheffe Theorem 2 (see, [3], p. 74), the introduced statistics is *unbiased*, *optimal* estimate for parametric function $\tau(\alpha)=(E_\alpha\xi,E_\alpha S_\psi(\xi))$, where E denotes the sign of mathematical statistics. The effective estimate is also optimal. But the reverse statement, generally speaking, is not true. Anyway, in our case the optimal estimate is also effective, which establishes the following

Theorem. For distribution (1.1)–(1.2) there is the unique parametric function, which has unbiased, effective estimate τ^*. Moreover,

$$\tau(\alpha)=(E_\alpha\xi,E_\alpha S_\psi(\xi)),\ \tau^*=(\bar{X}^n,\bar{S}_\psi(X^n)). \tag{1.4}$$

2⁰. Method of Analysis. Introduce the following *Conditions* (*R*):

a) $p_\alpha(x)$ is continuously differentiable by $\alpha\in A$ for any $x=0,1,2...$

b) The set $\{x\in R: p_\alpha(x)>0\}$ doesn't depend on α.

c) Denote $I_{i,j}(\alpha)=E_\alpha\left(\frac{\partial\ln p_\alpha(X_1)}{\partial\alpha_i}\cdot\frac{\partial\ln p_\alpha(X_1)}{\partial\alpha_j}\right), i,j=1,2,\ \alpha_1=\theta,\alpha_2=c.$

Then the matrix $I(\alpha)=(I_{i,j}(\alpha))$ is continuous by α and $\det I(\alpha)\neq 0$. Denote $A(\alpha)=(A_1(\alpha),A_2(\alpha)), A_{i,j}(\alpha)=\partial A_i(\alpha)/\partial\alpha_j,\ i,j=1,2,\ B'(\alpha)=\left(\frac{\partial B(\alpha)}{\partial\alpha_1},\frac{\partial B(\alpha)}{\partial\alpha_2}\right)$. Then

$$\begin{cases} A'(\alpha)=(A_{i,j}(\alpha))=\begin{pmatrix}1/\theta & 0\\ 0 & 1\end{pmatrix},\quad (A'(\alpha))^{-1}=\begin{pmatrix}\theta & 0\\ 0 & 1\end{pmatrix} \text{ is reverse to } A'(\alpha) \text{ matrix,}\\ \frac{\partial B(\alpha)}{\partial\alpha_1}=-\frac{1}{g(\alpha)}\cdot\frac{\partial g(\alpha)}{\partial\theta}=-\frac{1}{\theta}E_\alpha(\xi),\quad \frac{\partial B(\alpha)}{\partial\alpha_2}=-\frac{1}{g(\alpha)}\cdot\frac{\partial g(\alpha)}{\partial c}=-E_\alpha S_\psi(\xi). \end{cases} \tag{2.1}$$

The proof of Theorem is based on following known fact for exponential class (see, for instance, [2], p. 158), satisfying *Condition* (*R*):

$$\begin{cases} \text{The unique parametric function } \tau(\alpha), \text{ which has unbiased,}\\ \text{effective estimate, and the estimate } \tau^* \text{ itself, are defined from conditions}\\ \tau(\alpha)=-B'(\alpha)\cdot(A'(\alpha))^{-1},\ \tau^*=(n^{-1}S_1,n^{-1}S_2), \text{ where } S_j=\sum_{i=1}^n T_j(X_i),\ j=1,2. \end{cases} \tag{2.2}$$

Due to definition, τ^* is *effective* estimate for $\tau(\alpha)$ if the equality holds

$$D_\alpha \tau^* = \frac{1}{n} D(\alpha) \cdot I^{-1}(\alpha) \cdot D(\alpha)^T. \tag{2.3}$$

Here $D_\alpha \tau^* = E_\alpha (\tau^* - \tau(\theta))^T \cdot (\tau^* - \tau(\theta))$, $\tau_{ij}(\alpha) = \dfrac{\partial \tau_i(\alpha)}{\partial \alpha_j}$, $i, j = 1,2$, $D(\alpha) = (\tau_{ij}(\alpha))$, T denotes the transponation sign, $I^{-1}(\alpha)$ is the matrix reverse to $I(\alpha)$.

