TAMKANG JOURNAL OF MATHEMATICS
Volume 50, Number 3, 233-252, September 2019
doi:10.5556/j.tkjm.50.2019.3352
This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

THE EIGENVALUES' FUNCTION OF THE FAMILY OF STURM-LIOUVILLE OPERATORS AND THE INVERSE PROBLEMS

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Abstract. We study the direct and inverse problems for the family of Sturm-Liouville operators, generated by fixed potential q and the family of separated boundary conditions. We prove that the union of the spectra of all these operators can be represented as a smooth surface (as the values of a real analytic function of two variables), which has specific properties. We call this function "the eigenvalues function of the family of Sturm-Liouville operators (EVF)". From the properties of this function we select those, which are sufficient for a function of two variables be the EVF of a family of Sturm-Liouville operators.

1. Introduction and statement of the main results

Let us denote by $L(q,\alpha,\beta)$ the Sturm-Liouville boundary-value problem

$$\ell y \equiv -y'' + q(x)y = \mu y, \quad x \in (0,\pi),\ \mu \in \mathbb{C}, \tag{1.1}$$

$$y(0)\cos\alpha + y'(0)\sin\alpha = 0, \quad \alpha \in (0,\pi], \tag{1.2}$$

$$y(\pi)\cos\beta + y'(\pi)\sin\beta = 0, \quad \beta \in [0,\pi), \tag{1.3}$$

where q is a real-valued function, summable on $[0,\pi]$ (we write $q \in L^1_{\mathbb{R}}[0,\pi]$). By $L(q,\alpha,\beta)$ we also denote the self-adjoint operator, generated by problem (1.1)-(1.3) (see [1, 2, 3]). It is known, that under these conditions the spectra of the operator $L(q,\alpha,\beta)$ is discrete and consists of real, simple eigenvalues (see [1, 2, 3, 4]), which we denote by $\mu_n = \mu_n(q,\alpha,\beta) = \lambda_n^2(q,\alpha,\beta)$, $n = 0,1,2,\ldots$, emphasizing the dependence of μ_n on q, α and β. We assume that eigenvalues are enumerated in the increasing order, i.e.

$$\mu_0(q,\alpha,\beta) < \mu_1(q,\alpha,\beta) < \cdots < \mu_n(q,\alpha,\beta) < \ldots.$$

The existence, countability and asymptotic formulae for the eigenvalues of the operator $L(q,\alpha,\beta)$, in the cases of smooth q, were investigated in XIX and in the beginning of XX

2010 *Mathematics Subject Classification*. 34A55, 34B24, 47E05.
Key words and phrases. Direct and inverse Sturm-Liouville problems, dependence of eiganvalues on boundary conditions, necessary and sufficient counditions.

century (see, e.g., [5, 6, 7]). The dependence of μ_n on q was investigated in [8, 9, 10, 11] for $q \in L^2_{\mathbb{R}}[0,\pi]$ and we will not concern to this aspect. The dependence of μ_n on α and β usually studied (see e.g. [1, 2, 3, 12, 13, 14, 15]) in the following sense: the boundary conditions are separated into four cases, and results, in particular the asymptotics of the eigenvalues, are formulated separately for each case (more detailed list is in [16], page 386), namely:

$$1)\ \mu_n(q,\alpha,\beta) = n^2 + \frac{2}{\pi}\left(\cot\beta - \cot\alpha\right) + [q] + r_n\left(q,\alpha,\beta\right), \text{ if } \alpha,\beta \in (0,\pi), \tag{1.4}$$

$$2)\ \mu_n(q,\pi,\beta) = \left(n+\frac{1}{2}\right)^2 + \frac{2}{\pi}\cot\beta + [q] + r_n\left(q,\beta\right), \text{ if } \beta \in (0,\pi), \tag{1.5}$$

$$3)\ \mu_n(q,\alpha,0) = \left(n+\frac{1}{2}\right)^2 - \frac{2}{\pi}\cot\alpha + [q] + r_n\left(q,\alpha\right), \text{ if } \alpha \in (0,\pi), \tag{1.6}$$

$$4)\ \mu_n\left(q,\pi,0\right) = (n+1)^2 + [q] + r_n\left(q\right), \tag{1.7}$$

where $[q] = \dfrac{1}{\pi}\displaystyle\int_0^\pi q(t)\,dt$ and $r_n = o(1)$ when $n \to \infty$, but this estimate for r_n is not uniform in $\alpha, \beta \in [0,\pi]$ and we cannot obtain (1.5), (1.6) and (1.7) from (1.4) by passing to the limit when $\alpha \to \pi$ or $\beta \to 0$.

And we want to have one formula instead of this four.

Our first aim is to understand the nature of the dependence of eigenvalues $\mu_n(q,\alpha,\beta)$ on parameters α and β. In the paper [4] we have proved that the dependence of eigenvalues μ_n on α and β is smooth (analytic) and we have derived one new formula (see below (1.8)), which takes into account this smooth dependence, which contains all formulae (1.4)–(1.7) as the particular cases and in which the estimate of reminder is uniform with respect to α, β and q. More explicitly, in [4] we have proved the following assertion.

Theorem 1.1 ([4])**.** *The lowest eigenvalue $\mu_0(q,\alpha,\beta)$ has the property:* $\lim\limits_{\alpha\to 0}\mu_0(q,\alpha,\beta) = -\infty$, $\lim\limits_{\beta\to\pi}\mu_0(q,\alpha,\beta) = -\infty$.

For eigenvalues $\mu_n(q,\alpha,\beta)$, $n = 2,3,\ldots$, the formula

$$\lambda_n^2(q,\alpha,\beta) := \mu_n(q,\alpha,\beta) = \left[n + \delta_n(\alpha,\beta)\right]^2 + [q] + r_n(q,\alpha,\beta), \tag{1.8}$$

hold, where δ_n is the solution of the equation

$$\begin{aligned}\delta_n(\alpha,\beta) = {} & \frac{1}{\pi}\arccos\frac{\cos\alpha}{\sqrt{\left[n+\delta_n(\alpha,\beta)\right]^2\sin^2\alpha + \cos^2\alpha}} \\ & -\frac{1}{\pi}\arccos\frac{\cos\beta}{\sqrt{\left[n+\delta_n(\alpha,\beta)\right]^2\sin^2\beta + \cos^2\beta}},\end{aligned} \tag{1.9}$$

and $r_n = r_n(q,\alpha,\beta) = o(1)$, when $n \to \infty$, uniformly by $\alpha,\beta \in [0,\pi]$ and q from the bounded subsets of $L^1_{\mathbb{R}}[0,\pi]$ (we will write $q \in BL^1_{\mathbb{R}}[0,\pi]$).

This theorem has two parts. The first is connected with the behaviour of μ_0. These connections one of the reasons, why we take $\alpha \in (0,\pi]$ and $\beta \in [0,\pi)$.

The second connected with the formula $\mu_n(0,\alpha,\beta) = [n+\delta_n(\alpha,\beta)]^2$, $n \geq 2$, where for the first time was obtained the explicit formula for the zero potential. In many respect these advantages of formula (1.8) stipulated by introducing (in consideration) the sequence of functions $\{\delta_n(\alpha,\beta)\}_{n=2}^{\infty}$, which we define for $n \geq 2$ as

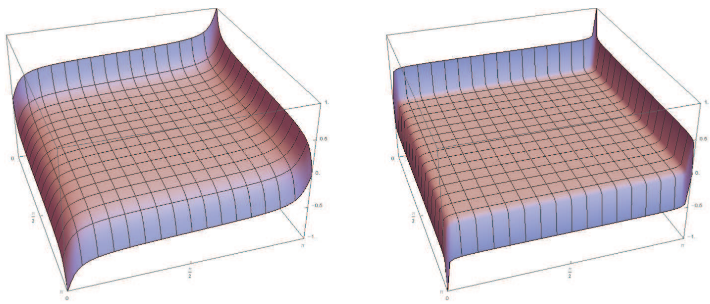
$$\delta_n(\alpha,\beta) := \sqrt{\mu_n(0,\alpha,\beta)} - n = \lambda_n(0,\alpha,\beta) - \lambda_n\left(0,\frac{\pi}{2},\frac{\pi}{2}\right),$$

i.e. δ_n is the "distance" between $\lambda_n(0,\alpha,\beta)$ and $\lambda_n(0,\frac{\pi}{2},\frac{\pi}{2}) = n$, (after we have proved, that δ_n satisfy (1.9)), and because of which the estimate of reminder term became uniform with respect to q, α and β.

It is easily follows from (1.9) (see for details [4]), that

$$\begin{aligned}
&1)\ \delta_n(\alpha,\beta) = \frac{\cot\beta - \cot\alpha}{\pi n} + O\left(\frac{1}{n^2}\right), \text{ when } \alpha,\beta \in (0,\pi), \\
&2)\ \delta_n(\pi,\beta) = \frac{1}{2} + \frac{\cot\beta}{\pi\left(n+\frac{1}{2}\right)} + O\left(\frac{1}{n^2}\right), \text{ when } \beta \in (0,\pi), \\
&3)\ \delta_n(\alpha,0) = \frac{1}{2} - \frac{\cot\alpha}{\pi\left(n+\frac{1}{2}\right)} + O\left(\frac{1}{n^2}\right), \text{ when } \alpha \in (0,\pi), \\
&4)\ \delta_n(\pi,0) = 1,
\end{aligned} \tag{1.10}$$

Although (1.9) is not a representation of $\delta_n(\alpha,\beta)$, but only an (transcendental) equation, it is sufficiently convenient for investigation. In particular, using the program “Wolfram Mathematica”, we construct the graphs of $\delta_n(\alpha,\beta)$ for different n. On Figure 1 we show the graphs of δ_{10} and δ_{100}.



When $n = 10$. When $n = 100$.

Figure 1: The graphs of the function $\delta_n(\alpha,\beta)$.

In what follows let $\varphi(x,\mu,\gamma)$ and $\psi(x,\mu,\delta)$ denote the solutions of (1.1), satisfying the initial conditions

$$\begin{aligned}\varphi(0,\mu,\gamma) &= \sin\gamma, \quad \varphi'(0,\mu,\gamma) = -\cos\gamma, \quad \gamma\in\mathbb{C},\\ \psi(\pi,\mu,\delta) &= \sin\delta, \quad \psi'(\pi,\mu,\delta) = -\cos\delta, \quad \delta\in\mathbb{C},\end{aligned}\tag{1.11}$$

correspondingly. The eigenvalues $\mu_n = \mu_n(q,\alpha,\beta)$,, $n = 0,1,2,\ldots$, of $L(q,\alpha,\beta)$ are the solutions of the equation

$$\Phi(\mu) = \Phi(\mu,\alpha,\beta) \stackrel{def}{=} \varphi(\pi,\mu,\alpha)\cos\beta + \varphi'(\pi,\mu,\alpha)\sin\beta = 0, \qquad \alpha\in(0,\pi].$$

The functions $\varphi_n(x) = \varphi(x,\mu_n,\alpha)$ and $\psi_n(x) = \psi(x,\mu_n,\beta)$, $n = 0,1,2,\ldots$, are the eigenfunctions, corresponding to the eigenvalue μ_n. Since the eigenvalues are simple, the eigenfunctions $\varphi_n(x)$ and $\psi_n(x)$ are linearly dependent, i.e. there exist constants $c_n = c_n(q,\alpha,\beta)$ such that

$$\varphi(x,\mu_n,\alpha) = c_n\psi(x,\mu_n,\beta).\tag{1.12}$$

The squares of the L^2-norms of these eigenfunctions:

$$a_n = a_n(q,\alpha,\beta) = \int_0^\pi |\varphi_n(x)|^2\,dx, \qquad b_n = b_n(q,\alpha,\beta) = \int_0^\pi |\psi_n(x)|^2\,dx,\tag{1.13}$$

are called the norming constants. In [4] we have proved that

$$\frac{\partial\mu(q,\alpha,\beta)}{\partial\alpha} = \frac{1}{a_n(q,\alpha,\beta)},\tag{1.14}$$

$$\frac{\partial\mu(q,\alpha,\beta)}{\partial\beta} = -\frac{1}{b_n(q,\alpha,\beta)}.\tag{1.15}$$

In order to investigate the dependence of the spectral data on parameters α and β in papers [17, 4, 18] we introduce the conception of "the eigenvalues function (EVF)". In order to give the definition, we note that arbitrary positive number γ we can represent in the form $\gamma = \alpha + \pi n$, where $\alpha\in(0,\pi]$ and $n = 0,1,2,\ldots$; and arbitrary $\delta\in(-\infty,\pi)$ we can represent as $\delta = \beta - \pi m$, where $\beta\in[0,\pi)$ and $m = 0,1,2,\ldots$

Let us note that here we assume q is fixed, so when we say "the function $\mu(q,\gamma,\delta)$ of two arguments" we understand that arguments are γ and δ.

Definition. The function $\mu(q,\gamma,\delta)$ of two arguments, defined on $(0,\infty)\times(-\infty,\pi)$ by formula

$$\mu(q,\gamma,\delta) = \mu(q,\alpha+\pi n,\beta-\pi m) \stackrel{def}{=} \mu_{n+m}(q,\alpha,\beta),\tag{1.16}$$

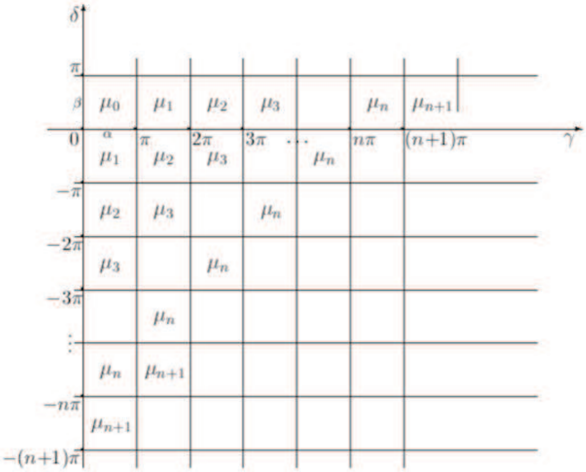


Figure 2: The domain $(0,\infty)\times(-\infty,\pi)$.

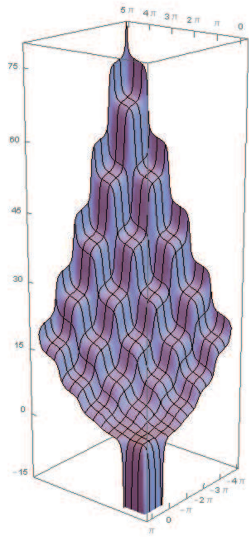


Figure 3: The graph of the function $\mu(0,\gamma,\delta)$.

where $\mu_k(q,\alpha,\beta)$, $k=0,1,2,\dots$, are the eigenvalues of $L(q,\alpha,\beta)$, enumerated in the increasing order, we shall call the eigenvalues function (EVF) of the family of the problems (operators) $\{L(q,\alpha,\beta),\alpha\in(0,\pi],\beta\in[0,\pi)\}$.

The meaning of this definition is easy to see on Figure 2. On Figure 3 we show the graph of EVF $\mu(0,\gamma,\delta)$, which contain part of μ_0.

Of course, these graph constructed for the case $q(x)\equiv 0$, but when we add $q(x)\neq 0$ the changes of graphs are not principal. Thus, we understand the nature of dependence of eigenvalues on parameters α and β in means of picture of graphs. More deep and strong understanding of this dependence we give in the next theorem.