Thus, the proof of Theorem consists in: verification of *Conditions* (*R*): evaluation of (2.2) and transformation of (2.2) to the form (1.4); verification of equality (2.3).

3^0. Proof of Theorem. Due to (2.1) and (2.2), easily verified that (2.2) and (1.4) coincide. Next, according to [2] (p. 158)

$$D_\alpha \tau^* = \frac{1}{n} D(\alpha)(A'(\alpha))^{-1}. \tag{3.1}$$

Since $\tau_{11}(\alpha) = \theta^{-1} D_\alpha \xi, \tau_{12}(\alpha) = \operatorname{cov}_\alpha(\xi, S_\psi(\xi)), \tau_{21}(\alpha) = \theta^{-1} \operatorname{cov}_\alpha(\xi, S_\psi(\xi))$, $\tau_{22}(\alpha) = D_\alpha S_\psi(\xi)$, where D is the sign of variation, cov – the sign of covariation, therefore, (3.1) takes the form

$$D_\alpha \tau^* = \frac{1}{n} \begin{pmatrix} D_\alpha \xi & \operatorname{cov}_\alpha(\xi, S_\psi(\xi)) \\ \theta^{-1} \operatorname{cov}_\alpha(\xi, S_\psi(\xi)) & D_\alpha S_\psi(\xi) \end{pmatrix}. \tag{3.2}$$

On the other hand (see [2], p. 159),

$$I(\alpha) = D(\alpha)^T A'(\alpha) = \frac{1}{\theta} \begin{pmatrix} \frac{1}{\theta} D_\alpha \xi & \operatorname{cov}_\alpha(\xi, S_\psi(\xi)) \\ \operatorname{cov}_\alpha(\xi, S_\psi(\xi)) & \theta \cdot D_\alpha S_\psi(\xi) \end{pmatrix}. \tag{3.3}$$

Due to (3.2), (3.3), the estimate (1.4) is effective. It remains to verify the fulfillment of *Condition* (*R*). Obviously, (a) and (b) take place. From Couchy–Shwartz Inequality we have

$$\det I(\alpha) = \frac{1}{\theta^2} \left(D_\alpha \xi \cdot D_\alpha S_\psi(\xi) - \operatorname{cov}_\alpha(\xi, S_\psi(\xi)) \right) > 0.$$

Thus, the condition (c) is fulfilled too.

The theorem is proved.

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Mathematics

ON THE RABIN'S SPEED-UP OF PROOFS FOR SOME SYSTEMS OF FIRST ORDER LOGIC

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In the paper a notion of ordinary theory is considered. It is proved that some systems of first order predicate calculus are ordinary. This property is used for a proof complexity comparison in the considered systems.

***Keywords*:** speed-up, proof complexity, predicate calculus, ordinary theory.

It is well known that investigations of proof complexities in propositional systems are very important due to their tight relation to the main problem of complexity theory: do the classes P and NP coincide? Besides, there is a close relation between the proof complexities in bounded arithmetic and propositional logic. Therefore it is useful to conduct comparative analysis of different formal systems to discover existing relations between them. Researchers in this particular area were used to divide systems to "stronger" and "weaker" ones. During the investigations in this direction it becomes very interesting to research the speed-up phenomena caused by existence of "stronger" formal theories. There are many results in this particular area. Some of them relate to the length of proofs, others – to the number of steps in proofs.

In some results a formula with speed-up is pointed out [1], in others – an infinite set of formulas the proof of which possesses the speed-up property [2]. We introduce such a generalization of the proof complexity notion that traditional characteristics of proof complexities – the number of steps and the length of the proof, satisfy our definition. Moreover, we consider such pairs of formal theories, for which the proof speed-up phenomena may be regarded as an analogue of Rabin's calculation speed-up. In the work we use the ordinary theory notion introduced in [3].

Definition 1. The theory Φ is called *ordinary* if there is a pair of recursively enumerable and effectively inseparable formula sets M_+^{Φ} and M_-^{Φ} of theory Φ and two algorithms A_1 and A_2, which for each formula α from Φ produce, respectively, formulas $A_1(\alpha)$ and $A_2(\alpha)$, such that the following conditions hold:

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1. $\alpha \in M_+^{\Phi}$, iff $\vdash_{\Phi} A_1(\alpha)$ and $\alpha \in M_-^{\Phi}$, iff $\vdash_{\Phi} A_2(\alpha)$;

2. For any formula β of Φ if $\vdash_{\Phi} \beta \vee A_1(\alpha)$ and $\vdash_{\Phi} A_2(\alpha)$, then $\beta \in M_+^{\Phi}$.