Theorem 1.2. *The EVF $\mu(q,\gamma,\delta)$ has the properties:*

P.1) *for arbitrary fixed $\beta \in [0,\pi)$ the function $\mu^{+}(\gamma) = \mu(q,\gamma,\beta)$ is strictly increasing on $(0,\infty)$ and its range of values is the whole real axis $(-\infty,\infty)$,*
for arbitrary fixed $\alpha \in (0,\pi]$ the function $\mu^{-}(\delta) = \mu(q,\alpha,\delta)$ is strongly decreasing on $(-\infty,\pi)$ and its range of values is $(-\infty,\infty)$;

P.2) *for arbitrary $(\gamma,\delta) \in (0,\infty) \times (-\infty,\pi)$ there exists a neighborhood $U_{\gamma,\delta} \subset \mathbb{C}^2$, on which defined one-valued analytic function $\tilde{\mu}(\tilde{\gamma},\tilde{\delta})$ (of two complex variables $\tilde{\gamma}$ and $\tilde{\delta}$), which coincide with $\mu(q,\gamma,\delta)$ for real values of arguments γ and δ (from $U_{\gamma,\delta}$). In other words, $\mu(q,\gamma,\delta)$ is a real analytic function on $(0,\infty) \times (-\infty,\pi)$;*

P.3) *for $\lambda_n(q,\alpha,\beta)$ ($\lambda_n^2 = \mu_n$) the following asymptotic (when $n \to \infty$) formulae hold:*

$$\lambda_n = n + \delta_n(\alpha,\beta) + \frac{[q]}{2[n+\delta_n(\alpha,\beta)]} + l_n,$$

where the reminders $l_n = l_n(q,\alpha,\beta)$ are such that $l_n = o\left(\frac{1}{n}\right)$, when $n \to \infty$, uniformly by $\alpha,\beta \in [0,\pi]$ and $q \in BL^1_{\mathbb{R}}[0,\pi]$, and the function $l(\cdot)$, defined by formula

$$l(x) = \sum_{n=1}^{\infty} l_n \sin nx,$$

is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$, i.e. $l \in AC(0,2\pi)$, for each $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$ and each $q \in L^1_{\mathbb{R}}[0,\pi]$;

P.4$_+$) *for norming constant $a_n = a_n(q,\alpha,\beta)$ the following asymptotic formula hold:*

$$\begin{aligned} a_n(q,\alpha,\beta) &= \left[\left.\frac{\partial \mu(\gamma,\beta)}{\partial \gamma}\right|_{\gamma=\alpha+\pi n} \right]^{-1} \\ &= \frac{\pi}{2}\left[1 + \frac{2\,æ_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]} + r_n\right]\sin^2\alpha \\ &\quad + \frac{\pi}{2\left[n+\delta_n(\alpha,\beta)\right]^2}\left[1 + \frac{2\,æ_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]} + \tilde{r}_n\right]\cos^2\alpha, \end{aligned}$$

where

$$æ_n = æ_n(q,\alpha,\beta) = -\frac{1}{2}\int_0^{\pi} (\pi - t)\, q(t) \sin 2\left[n + \delta_n(\alpha,\beta)\right] t\, dt,$$

$r_n = r_n(q,\alpha,\beta) = O\left(\frac{1}{n^2}\right)$ *and* $\tilde{r}_n = \tilde{r}_n(q,\alpha,\beta) = O\left(\frac{1}{n^2}\right)$, *when* $n \to \infty$, *uniformly by* $\alpha, \beta \in [0,\pi]$ *and* $q \in BL^1_{\mathbb{R}}[0,\pi]$. *For arbitrary* $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$ *the function*

$$k(x) := \sum_{n=2}^{\infty} \frac{æ_n}{n+\delta_n(\alpha,\beta)} \cos\left[n+\delta_n(\alpha,\beta)\right] x$$

is absolutely continuous function on arbitrary segment $[a,b] \subset (0,2\pi)$, *i.e.* $k \in AC(0,2\pi)$;

P.4$_-$) *for norming constant* $b_n = b_n(q,\alpha,\beta)$ *the following asymptotic formula hold:*

$$b_n(q,\alpha,\beta) = -\left[\left.\frac{\partial\mu(\alpha,\delta)}{\partial\delta}\right|_{\delta=\beta-\pi n}\right]^{-1}$$
$$= \frac{\pi}{2}\left[1+\frac{2æ_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]}+p_n\right]\sin^2\beta + \frac{\pi}{2\left[n+\delta_n(\alpha,\beta)\right]^2}\left[1+\frac{2æ_n(q,\alpha,\beta)}{\pi\left[n+\delta(\alpha,\beta)\right]}\tilde{p}_n\right]\cos^2\beta$$

where $p_n = p_n(q,\alpha,\beta) = O\left(\frac{1}{n^2}\right)$ *and* $\tilde{p}_n = \tilde{p}_n(q,\alpha,\beta) = O\left(\frac{1}{n^2}\right)$, *when* $n \to \infty$, *uniformly in* $\alpha, \beta \in [0,\pi]$ *and* $q \in BL^1_{\mathbb{R}}[0,\pi]$;

P.5$_+$) *for arbitrary* $\varepsilon \in (0,\pi)$, $\varepsilon \neq \alpha$, $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$

$$\frac{1}{a_n(q,\alpha,\beta)} = \left.\frac{\partial\mu(\gamma,\beta)}{\partial\gamma}\right|_{\gamma=\alpha+\pi n}$$
$$= \begin{cases} \dfrac{\sin\varepsilon}{\sin\alpha}\dfrac{\mu_n(\alpha,\beta)-\mu_n(\varepsilon,\beta)}{\sin(\alpha-\varepsilon)}\displaystyle\prod_{\substack{k=0\\k\neq n}}^{\infty}\dfrac{\mu_k(\varepsilon,\beta)-\mu_n(\alpha,\beta)}{\mu_k(\alpha,\beta)-\mu_n(\alpha,\beta)}, & \text{if } \alpha,\beta \in (0,\pi), \\ \dfrac{\pi\left(n+\frac{1}{2}\right)^2}{4n^2}\cdot\dfrac{\left[\mu_0(\varepsilon,\beta)-\mu_n(\pi,\beta)\right]\cdot\left[\mu_n(\pi,\beta)-\mu_n(\varepsilon,\beta)\right]}{\mu_0(\pi,\beta)-\mu_n(\pi,\beta)}\cdot & \\ \quad\cdot\displaystyle\prod_{\substack{k=1\\k\neq n}}^{\infty}\dfrac{\left(k+\frac{1}{2}\right)^2}{k^2}\cdot\dfrac{\mu_k(\varepsilon,\beta)-\mu_n(\pi,\beta)}{\mu_k(\pi,\beta)-\mu_n(\pi,\beta)}, & \text{if } \alpha=\pi, \beta\in(0,\pi) \text{ and } n\neq 0, \\ \dfrac{\pi}{4}\left[\mu_0(\pi,\beta)-\mu_0(\varepsilon,\beta)\right]\cdot & \\ \quad\cdot\displaystyle\prod_{k=1}^{\infty}\dfrac{\left(k+\frac{1}{2}\right)^2}{k^2}\cdot\dfrac{\mu_k(\varepsilon,\beta)-\mu_0(\pi,\beta)}{\mu_k(\pi,\beta)-\mu_0(\pi,\beta)}, & \text{if } \alpha=\pi, \beta\in(0,\pi) \text{ and } n=0, \\ \dfrac{(n+1)^2}{\left(n+\frac{1}{2}\right)^2}\dfrac{\mu_n(\pi,0)-\mu_n(\varepsilon,0)}{\pi}\displaystyle\prod_{\substack{k=0\\k\neq n}}^{\infty}\dfrac{(k+1)^2}{\left(k+\frac{1}{2}\right)^2}\cdot\dfrac{\mu_k(\varepsilon,0)-\mu_n(\pi,0)}{\mu_k(\pi,0)-\mu_n(\pi,0)}, & \text{if } \alpha=\pi, \beta=0; \end{cases}$$