It is necessary to stress the main purpose of introducing the special notion of ordinary theory. If sets of provable and disprovable formulas of some formal theory form a pair of effectively inseparable enumerable sets then such a theory is an ordinary one. Different formal systems of full arithmetic and Robinson's arithmetic are examples of such theories. Indeed, in this case as a pair of recursively enumerable effectively inseparable formula sets M_+^{Φ} and M_-^{Φ} it's enough to take, correspondingly, the set of provable and disprovable formulas, and as algorithms $A_1(\alpha)$ and $A_2(\alpha)$ – such ones, that $A_1(\alpha) = \alpha$ and $A_2(\alpha) = \neg\alpha$. However, in case of predicate calculus the situation is quite different: sets of provable and disprovable formulas are recursively separable, for instance, by set of formulas identically true in classical sense on two-item models. Ordinarily of predicate calculus is can be proved by the well-known method of embedding Robinson's arithmetic into the predicate calculus (see, for example, [4]). The notion of ordinary theory is important for studying the speed-up.

Further we consider such pairs of formal systems, one of which is derived from the other one by adding a formula not provable in the first formal system.

Definition 2. Theory Ψ is said to be an *extension* of theory Φ (denoted as $\Psi \supseteq \Phi$), if any formula of Φ and any proof in this system are, respectively, a formula and a proof in the theory Ψ.

A notion of proof complexity is introduced in [3] by analogy with Blum's general concept of calculation complexity.

Definition 3. Denote by $C_{\Phi}(\alpha)$ the minimal proof complexity of formula α in the system Φ, where $C(\alpha)$ is such a general recursive function that for each n the equation $C(\alpha) = n$ has only finite number of solutions and there is an algorithm that generates the set of all solutions of this equation for every n.

The following statement was proved in [3].

Main Theorem. Let Φ_1 be an ordinary theory, α be such a formula of Φ_1 that $\alpha \notin M_+^{\Phi_1}$ and $\alpha \notin M_-^{\Phi_1}$. Further let Φ_2 be such an extension of Φ_1 that $\vdash_{\Phi_2} \alpha$. Then for every general recursive function f there is such a number n_0 that for any n, greater than n_0, there is such a formula α_n that $C_{\Phi_2}(\alpha_n) \le n$ and $C_{\Phi_1}(\alpha_n) > f(n)$.

The proof of this theorem is based on a difficult digitalization method allowing to construct the necessary formula for every $n > n_0$.

It is proved in [3] that if Φ_1 and Φ_2 are such arithmetical or Hilbert type pure predicate systems, that $\Phi_1 \supseteq \Phi_2$, then the statement of the Theorem holds. We show that for some new systems of pure predicate calculus the result holds as well.

Let HP_C, HP_I, HP_M, SP_C, SP_I, SP_M, NP_C, NP_I, NP_M, SP_C^-, SP_I^-, SP_M^-, RP_C, RP_I, RP_M be Hilbert-type (H), sequent (S), natural (N), cut-free sequent (S^-) and resolution (R) systems of pure predicate calculus, respectively, based on classical ($_C$), intuitionistic ($_I$) and minimal ($_M$) logic. Hilbert type systems, sequent systems and cut-free sequent systems for classical and intuitionistic logics are well-known (see, for example, [4]). Natural and resolution systems for intuitionistic logic are defined in [5], other systems are defined in [6, 7]. It is easy to see that all these theories are ordinary in the meaning defined earlier and the statement of the Main Theorem is also valid for every pair Φ_1 and Φ_2 of the above mentioned systems, for which $\Phi_1 \supseteq \Phi_2$. But we can also prove the Main Theorem for other pairs.

Definition 4. Theory Ψ is a *strong extension* of theory Φ (denoted as $\Psi \supset \Phi$), if for any object (formula, sequence, formula set) provable (refutable) in Φ a corresponding provable (refutable) object may be pointed out in Ψ.

Theorem. Let Φ_1 be an ordinary theory and α be such a formula of Φ_1 that $\alpha \notin M_+^{\Phi_1}$ and $\alpha \notin M_-^{\Phi_1}$. Assume that Φ_2 is a strong extension of Φ_1, and there exists an algorithm that for every proof or refutation in Φ_1 constructs a proof or refutation of the corresponding object in Φ_2. Then for every general recursive function f there is such a number n_0 that for any n, greater than n_0, there is a formula α_n, such that $C_{\Phi_2}(\alpha_n) \leq n$ and $C_{\Phi_1}(\alpha_n) > f(n)$.

Corollary. The statement of Main Theorem holds for every pair of the above mentioned systems with lower indices M and I, M and C, I and C.

The proof follows from the Main Theorem and the results from [5–7], where the algorithms producing the proof in some system based on a proof given in another system are constructed.

Summarizing the said above, one can conclude that for a quite wide classes of formulas the proof complexities in "weaker" systems can be much higher than the proof complexities of same formulas in "stronger" formal systems.

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Physics

A COMBINED EFFECT OF ELECTROSTATIC FIELD AND HYDROSTATIC PRESSURE ON THE STABILITY OF BILAYER LIPID MEMBRANES

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The paper is devoted to the experimental research of bilayer lipid membranes stability (BLM) at joint action of an electrostatic field and hydrostatic pressure. It is shown, that with increase in a potential difference on BLM the average time of a life of a membrane decreases. Presence of hydrostatic pressure increasing leads to additional reduction of average time of the membrane life.

Keywords: BLM, lifetime, difference of potential, hydrostatic pressure.

The issue of cell membrane stability is central in membranology [1]. The extreme complexity of cell membranes urges to study this problem using a model, a bilayer lipid membrane (BLM). As known, the membrane is often impacted both by electric forces and hydrostatic pressure [2]. For this reason studying a combined effect of electrostatic field and hydrostatic pressure on BLM is of interest. This experiment is a based research focused on a combined effect of hydrostatic pressure and transmembrane potential difference on the BLM stability. As a parameter characterizing the level of stability of BLM, we assumed a lifetime of BLM at specified values of electrostatic field and hydrostatic pressure [3].

Research and results. The experiments were performed on BLM obtained from phosphatydiserine, earlier suspended in nonane. BLM was formed by the method of Muller et al. [4] on a hole with a 1 *mm* diameter in a PTFE (polytetrafluorethylene) cell. On the both sides of the membrane 0,1 *M* NaCl solution (pH=6,1) was positioned. All the experiments ran at 20^0C. The potential difference was applied to BLM by chlorine-silver electrodes, connected to ADT (NI USB-6008) and regulated by computer. The voltage varied between 150 and 350 *mV* with a pace 50 *mV*. Hydrostatic pressure was reached adding NaCl solution to one of two compartments of the cell. The BLM lifetime was determined applying a computer program.

In the first instance, in mean lifetime of BLM change was studied depending on the voltage increase in the absence of hydrostatic pressure (Fig., curve 1). As seen from Fig., the effect of electrostatic field brings to a drastic mean lifetime reduction. The loss of BLM stability in electric field is connected with formation of through hydrophilic pores [3, 5]. Pores in BLM arise spontaneously, and then as

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a result of a random change in size, reach some critical size, after which the BLM looses its stability. Mean lifetime of BLM exponentially reduces with an increase in transmembrane voltage [6, 7].