P.5$_-$) *for arbitrary* $\eta \in [0,\pi)$, $\eta \neq \beta$, $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$

$$\frac{1}{b_n(q,\alpha,\beta)} = -\left.\frac{\partial\mu(\alpha,\delta)}{\partial\delta}\right|_{\delta=\beta-\pi n}$$

$$
= \begin{cases}
\dfrac{\sin\eta}{\sin\beta} \cdot \dfrac{\mu_n(\alpha,\beta)-\mu_n(\alpha,\eta)}{\sin(\beta-\eta)} \prod\limits_{\substack{k=0\\k\neq n}}^{\infty} \dfrac{\mu_k(\alpha,\eta)-\mu_n(\alpha,\beta)}{\mu_k(\alpha,\beta)-\mu_n(\alpha,\beta)}, \; if \;\; \alpha,\beta\in(0,\pi),\\
\dfrac{\pi\left(n+\frac{1}{2}\right)^2}{4n^2} \cdot \dfrac{\left[\mu_n(\alpha,0)-\mu_n(\alpha,\eta)\right]\cdot\left[\mu_0(\alpha,\eta)-\mu_n(\alpha,0)\right]}{\mu_0(\alpha,0)-\mu_n(\alpha,0)} \cdot \\
\qquad \cdot \prod\limits_{\substack{k=1\\k\neq n}}^{\infty} \dfrac{\left(k+\frac{1}{2}\right)^2}{k^2} \cdot \dfrac{\mu_k(\alpha,\eta)-\mu_n(\alpha,0)}{\mu_k(\alpha,0)-\mu_n(\alpha,0)}, \; if \;\; \alpha=\pi, \beta\in(0,\pi), n\neq 0,\\
\dfrac{\pi}{4}\left[\mu_0(\alpha,0)-\mu_0(\alpha,\eta)\right]\cdot \\
\qquad \cdot \prod\limits_{k=1}^{\infty} \dfrac{\left(k+\frac{1}{2}\right)^2}{k^2} \cdot \dfrac{\mu_k(\alpha,\eta)-\mu_n(\alpha,0)}{\mu_k(\alpha,0)-\mu_n(\alpha,0)}, \; if \; \alpha=\pi, \beta\in(0,\pi), n=0,\\
\dfrac{(n+1)^2}{\left(n+\frac{1}{2}\right)^2} \dfrac{\mu_n(\pi,0)-\mu_n(\pi,\eta)}{\pi} \prod\limits_{\substack{k=0\\k\neq n}}^{\infty} \dfrac{(k+1)^2}{\left(k+\frac{1}{2}\right)^2} \cdot \dfrac{\mu_k(\pi,\eta)-\mu_n(\pi,0)}{\mu_k(\pi,0)-\mu_n(\pi,0)}, \; if \; \alpha=\pi, \beta=0;
\end{cases}
$$

P.6$_+$) *for* $\alpha(0,\pi)$

$$\sum_{n=0}^{\infty}\left\{\frac{\sin^2\alpha}{a_n(q,\alpha,\beta)}-\frac{1}{a_n^0}\right\}=\cot\alpha,$$

where $a_0^0=a_0\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)=\pi$, *and* $a_n^0=a_n\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)=\frac{\pi}{2}$, *for* $n=1,2,\dots,$;

P.6$_-$) *for* $\beta(0,\pi)$

$$\sum_{n=0}^{\infty}\left\{\frac{\sin^2\beta}{b_n(q,\alpha,\beta)}-\frac{1}{b_n^0}\right\}=-\cot\beta,$$

where $b_0^0=b_0\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)=\pi$, *and* $b_n^0=b_n\left(0,\frac{\pi}{2},\frac{\pi}{2}\right)=\frac{\pi}{2}$, *for* $n=1,2,\dots,$;

P.7) *for arbitrary* $\alpha,\beta\in(0,\pi)$, *the following formulae hold:*

$$
\begin{aligned}
&a_n(q,\alpha,\beta)\cdot b_n(q,\alpha,\beta)\\
&=\begin{cases}
\left[\pi\sin\alpha\sin\beta\prod\limits_{k=1}^{\infty}\dfrac{\mu_k(q,\alpha,\beta)-\mu_0(q,\alpha,\beta)}{k^2}\right]^2, \; if \;\; n=0,\\
\left[\dfrac{\pi}{n^2}\left[\mu_0(q,\alpha,\beta)-\mu_n(q,\alpha,\beta)\right]\sin\alpha\sin\beta\prod\limits_{\substack{k=1\\k\neq n}}^{\infty}\dfrac{\mu_k(q,\alpha,\beta)-\mu_n(q,\alpha,\beta)}{k^2}\right]^2, \; if \;\; n\neq 0,
\end{cases}
\end{aligned}
\tag{1.27}
$$

Theorem 1.2 corresponds to the direct Sturm-Liouville problems. In this theorem we collect some results, which was obtained separately in our previous paper and some new. We need in such a statement of Theorem 1.2, since namely these 7 properties are necessary and sufficient for a function to be the EVF (see below).

The inverse problem.

By formula (1.16) we construct the map

$$L^1_{\mathbb{R}}[0,\pi]\ni q\to\mu(\cdot,\cdot), \tag{1.28}$$

where we state the correspondence between $L^1_{\mathbb{R}}[0,\pi]$ and surfaces $\mu(\cdot,\cdot)$, defined on $(0,\infty)\times(-\infty,\pi)$, which have the properties $P.1)-P.7)$ of Theorem 1.2. One of the principal result of our work (it is our second aim) is that the map (1.28) is one to one, or, in other words, the properties described in Theorem 1.2 not only necessary but also sufficient for a function, possessing these properties, to be the EVF of a family of operators $\{L(q,\alpha,\beta),\alpha\in(0,\pi],\beta\in[0,\pi)\}$.

To make more precise what we keep in mind, we formulate the assertion.

Theorem 1.3. *Let a function $\nu(\cdot,\cdot)$ of two variables γ and δ, where γ changes on $(0,\infty)$ and δ changes on $(-\infty,\pi)$, has the property*

$$\nu\left(\alpha+\pi n,\beta\right)=\nu\left(\alpha,\beta-\pi n\right)=\nu\left(\alpha+\pi k,\beta-\pi m\right)$$

for arbitrary $\alpha\in(0,\pi]$, $\beta\in[0,\pi)$ and arbitrary $k,m,n=0,1,2,\ldots$, such that $k+m=n$. This value of function $\nu(\cdot,\cdot)$ we will denote by $\nu_n(\alpha,\beta)$, i.e.

$$\nu_n(\alpha,\beta)\overset{def}{:=}\nu\left(\alpha+\pi n,\beta\right)=\nu\left(\alpha,\beta-\pi n\right)=\nu\left(\alpha+\pi k,\beta-\pi m\right). \tag{1.29}$$

That means $\nu(\cdot,\cdot)$ corresponds to Figure 2), where instead of μ_n state ν_n.

Besides, let function $\nu(\cdot,\cdot)$ has the properties:

P.1. For each fixed $\delta\in(-\infty,\pi)$ function $\nu(\cdot,\delta)$ is strictly increasing by γ on $(0,\infty)$ and for arbitrary $\beta\in[0,\pi)$ the range of $\nu\left(\cdot,\beta\right)$ is the whole real axis $(-\infty,\infty)$.

For each fixed $\gamma\in(0,\infty)$ function $\nu\left(\gamma,\cdot\right)$ is strictly decreasing by δ on $(-\infty,\pi)$ and for arbitrary $\alpha\in(0,\pi]$ the range of $\nu(\alpha,\cdot)$ is the whole real axis $(-\infty,\infty)$.

P.2. $\nu(\cdot,\cdot)$ is a real analytic function on $(0,\infty)\times(-\infty,\pi)$.

P.3. There exist a constant c and a sequence $\{l_n\}_{n=2}^{\infty}$ such that the asymptotics $(n\to\infty)$

$$\sqrt{\nu_n(\alpha,\beta)}=n+\delta_n(\alpha,\beta)+\frac{c}{2[n+\delta_n(\alpha,\beta)]}+l_n \tag{1.30a}$$

hold, where

$$l_n=o\left(\frac{1}{n}\right), \tag{1.30b}$$

and the function

$$l(x):=\sum_{n=2}^{\infty}l_n\sin[n+\delta_n(\alpha,\beta)]\,x \tag{1.30c}$$

is absolutely continuous on arbitrary $[a,b]\subset(0,2\pi)$, i.e.

$$l\in AC\,(0,2\pi). \tag{1.30d}$$

P.4$_+$.

$$\left[\left.\frac{\partial\nu(\gamma,\beta)}{\partial\gamma}\right|_{\gamma=\alpha+\pi n}\right]^{-1}=\frac{\pi}{2}[1+s_{n1}]\sin^2\alpha+\frac{\pi}{2\left[n+\delta_n(\alpha,\beta)\right]^2}[1+s_{n2}]\cos^2\alpha,$$

where

$$s_{ni}=o\left(\frac{1}{n}\right),$$

and the functions $(i=1,2)$

$$S_i(x):=\sum_{n=2}^{\infty}s_{ni}\cos\left[n+\delta_n\left(\alpha,\beta\right)\right]x$$

are absolutely continuous on arbitrary segment $[a,b]\subset(0,2\pi)$, *i.e.* $S_i\in AC\,(0,2\pi)$.

P.4$_-$.

$$\left[\left.\frac{\partial\nu(\alpha,\delta)}{\partial\delta}\right|_{\delta=\beta-\pi n}\right]^{-1}=-\frac{\pi}{2}\left[1+p_{n1}\right]\sin^2\beta-\frac{\pi}{2\left[n+\delta_n(\alpha,\beta)\right]^2}\left[1+p_{n2}\right]\cos^2\beta,$$

where

$$p_{ni}=o\left(\frac{1}{n}\right),$$

and the functions $(i=1,2)$

$$P_i(x):=\sum_{n=2}^{\infty}p_{ni}\cos\left[n+\delta_n\left(\alpha,\beta\right)\right]x$$

are absolutely continuous on arbitrary segment $[a,b]\subset(0,2\pi)$, *i.e.*

$$P_i\in AC\,(0,2\pi)\,.$$

P.5$_+$. For arbitrary $\alpha,\beta\in(0,\pi)$ *and arbitrary* $\varepsilon\in(0,\pi)$, $\varepsilon\neq\alpha$,

$$\left.\frac{\partial\nu(\gamma,\beta)}{\partial\gamma}\right|_{\gamma=\alpha+\pi n}=\frac{\sin\varepsilon}{\sin\alpha}\frac{\nu_n(\alpha,\beta)-\nu_n(\varepsilon,\beta)}{\sin(\alpha-\varepsilon)}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\nu_k(\varepsilon,\beta)-\nu_n(\alpha,\beta)}{\nu_k(\alpha,\beta)-\nu_n(\alpha,\beta)}.\tag{1.33}$$

P.5$_-$. For arbitrary $\alpha,\beta\in(0,\pi)$ *and arbitrary* $\eta\in[0,\pi)$, $\eta\neq\beta$,

$$\left.\frac{\partial\nu(\alpha,\delta)}{\partial\delta}\right|_{\delta=\beta-\pi n}=\frac{\sin\eta}{\sin\beta}\cdot\frac{\nu_n(\alpha,\beta)-\nu_n(\alpha,\eta)}{\sin(\beta-\eta)}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\nu_k(\alpha,\eta)-\nu_n(\alpha,\beta)}{\nu_k(\alpha,\beta)-\nu_n(\alpha,\beta)}.$$

P.6$_+$. If $\alpha\in(0,\pi)$

$$\frac{\partial\nu_0(\alpha,\beta)}{\partial\alpha}\sin^2\alpha-\frac{1}{\pi}+\sum_{n=1}^{\infty}\left\{\frac{\partial\nu_n(\alpha,\beta)}{\partial\alpha}\sin^2\alpha-\frac{2}{\pi}\right\}=\cot\alpha.$$

P.6$_-$. *If* $\beta \in (0,\pi)$

$$\frac{\partial \nu_0(\alpha,\beta)}{\partial \beta}\sin^2\beta + \frac{1}{\pi} + \sum_{n=1}^{\infty}\left\{\frac{\partial \nu_n(\alpha,\beta)}{\partial \beta}\sin^2\beta + \frac{2}{\pi}\right\} = \cot\beta. \tag{1.34}$$

P.7. *For arbitrary* $\alpha,\beta \in (0,\pi)$

$$-\frac{\partial \nu_n(\alpha,\beta)}{\partial \alpha}\frac{\partial \nu_n(\alpha,\beta)}{\partial \beta} = \begin{cases} \left[\pi \sin\alpha \sin\beta \prod_{k=1}^{\infty} \dfrac{\mu_k(q,\alpha,\beta)-\mu_0(q,\alpha,\beta)}{k^2}\right]^{-2}, & \text{if } n=0 \\ \left[\dfrac{\pi}{n^2}\left[\mu_0(q,\alpha,\beta)-\mu_n(q,\alpha,\beta)\right]\sin\alpha\sin\beta \prod_{\substack{k=1\\k\neq n}}^{\infty} \dfrac{\mu_k(q,\alpha,\beta)-\mu_n(q,\alpha,\beta)}{k^2}\right]^{-2}, & \text{if } n\neq 0. \end{cases} \tag{1.35}$$

Then there exists unique function $q \in L^1_{\mathbb{R}}[0,\pi]$ *(unique in the sense of* L^1*), such that*

$$\mu_n(q,\alpha,\beta) = \nu_n(\alpha,\beta) \tag{1.36}$$

for all $(\alpha,\beta) \in (0,\pi] \times [0,\pi)$ *and all* $n = 0,1,2,\ldots$, *i.e.* $\nu(\cdot,\cdot)$ *is EVF of the family of operators* $\{L(q,\alpha,\beta);\alpha \in (0,\pi], \beta \in [0,\pi)\}$.

The organization of our paper is as follows: in Section 2 we give the proof of Theorem 1.2, in Section 3 we give some auxiliary results, which must help to prove Theorem 1.3, and in Section 4 we give the proof of Theorem 1.3.

2. The proof of Theorem 1.2.

P.1) The strict increasing of function $\mu^+(\gamma)$ follows from relation (1.14) and the strict decreasing of function $\mu^-(\delta)$ follows from relation (1.15). The proof that the ranges of values of $\mu^+(\cdot)$ and $\mu^-(\cdot)$ are the whole real axis $(-\infty,\infty)$ there is in [4] (see [4], page 289). From this property follows that $\mu_0(q,\alpha,\beta) \to -\infty$ when $\alpha \to 0$ and $\mu_0(q,\alpha,\beta) \to -\infty$ when $\beta \to \pi$. From this property follows also the alternation of eigenvalues $\{\mu_n(q,\alpha,\beta)\}$ and $\{\mu_n(q,\alpha_1,\beta)\}$, if $0 < \alpha < \alpha_1 \le \pi$, and also the alternation of eigenvalues $\{\mu_n(q,\alpha,\beta)\}$ and $\{\mu_n(q,\alpha,\beta_1)\}$, if $0 \le \beta < \beta_1 < \pi$.