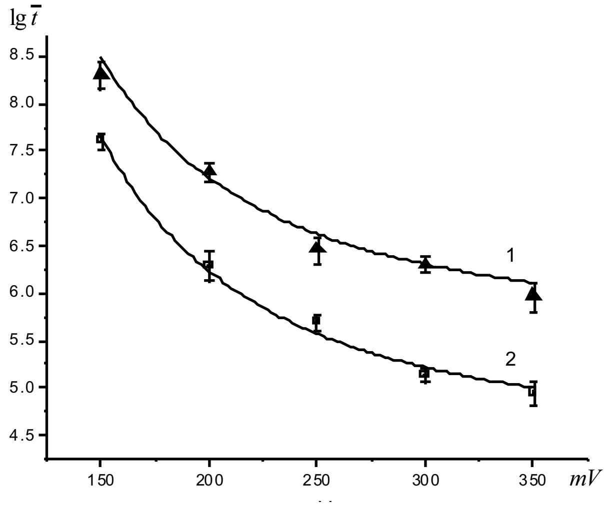
Then we studied the impact of the potential difference on the mean lifetime of BLM in the presence of given hydrostatic pressure p =12,64 *Pa* (Fig., curve 2). This figure demonstrates that dependence of mean lifetime of BLM on potential (curves 1 and 2) is similar. However, the second curve runs lower, which means that the mean lifetime of BLM reduces at hydrostatic pressure.

The loss of BLM stability under hydrostatic pressure can be predetermined by change either in BLM tension [8] or in the number of malformations on BLM [6]. To find out which of the two factors predetermines the loss of BLM stability, let's fit the experimental data with equations for the mean lifetime of BLM [6], employing a least-squares technique:

$$\ln \bar{t}\,(\varphi) = A + \frac{B}{1 + \dfrac{C\varphi^2}{2\sigma}}, \qquad (1)$$

$$A = \ln \frac{(kT)^{3/2}}{4\pi n D\gamma \left(\sigma + \dfrac{C\varphi^2}{2} \right)^{1/2}}, \qquad B = \frac{\pi\gamma^2}{\sigma kT},$$

where σ is the surface tension of BLM (σ $2 \cdot 10^{-3}$ *N/m* [5]); γ is the linear tension of a pore edge in BLM; n is the number of malformations on the membrane; D is the coefficient of malformation diffusion in the radial space; φ is the difference of potential applied to membrane; k is the Boltzmann's constant; C is the reduced capacitance determined by a correlation $C = C_0(\varepsilon_w / \varepsilon_m - 1)$, where $C_0 = \varepsilon_0 \varepsilon_m / h$ is the specific electric capacitance of the membrane; ε_w is the dielectric water transparency, and ε_m is the dielectric permeability of BLM.



Reduction of mean lifetime of BLM at increasing potential difference: 1 – in the absence of hydrostatic pressure difference on the membrane; 2 – in the presence of hydrostatic pressure p=12,64 *Pa*. Dots indicate experimental data, continuous line – theoretical curves, drawn according to equation (1) through a least-squares technique.

Collating theoretical lines with experimental points allowed determination of A and B parameters. As indicated above, in the both cases the values of B parameter practically coincide (B=2,50 on curve 1 and B=2,26 on curve 2), whereas A parameters differ by values (A=5,57 on curve 1, A=4,40 on curve 2). The analysis of equation (1) and its collation with data obtained for A and B indicates that reduction of mean lifetime of BLM in the presence of hydrostatic pressure cannot be connected with changes in surface (σ) or linear (γ) tensions. One can suppose that reduction of mean lifetime of BLM at hydrostatic pressure should be connected with the increase in the number of malformations (n) on BLM. It also proves that the difference of BLM hydrostatic pressure leads to the expansion of its area [4].

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ՄԱԹԵՄԱՏԻԿԱ

Աշխատանքում ընդհանրացված են Վայերշտրասի արտադրիչները և ապացուցված է համապատասխան անվերջ արտադրյալների զուգամիտության թեորեմը: Ապացուցված են Բլյաշկեի տիպի ֆունկցիաների ներկայացումներ Վայերշտրասի տիպի և Բլյաշկեի ֆունկցիաների միջոցով: Բերված է Բլյաշկեի տիպի ֆունկցիաների կառուցման և դրանց զուգամիտությունն ապացուցելու մեթոդ: Ստացված են որոշ առնչություններ Բլյաշկեի տիպի արտադրյալների միջև:

Աշխատանքում առաջարկվում է երկրորդ կարգի որոշ դասի մատրից-ֆունկցիաների ֆակտորացման եղանակ: Այդ մատրից-ֆունկցիաների ստորին ձախ տարրը գրվում է մատրից-ֆունկցիայի մնացած երեք տարրերի և երկու մերոմորֆ ֆունկցիաների համակցությամբ, որոնք համապատասխանաբար որոշված են միավոր շրջանագծի արտաքին և ներքին տիրույթներում:

Աշխատանքում ուսումնասիրվում է Սոբոլևի հավասարումների տիպի հավասարումների մի դասի համար հետևյալ սկզբնական-եզրային խնդիրը.

$$
\begin{cases}
\dfrac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = f(t,x), \quad t>0,\ x=(x_1,\ldots,x_n)\in\Omega\subset \mathrm{R}^n, \\
u\big|_{\partial\Omega}=0, \\
(Lu)(0,x)=g(x), \qquad x\in\Omega,
\end{cases}
$$

որտեղ L-ը և M-ը 2-րդ կարգի դիֆերենցիալ օպերատորներ են: Ապացուցվում է, որ որոշակի պայմանների դեպքում համապատասխան ֆունցիոնալ տարածությունում այդ խնդիրն ունի լուծում և այն էլ միակը:

Աշխատանքում դիտարկվում է Նեյմանի խնդիրը հետևյալ հավասարման համար. $Lu \equiv (t^{\alpha}u'')'' + au = f$, որտեղ $0\le\alpha\le 4$, $t\in[0,b]$, $f\in L_2(0,b)$: Նախ սահմանվում են Սոբոլևի կշռային W_α^2 տարածությունը և այդ հավասարման ընդհանրացված լուծումը: Այնուհետև հետազոտվում են ընդհանրացված լուծման գոյության և միակության հարցերը, ինչպես նաև տրվում են համապատասխան օպերատորի սպեկտրի և որոշման տիրույթի նկարագրերը:

Խ. Լ. Վարդանյան. Չափավոր աճի բաշխումներով ծնված ռեգրեսիոն մոդելներ
էջ. 27–31

Դիտարկվում է կենսաինֆորմատիկայում առաջ եկած չափավոր աճի երկպարամետրական բաշխման միջոցով ծնված ռեգրեսիոն մոդել: Ապացուցվում է մոդելի պարամետրերի նվազագույն քառակուսիների գնահատականների ունակությունը թույլ իմաստով: Ստացված են պարամետրերի նվազագույն քառակուսիների գնահատականների և գաուսյան աղմուկի դիսպերսիայի գնահատականների բաշխումները, որոնք կարող են օգտագործվել մոդելի պարամետրերին վերաբերող վարկածների ստուգման համար:

ՄԵԽԱՆԻԿԱ

Ռ. Շարիֆյան. Իզոտրոպ սալերի կայունության կորուստը, երբ սալի երկու կողմերը ազատ հենված են, իսկ մյուս երկուսը ամրակցված են առաձգական հողակապով
էջ. 32–36

Ներկայացվում են կայունության կորստի կրիտիկական բեռները իզոտրոպ ուղղանկյուն սալերի համար, երբ սալի երկու կողմերը ազատ հենված են, իսկ մյուս երկուսն ամրակցված են առաձգական հողակապով: Կիրառվում է վերլուծական եղանակ՝ հիմնված Լևիի աշխատանքի վրա: Կատարված են համեմատություններ գրականությունից հայտնի մասնավոր դեպքերի հետ: Ուսումնասիրված է կրիտիկական բեռի կախվածությունն առաձգական հողակապի բնութագրիչներից:

ԻՆՖՈՐՄԱՏԻԿԱ

Ա. Հ. Առաքելյան. Պոլիմորֆ λ-թերմերի տիպային կոռեկտության մասին: 2
էջ. 37–46

Աշխատանքում դիտարկվում են պոլիմորֆ λ-թերմերը, որոնցում չկա ինֆորմացիա փոփոխականների տիպերի մասին: Աշխատանքի նպատակն է ապացուցել, որ [1]-ում ներկայացված փոփոխականների տիպի արտածման ալգորիթմը տիպայնացնում է այդպիսի թերմերն ամենաընդհանուր ձևով:

ՖԻԶԻԿԱ

Ա. Լ. Մխիթարյան. Վակուումային ֆլուկտուացիաները կոմպակտ չափերով կոսմոլոգիական մոդելներում
էջ. 47–53