It is well-known that alternation of the two spectra corresponding to the points (α,β) and (α_1,β) [or (α,β) and (α,β_1)] it is one of necessary and sufficient conditions for solvability the inverse problem by two spectra (see Borg [19], Marchenko [16], Krein [20], Levitan [21], Gasymov-Levitan [22], Zhikov [13] and others). Now we see that alternation is the corollary of increasing of function μ^+ or decreasing of function μ^-.

P.2) The analyticity of EVF is also proved in [4] (see [4], page 288). It is a new property. Here we achieve one of the main part of our first aim: the dependence of eigenvalues on parameters α and β is analytic and the introduction of EVF shows, in particular, how μ_n analytically turn into μ_{n-1} or μ_{n+1} by edges of the square. It also let us in Theorem 1.3 state the conditions $P.5_{\pm}$ only for interior points of the square $(0,\pi]\times[0,\pi)$.

P.3) The asymptotics of eigenvalues for the cases $\alpha,\beta\in(0,\pi)$ (the interior points) and $\alpha=\pi,\beta=0$ (the corner) proved in [23]. For the cases $\alpha=\pi,\beta\in(0,\pi)$ and $\alpha\in(0,\pi),\beta=0$ (the open edges) it is proved in [29].

P.4$_{\pm}$) The asymptotics of norming constants for the cases $\alpha,\beta\in(0,\pi)$ and $\alpha=\pi,\beta=0$ obtained in [30], for the cases $\alpha=\pi,\beta\in(0,\pi)$ and $\alpha\in(0,\pi),\beta=0$ it is obtained in [29].

P.5$_{\pm}$) The representations of norming constants by two spectra was obtained in [24].

P.6$_{\pm}$) These properties were obtained in [25] and [26], see also [27] and[28].

P.7) Here we give the proof of $P.7$).

Let us write that $\varphi(x,\mu,\alpha)=\varphi(x,\mu)$ (see (1.11)) is the solution of (1.1)

$$-\varphi''(x,\mu)+q(x)\varphi(x,\mu)=\mu\varphi(x,\mu),\qquad \text{(a.e. on } (0,\pi)) \tag{2.1}$$

and let us to take the derivative by μ $\left(\cdot=\dfrac{\partial}{\partial\mu}\right)$ of the last equality (2.1):

$$-\dot\varphi''(x,\mu)+q(x)\dot\varphi(x,\mu)=\varphi(x,\mu)+\mu\dot\varphi(x,\mu). \tag{2.2}$$

If we multiply two hands of (2.1) by $\dot\varphi(x,\mu)$, two hands of (2.2) by $\varphi(x,\mu)$, and subtract from the second equality the first, we will obtain

$$\varphi''(x,\mu)\dot\varphi(x,\mu)-\dot\varphi''(x,\mu)\varphi(x,\mu)\equiv\frac{d}{dx}\left[\varphi'(x,\mu)\dot\varphi(x,\mu)-\dot\varphi'(x,\mu)\varphi(x,\mu)\right]=\varphi^2(x,\mu).$$

Integrating the last equality from 0 to π, we obtain

$$\int_0^\pi\varphi^2(x,\mu,\alpha)\,dx=\varphi'(\pi,\mu,\alpha)\dot\varphi(\pi,\mu,\alpha)-\dot\varphi'(\pi,\mu,\alpha)\varphi(\pi,\mu,\alpha). \tag{2.3}$$

It is easy to see that the value of the expression $\varphi'\dot\varphi-\dot\varphi'\varphi$ in 0 equals 0. From (2.3), (1.12)–(1.13) follow that

$$\begin{aligned}a_n=\int_0^\pi\varphi^2(x,\mu_n,\alpha)\,dx&=c_n\psi'(\pi,\mu_n)\dot\varphi(\pi,\mu_n)-\dot\varphi'(\pi,\mu_n)c_n\psi(\pi,\mu_n)\\&=+c_n\left[-\cos\beta\dot\varphi(\pi,\mu_n)-\dot\varphi'(\pi,\mu_n)\sin\beta\right]\end{aligned}$$

$$= -c_n \frac{\partial}{\partial \mu}\left[\varphi(\pi,\mu_n)\cos\beta + \varphi'(\pi,\mu_n)\sin\beta\right] = -c_n\dot{\Phi}(\mu_n,\alpha,\beta). \quad (2.4)$$

From the other hand, it follow from (1.12) that $a_n = c_n^2 b_n$ and, consequently,

$$b_n = -\frac{1}{c_n}\dot{\Phi}(\mu_n). \quad (2.5)$$

If we multiply (2.4) and (2.5), we will obtain

$$a_n b_n = \left[\dot{\Phi}(\mu_n)\right]^2. \quad (2.6)$$

In paper [24] we have obtained the representation of $\dot{\Phi}(\mu_n)$ in form of infinite product (see [24], page 5):

$$\dot{\Phi}(\mu_n) = \begin{cases} -\pi\sin\alpha\sin\beta \prod_{k=1}^{\infty} \dfrac{\mu_k(q,\alpha,\beta)-\mu_0(q,\alpha,\beta)}{k^2}, & \text{if } \ n=0, \\ -\dfrac{\pi}{n^2}\left[\mu_0(q,\alpha,\beta)-\mu_n(q,\alpha,\beta)\right]\sin\alpha\sin\beta \prod_{\substack{k=1 \\ k\neq n}}^{\infty} \dfrac{\mu_k(q,\alpha,\beta)-\mu_n(q,\alpha,\beta)}{k^2}, & \text{if } \ n\neq 0, \end{cases}$$

Together with (2.6) it gives formula (1.27). Thus, *P*.7) is proved.

3. Auxiliary results

In many papers (see, e.g., [13], [15], [22]), where considered the cases $\sin\alpha \neq 0$ and $\sin\beta \neq 0$, under norming constants understand the quantities $\tilde{a}_n = \dfrac{a_n}{\sin^2\alpha}$ and $\tilde{b}_n = \dfrac{b_n}{\sin^2\beta}$, $n = 0,1,2,\dots$.

In [26] the following results were obtained:

Theorem 3.1 ([26])**.** *For a real increasing sequence $\{\mu_n\}_{n=0}^{\infty} = \{\lambda_n^2\}_{n=0}^{\infty}$ and a positive sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ to be the set of eigenvalues and the set of norming constants respectively for boundary-value problem $L(q,\alpha,\beta)$ with a $q \in L^1_{\mathbb{R}}[0,\pi]$ and fixed $\alpha,\beta \in (0,\pi)$ it is necessary and sufficient that the following relations hold:*

1) *the sequence $\{\lambda_n\}_{n=0}^{\infty}$ has asymptotic form*

$$\lambda_n = n + \frac{\omega}{n} + l_n, \quad (3.1a)$$

where $\omega = const$,

$$l_n = o\left(\frac{1}{n}\right), \quad when \quad n \to \infty, \quad (3.1b)$$

and the function $l(\cdot)$, defined by formula

$$l(x) := \sum_{n=1}^{\infty} l_n \sin nx, \tag{3.1c}$$

is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$, i.e.

$$l \in AC(0,2\pi); \tag{3.1d}$$

2) *the sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ has asymptotic form*

$$\tilde{a}_n = \frac{\pi}{2} + s_n, \tag{3.2a}$$

where

$$s_n = o\left(\frac{1}{n}\right), \quad when \quad n \to \infty, \tag{3.2b}$$

and the function $s(\cdot)$, defined by formula

$$s(x) := \sum_{n=1}^{\infty} s_n \cos nx, \tag{3.2c}$$

is absolutely continuous on arbitrary segment $[a,b] \subset (0,2\pi)$, i.e.

$$s \in AC(0,2\pi); \tag{3.2d}$$

3)

$$\frac{1}{\tilde{a}_0} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{1}{\tilde{a}_n} - \frac{2}{\pi}\right) = \cot\alpha, \tag{3.3}$$

4)

$$\frac{\tilde{a}_0}{\left(\pi \prod_{k=1}^{\infty} \frac{\mu_k - \mu_0}{k^2}\right)^2} - \frac{1}{\pi} + \sum_{n=1}^{\infty} \left(\frac{\tilde{a}_n n^4}{\left(\pi [\mu_0 - \mu_n] \prod_{\substack{k=1 \\ k \neq n}}^{\infty} \frac{\mu_k - \mu_n}{k^2}\right)^2} - \frac{2}{\pi} \right) = -\cot\beta. \tag{3.4}$$

As the corollary of the sufficient part of this theorem we want to have the following theorem:

Theorem 3.2. *If a function $\nu(\cdot,\cdot)$ satisfies the conditions of Theorem 1.3, then for each point $(\alpha,\beta) \in (0,\pi) \times (0,\pi)$ there exists unique function $q \in L^1_{\mathbb{R}}[0,\pi]$, s.t.*

$$\mu_n(q,\alpha,\beta) = \nu_n(\alpha,\beta), \tag{3.5}$$

$$\frac{1}{a_n(q,\alpha,\beta)} = \left.\frac{\partial \nu(\gamma,\beta)}{\partial \gamma}\right|_{\gamma=\alpha+\pi n} = \frac{\partial \nu_n(\alpha,\beta)}{\partial \alpha}, \tag{3.6}$$

$$-\frac{1}{b_n(q,\alpha,\beta)} = \left.\frac{\partial \nu(\alpha,\delta)}{\partial \delta}\right|_{\delta=\beta-\pi n} = \frac{\partial \nu_n(\alpha,\beta)}{\partial \beta}, \tag{3.7}$$

for all $n = 0,1,2,\ldots$.

Indeed, since $\nu(\gamma,\delta)$ is increasing by γ (property 1), then the sequence $\{\nu_n\}_{n=0}^{\infty}$ defined in (1.29) by

$$\nu_n(\alpha,\beta)=\nu(\alpha+\pi n,\beta)=\nu(\alpha,\beta-\pi n),\qquad n=0,1,2,\dots,$$

is a real increasing sequence, and since for $\alpha,\beta\in(0,\pi)$, $\delta_n(\alpha,\beta)=O\left(\frac{1}{n}\right)$ (see (1.10)), then (according to the property $P.3$ $\sqrt{\nu_n}$ can be represented in the form

$$\lambda_n=\sqrt{\nu_n}=n+\frac{\omega}{n}+l_n,$$

where $\omega=const=\dfrac{\cot\beta-\cot\alpha}{2}+\dfrac{c}{2}$, and l_n satisfy to the conditions (3.1b)–(3.1d);

If we take as the sequence $\tilde{a}_n$

$$\tilde{a}_n=\left[\frac{\partial\nu(\gamma,\beta)}{\partial\gamma}\bigg|_{\gamma=\alpha+\pi n}\right]^{-1}\frac{1}{\sin^2\alpha}$$

then this sequence $\{\tilde{a}_n\}_{n=0}^{\infty}$ will satisfy to the conditions (3.2a)–(3.2d) (according to property $P.4_+$);

According to the property $P.6_+$ of Theorem 1.3 we have also the condition (3.3) of Theorem 3.1;

If we define the quantity $\dfrac{\partial\nu_n(\alpha,\beta)}{\partial\beta}$ from $P.7$ and substitute into equation (1.34) of $P.6_-$, then we will obtain the condition (3.4).

Thus all the conditions of Theorem 3.1 are satisfied. So there exists $q\in L^1_{\mathbb{R}}[0,\pi]$, s.t. the connections (3.5)–(3.7) hold for arbitrary $\alpha,\beta\in(0,\pi)$. Here arises a question: is the function q still remaining the same, when we change α and β? The answer is "yes". It will be proved in Section 4.

Now we formulate a theorem about the solution of the inverse Sturm-Liouville problem "by two spectra".

Theorem 3.3.

1) *Let a function $\nu(\cdot,\cdot)$ satisfy the conditions of Theorem 1.3. Then for arbitrary two points (α,β) and (α_1,β), such that $\alpha\neq\alpha_1$ and $\alpha,\alpha_1,\beta\in(0,\pi)$ there exists a unique function $q\in L^1_{\mathbb{R}}[0,\pi]$, such that*

$$\nu_n(\alpha,\beta)=\mu_n(q,\alpha,\beta),$$
$$\nu_n(\alpha_1,\beta)=\mu_n(q,\alpha_1,\beta),$$

for all $n=0,1,2,\dots$.

2) *Analogously, for arbitrary two points* (α,β) *and* (α,β_1), *such that* $\beta\neq\beta_1$ *and* $\alpha,\beta,\beta_1\in(0,\pi)$ *there exists a unique function* $q\in L^1_{\mathbb{R}}[0,\pi]$, *such that*

$$\nu_n(\alpha,\beta)=\mu_n(q,\alpha,\beta),$$
$$\nu_n(\alpha,\beta_1)=\mu_n(q,\alpha,\beta_1),$$

for all $n=0,1,2,\dots$.

This theorem follows from the theorem of Marchenko, which have been proved in [16] (pp.386–394) and which can be rewritten in the following statement:

Theorem 3.4. *Let* $\alpha,\alpha_1,\beta,\beta_1\in(0,\pi)$ *and* c *is some constant.*

1) *Let two sequences* ν_n *and* $\tilde\nu_n$ *alternate and have the asymptotics* (*when* $n\to\infty$)

$$\sqrt{\nu_n}=n+\frac{1}{\pi n}\left[\cot\beta-\cot\alpha+c\right]+o\left(\frac{1}{n}\right)$$
$$\sqrt{\tilde\nu_n}=n+\frac{1}{\pi n}\left[\cot\beta-\cot\alpha_1+c\right]+o\left(\frac{1}{n}\right)$$

Then there exists a unique $q\in L^1_{\mathbb{R}}[0,\pi]$, *such that*

$$\nu_n=\mu_n(q,\alpha,\beta),$$
$$\tilde\nu_n=\mu_n(q,\alpha_1,\beta),$$

for all $n=0,1,2,\dots$.

2) *Let two sequences* ν_n *and* $\tilde\nu_n$ *alternate and have the asymptotics (when* $n\to\infty$*)*

$$\sqrt{\nu_n}=n+\frac{1}{\pi n}\left[\cot\beta-\cot\alpha+c\right]+o\left(\frac{1}{n}\right)$$
$$\sqrt{\tilde\nu_n}=n+\frac{1}{\pi n}\left[\cot\beta_1-\cot\alpha+c\right]+o\left(\frac{1}{n}\right)$$

Then there exists a unique $q\in L^1_{\mathbb{R}}[0,\pi]$, *such that*

$$\nu_n=\mu_n(q,\alpha,\beta),$$
$$\tilde\nu_n=\mu_n(q,\alpha,\beta_1),$$

for all $n=0,1,2,\dots$.

Let us give a statement of the famous uniqueness theorem of Marchenko.

Theorem 3.5 (Marchenko [16])**.** *Let $\alpha,\beta\in(0,\pi)$ and $q\in L^1_{\mathbb{R}}[0,\pi]$. If*

$$\mu_n(q,\alpha,\beta)=\mu_n(\tilde q,\tilde\alpha,\tilde\beta)$$

and

$$a_n(q,\alpha,\beta)=a_n(\tilde q,\tilde\alpha,\tilde\beta)$$
$$(\text{or } b_n(q,\alpha,\beta)=b_n(\tilde q,\tilde\alpha,\tilde\beta)) \tag{3.8}$$

for all $n=0,1,2,\ldots$, then $q(x)=\tilde q(x)$ a.e., $\alpha=\tilde\alpha$ and $\beta=\tilde\beta$.

4. The proof of Theorem 1.3

The plan of the proof of Theorem 1.3 is the following. In open square $(0,\pi)\times(0,\pi)$ we take two arbitrary different points (α,β) and (α_1,β_1).