Ուսումնասիրվում են սկալյար դաշտի քվանտային երևույթները աստիճանային մասշտաբային ֆակտորով Ֆրիդման–Ռոբերթսոն–Ուոկերի կոսմոլոգիական մոդելներում, որոնք ունեն $R^p \times (S^1)^q$ տարածական տոպոլոգիա: Ստացված են Վայթմանի դրական հաճախային ֆունկցիայի անդրադարձ բանաձևեր դաշտի քառակուսու և էներգիայի միջինների համար:

Ա. Մ. Ջեյրանյան. **Ոչ գծային-դիսպերսիոն սիմիլարիտոնի սպեկտրային լայնությունը և տևողությունը** էջ. 54–57

Առանց ուժեղացման (պասսիվ) միամոդ օպտիկական մանրաթելային լուսատարում ձևավորված ոչ գծային-դիսպերսիոն սիմիլարիտոնի սպեկտրային առանձնահատկությունների ուսումնասիրության արդյունքում ցույց է տրվել, որ այդպիսի սիմիլարիտոնի սպեկտրային լայնությունը կախված է բացառապես իմպուլսի հզորությունից: Ոչ գծային-դիսպերսիոն սիմիլարիտոնի այս հատկությունը կարող է օգտագործվել ֆեմտովայրկյանային սանդղակում իմպուլսների տևողությունների չափման նպատակով՝ որպես այլընտրանք ավտոկոռելյացիոն մեթոդին:

ՀԱՂՈՐԴՈՒՄՆԵՐ

Հ. Գ. Հովհաննիսյան. **Արդյունավետ գնահատականներ չափավոր աճի բաշխումով ծնված մոդելի համար** էջ. 58–60

Աշխատանքում դիտարկված է մի երկպարամետրական մոդել, որը ծնված է չափավոր աճի բաշխումով: Ստացված է անհայտ պարամետրերի պարամետրական ֆունկցիա, որի համար գոյություն ունի անշեղելի, արդյունավետ գնահատական: Կառուցված է նշված գնահատականը:

Ա. Ա. Չուբարյան, Հ. Ռ. Բոլիբեկյան. **Առաջին կարգի տրամաբանության որոշ համակարգերում արտածումների Ռաբինի արագացման մասին** էջ. 61–63

Հոդվածում ուսումնասիրվում է բնական տեսության գաղափարը: Ապացուցված է, որ առաջին կարգի պրեդիկատների տեսության մի շարք համակարգեր բնական են: Այդ հատկությունը օգտագործված է դիտարկվող համակարգերում արտածումների բարդությունների համեմատության համար:

Հ. Կ. Գևորգյան. **Էլեկտրաստատիկ դաշտի և հիդրոստատիկ ճնշման համատեղ ազդեցությունը երկշերտ լիպիդային թաղանթների կայունության վրա** էջ. 64–66

Աշխատանքը նվիրված է երկշերտ լիպիդային թաղանթների (ԵԼԹ) կայունության փորձարարական ուսումնասիրությանը էլեկտրաստատիկ դաշտի և հիդրոստատիկ ճնշման համատեղ առկայության պայմաններում: Ցույց է տրվել, որ ԵԼԹ-ի վրա պոտենցիալների տարբերության աճի դեպքում թաղանթի կյանքի միջին տևողությունը կարճանում է: Հիդրոստատիկ ճնշման առկայությունը հանգեցնում է ԵԼԹ-ի կյանքի միջին տևողության լրացուցիչ կրճատման:

АННОТАЦИИ

МАТЕМАТИКА

***Г. В. Микаелян, З. С. Микаелян.* Функции типа Вейерштрасса и Бляшке** стр. 3–8

В данной статье обобщены множители Вейерштрасса и доказана теорема сходимости соответствующих бесконечных произведений. Установлены представления функций типа Бляшке через функции типа Вейерштрасса и функции Бляшке. Указаны способ построения новых произведений типа Бляшке и метод доказательства их сходимости. Установлены некоторые соотношения между произведениями типа Бляшке.

***А. В. Саргсян.* О факторизации одного класса матриц-функций второго порядка** стр. 9–15

В работе предлагается метод построения факторизации одного класса матриц-функций второго порядка. Левый нижний элемент этих матриц-функций записывается с помощью комбинаций остальных трех элементов и двух мероморфных функций, которые определяются соответственно внутри и вне единичного круга.