According to Theorem 3.2 for arbitrary point $(\alpha,\beta)\in(0,\pi)\times(0,\pi)$ there exists a unique function $q\in L^1_{\mathbb{R}}[0,\pi]$, such that the connections (3.5)–(3.7) hold. Correspondingly, for (α_1,β_1) there exists $q_1\in L^1_{\mathbb{R}}[0,\pi]$, such that the connections

$$\begin{aligned}
&\mu_n(q_1,\alpha_1,\beta_1)=\nu_n(\alpha_1,\beta_1),\\
&\frac{1}{\tilde a_n(q_1,\alpha_1,\beta_1)\sin^2\alpha}=\frac{1}{a_n(q_1,\alpha_1,\beta_1)}=\left.\frac{\partial\nu(\gamma,\beta_1)}{\partial\gamma}\right|_{\gamma=\alpha_1+\pi n},\\
&-\frac{1}{\tilde b_n(q,\alpha,\beta)\sin^2\beta}=-\frac{1}{b_n(q,\alpha,\beta)}=\left.\frac{\partial\nu(\alpha,\delta)}{\partial\delta}\right|_{\delta=\beta-\pi n},
\end{aligned}$$

hold. If we prove that $q(x)=q_1(x)$, a.e. on $(0,\pi)$, it will mean that all the points of open square $(0,\pi)\times(0,\pi)$ "generate" the same function q, such that connections (3.5)–(3.7) hold for all $(\alpha,\beta)\in(0,\pi)\times(0,\pi)$. Now, if by this unique q we construct the EVF $\mu(q,\gamma,\delta)$ of a family $\{L(q,\alpha,\beta),\alpha\in(0,\pi],\beta\in[0,\pi)\}$, then, since the equalities

$$\mu_n(q,\alpha,\beta)=\nu_n(\alpha,\beta)$$

will be true for all $\alpha,\beta\in(0,\pi)$ and all $n=0,1,2,\ldots$, then the constructed EVF $\mu(q,\cdot,\cdot)$ will coincide with $\nu(\cdot,\cdot)$ an all open square $(\pi k,\pi(k+1))\times(-\pi m,-\pi(m-1))$, $k,m=0,1,2,\ldots$ (see Figure 2), which lie inside of $(0,\infty)\times(-\infty,\pi)$. Since $\mu(q,\cdot,\cdot)$ and $\nu(\cdot,\cdot)$ are continuous (more, analytic!), they will coincide also on the closure of these squares, and therefore on the all domain $(0,\infty)\times(-\infty,\pi)$. Thus, the proof of Theorem 1.3 reduced to prove that $q(x)=q_1(x)$, a.e. on $(0,\pi)$. With this aim we consider the point (α_1,β), for which, according to the Theorem 3.2 there exists a unique function $q_2\in L^1_{\mathbb{R}}[0,\pi]$, such that

$$\nu_n(\alpha_1,\beta)=\mu_n(q_2,\alpha_1,\beta), \tag{4.1}$$

$$\left.\frac{\partial \nu(\gamma,\beta)}{\partial \gamma}\right|_{\gamma=\alpha_1+\pi n} = \frac{1}{a_n(q_2,\alpha_1,\beta)}, \tag{4.2}$$

$$\left.\frac{\partial \nu(\alpha_1,\delta)}{\partial \delta}\right|_{\delta=\beta-\pi n} = -\frac{1}{b_n(q_2,\alpha_1,\beta)}, \tag{4.3}$$

for all $n = 0,1,2,\ldots$.

Now we will use Theorem 3.3 to prove that from the one hand $q(x) = q_2(x)$ and from the other hand $q_1(x) = q_2(x)$ and, therefore, $q(x) = q_1(x)$.

Indeed, according to Theorem 3.3, there exists a unique function $\tilde{q} \in L^1_{\mathbb{R}}[0,\pi]$, such that

$$\nu_n(\alpha,\beta) = \mu_n(\tilde{q},\alpha,\beta), \tag{4.4}$$

$$\nu_n(\alpha_1,\beta) = \mu_n(\tilde{q},\alpha_1,\beta). \tag{4.5}$$

If we construct by these two sequences the expression

$$\frac{\sin\alpha_1}{\sin\alpha}\,\frac{\nu_n(\alpha,\beta)-\nu_n(\alpha_1,\beta)}{\sin(\alpha-\alpha_1)}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\nu_k(\alpha_1,\beta)-\nu_n(\alpha,\beta)}{\nu_k(\alpha,\beta)-\nu_n(\alpha,\beta)}, \tag{4.6}$$

we obtain, according to the property $P.5_+$ of Theorem 1.3, $\left.\frac{\partial \nu(\gamma,\beta)}{\partial\gamma}\right|_{\gamma=\alpha+\pi n}$ and, from the other hand, according to (3.6), $\frac{1}{a_n(q,\alpha,\beta)}$. But, since there exists $\tilde{q} \in L^1_{\mathbb{R}}[0,\pi]$, such that (4.4) and (4.5) hold, then according to Theorem 1.2 (property $P.5_+$)) the same expression (4.6) equals $\frac{1}{a_n(\tilde{q},\alpha,\beta)}$. Thus we have

$$a_n(q,\alpha,\beta) = a_n(\tilde{q},\alpha,\beta)$$

and

$$\nu_n(\alpha,\beta) = \mu_n(q,\alpha,\beta) = \mu_n(\tilde{q},\alpha,\beta)$$

for all $n = 0,1,2,\ldots$. By Theorem 3.5 it follows that $q(x) = \tilde{q}(x)$, a.e. on $(0,\pi)$.

On the other hand, if by the same two sequences (4.4) and (4.5) we construct expression

$$\frac{\sin\alpha}{\sin\alpha_1}\,\frac{\nu_n(\alpha_1,\beta)-\nu_n(\alpha,\beta)}{\sin(\alpha_1-\alpha)}\prod_{\substack{k=0\\k\neq n}}^{\infty}\frac{\nu_k(\alpha,\beta)-\nu_n(\alpha_1,\beta)}{\nu_k(\alpha_1,\beta)-\nu_n(\alpha_1,\beta)}, \tag{4.7}$$

we obtain, according the property $P.5_+$ of Theorem 1.3, $\left.\frac{\partial \nu(\gamma,\beta)}{\partial\gamma}\right|_{\gamma=\alpha_1+\pi n}$, which, from the other hand, according to (4.2), equals $\frac{1}{a_n(q_2,\alpha_1,\beta)}$.

In the same time if we change in (4.7) $\nu_n(\alpha,\beta)$ by $\mu_n(\tilde{q},\alpha,\beta)$ and $\nu_n(\alpha_1,\beta)$ by $\mu_n(\tilde{q},\alpha_1,\beta)$, according to (4.4) and (4.5), we obtain, that (4.7) equals $\frac{1}{a_n(\tilde{q},\alpha_1,\beta)}$, according to property $P.5_+$) of Theorem 1.2. So

$$\frac{1}{a_n(q_2,\alpha_1,\beta)} = \frac{1}{a_n(\tilde{q},\alpha_1,\beta)}. \tag{4.8}$$

From (4.1) and (4.5) we have

$$\mu_n(q_2, \alpha_1, \beta) = \mu_n(\tilde{q}, \alpha_1, \beta). \tag{4.9}$$

Thus, from (4.8) and (4.9), according theorem of Marchenko (Theorem 3.5),

$$q_2(x) = \tilde{q}(x).$$

We obtain, that $q(x) = q_2(x)$.

If we repeat all these steps for the couple q_1 and q_2 (using the properties $P.5_{-}$) and $P.5_{-}$ in Theorems 1.2 and 1.3 and the second parts of Theorems 3.3, 3.5, we will obtain that $q_1(x) = q_2(x)$ a.e. on $[0, \pi]$. Thus, Theorem 1.3 is proved.

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