***Сиаваш Гарбаниан.* Краевая задача для псевдопараболических уравнений** стр. 16–21

В работе исследуется начально-краевая задача для уравнения типа Соболева

$$\begin{cases} \dfrac{\partial}{\partial t} L(u(t,x)) + M(u(t,x)) = f(t,x), \quad t>0,\ x=(x_1,\ldots,x_n)\in\Omega\subset \mathrm{R}^n, \\ u\big|_{\partial\Omega} = 0, \\ (Lu)(0,x) = g(x), \qquad x\in\Omega, \end{cases}$$

где L и M – дифференциальные операторы второго порядка. Доказывается, что если удовлетворяются некоторые условия, то эта задача в соответствующем функциональном пространстве имеет единственное решение.

***Л. П. Тепоян, Дарюш Калванд.* Задача Неймана для обыкновенного вырождающегося дифференциального уравнения четвертого порядка** стр. 22–26

В работе рассматривается задача Неймана для уравнения $Lu \equiv (t^\alpha u'')'' + au = f$, где $0 \le \alpha \le 4$, $t\in[0,b]$, $f\in L_2(0,b)$. Сначала мы определяем весовое пространство Соболева W_α^2 и обобщенное решение для этого уравнения. Затем изучается вопрос существования и единственности обобщенного решения, а также даются описания спектра и области определения для соответствующего оператора.

Рассматривается регрессионная модель, порожденная двухпараметрическим распределением умеренного роста, которое возникает в биоинформатике. Доказывается состоятельность в слабом смысле оценок наименьших квадратов параметров модели. Получены распределения оценок наименьших квадратов параметров и оценки дисперсии гауссовского шума, которые могут быть использованы для проверки гипотез относительно параметров модели.

МЕХАНИКА

С помощью аналитического метода Леви определены критические нагрузки потери устойчивости прямоугольных пластин, когда две противоположные стороны пластины свободно оперты, а две другие закреплены посредством упругого шарнира. Изучена зависимость критических нагрузок от параметров упругого закрепления. Для частных случаев установлено соответствие с известными из литературы результатами.

ИНФОРМАТИКА

В работе рассматриваются полиморфные λ-термы, в которых отсутствует информация о типах переменных. Цель даной работы – доказать, что представленный в [1] алгоритм типизации выводит самый общий тип таких термов.

ФИЗИКА

Исследованы квантовые эффекты скалярного поля в космологических моделях Фридмана–Робертсона–Уокера со степенным масштабным фактором и с пространственной топологией $R^p \times (S^1)^q$. Получены рекуррентные формулы для положительно-частотной функции Вайтмана и плотности энергии.

На основе исследований спектральных особенностей нелинейно-дисперсионного симиляритона, сформированного в одномодовом оптическом волоконном световоде без усиления (в пассивном волоконном световоде), показано, что спектральная ширина такого симиляритона обусловлена исключительно исходной мощностью импульса. Эта особенность нелинейно-дисперсионного симиляритона может быть использована для измерения длительностей импульсов в фемтосекундном временном масштабе в качестве альтернативы автокорреляционному методу.

СООБЩЕНИЯ

В настоящей заметке рассмотрена двухпараметрическая модель, порожденная распределением умеренного роста. Получена параметрическая функция неизвестных параметров, для которой существует несмещенная эффективная оценка. Найдена также сама оценка.

В статье рассматривается понятие стандартной теории. Для ряда систем исчисления предикатов первого порядка доказано, что они являются стандартными. На основе понятия стандартной теории проведен анализ сложности выводов в указанных системах.

Работа посвящена экспериментальному исследованию устойчивости бислойной липидной мембраны (БЛМ) при совместном действии электростатического поля и гидростатического давления. Показано, что с увеличением разности потенциалов на БЛМ среднее время жизни мембраны уменьшается. Наличие гидростатического давления приводит к дополнительному уменьшению среднего времени жизни БЛМ.

Խմբագրության հասցեն. Երևան, 0025, Ալ. Մանուկյան, 1

Թողարկման պատասխանատու՝ Դավիդյան Ի.Գ.

Լրատվության գործունեություն իրականացնող՝ ԵՊՀ ՊՈԱԿ,
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