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Integral representations in general weighted Bergman spaces

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We introduce "fractional" ω-integro-differentiation for functions holomorphic in the upper half-plane. It gives us a tool to construct Cauchy–Bergman type kernels associated with the weights ω. Some estimates of the kernels enable us to obtain reproducing integral formulas for Bergman spaces with general weights which may decrease to zero with arbitrary rate near the origin. Accordingly, such Bergman functions have arbitrary growth near the real axis.

Keywords: Bergman type kernels; ω-Integration; ω-Differentiation; General weighted Bergman spaces

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1. Introduction

Let $\mathbb{R}^2_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the upper half-plane on the complex plane and $H(\mathbb{R}^2_+)$ be the set of all holomorphic functions in $\mathbb{R}^2_+$. For $0 < p < \infty$ let $H^p = H^p(\mathbb{R}^2_+)$ be the usual Hardy space on $\mathbb{R}^2_+$. Denote by L^p_ω the collection of those functions $f(z)$ measurable in $\mathbb{R}^2_+$, for which the (quasi-)norm

$$\|f\|_{p,\omega} = \left(\iint_{\mathbb{R}^2_+} |f(x+iy)|^p \omega(2y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$$

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is finite, where $0 < p < \infty$ and ω is a weight function. For the subspace of L^p_ω consisting of holomorphic functions let $H^p_\omega = H(\mathbb{R}^2_+) \cap L^p_\omega$.

General weighted Bergman spaces have been studied, for example, in [1–8] (see also references therein) in the context of the unit disk, unit ball or polydisk in $\mathbb{C}^n$, while general weighted Bergman spaces on the half-plane are studied much lesser. Following Shields and Williams [7], all these authors consider weight functions satisfying certain restrictions on regularity of growth. Therefore, their techniques are closely related to those of the weighted Bergman spaces H^p_α with standard power weights $\omega(r) = (1-r)^{\alpha-1}$ for the unit disk (ball or polydisk) and $\omega(y) = y^{\alpha-1}$ $(\alpha > 0)$ for the upper half-plane (half-space). More general weight functions are studied in [9,10] for mixed norm spaces with $1 \le p \le 2$ on tube domains. Note that Fourier–Plancherel techniques are mainly used in [9,10], so the proofs in [9,10] strongly depended on the assumption that $1 \le p \le 2$.

In contrast to [9,10], our proofs are essentially based on the techniques of "fractional" integro-differentiation associated with a weight function ω, as well as on estimates of Cauchy–Bergman type kernels K_ω. This makes it possible to obtain the results for all $p, 1 \le p < \infty$. In the present article, we first define the class W of radial weight functions $\omega(y)$ which *may decrease to zero with arbitrary rate* as $y \to +0$. Accordingly, we deal with much wider spaces H^p_ω whose members have arbitrary growth near the real axis. Then we introduce "fractional" ω-integro-differentiation for functions holomorphic in the upper half-plane. It gives us a tool to construct Cauchy–Bergman type kernels K_ω associated with the weights ω. Then some estimates of the kernels enable us in Theorem 1 to obtain reproducing integral formulas for Bergman spaces H^p_ω $(1 \le p < \infty)$ with the general weights from W. Furthermore, we give in Theorem 2 another characterization of H^2_ω by means of a convolution type integral formula. To this end, we solve an integral equation of the first kind.

1.1. *Main theorems*

Throughout the article, $z = x + iy$, $\zeta = \xi + i\eta$, $f_y(x) = f(x + iy)$ and the letters $C(\alpha, \beta, \omega, \ldots)$, C_α etc. stand for positive different constants depending only on the parameters indicated. The notation $A \approx B$ means that the quotient of two quantities A and B is bounded above and below by some positive constants when the variable varies. For any p, $1 \le p \le \infty$, we define the conjugate index p' as $p' = p/(p-1)$. We shall write $T : X \longrightarrow Y$, if T is a bounded operator mapping X to Y, i.e., $\|Tf\|_Y \le C\|f\|_X$ $\forall f \in X$.

Definition The function $\omega(x)$ positive and continuous on $(0, \infty)$ is said to belong to the class $W(= W_{\delta,\alpha})$ if there exist $\delta, \alpha > 0$ such that $\omega(x) = O(x^{\delta-1})$ as $x \to +0$, and $\omega(x) \approx x^{\alpha-1}$ as $x \to +\infty$.

The last condition at the infinity may be weakened, but it is without importance for us here. The following typical weight functions are in W:

$$x^{\alpha-1}, \quad e^{-1/x}, \quad x^{\alpha-1}e^{-\beta/x}, \quad \exp\bigl(-e^{1/x}\bigr), \quad \exp\bigl(-\exp\bigl(e^{1/x}\bigr)\bigr) \quad \text{etc.},$$

where $\alpha, \beta > 0$. We now formulate the main theorems of the article.

THEOREM 1 *If $1 \le p < \infty, \omega \in W$, then any function $f \in H^p_\omega$ is representable in the form*

$$f(z) = \frac{1}{\pi} \iint_{\mathbb{R}^2_+} f(\zeta)\, K_\omega(z, \zeta)\, \omega(2\eta)\, \mathrm{d}\xi \mathrm{d}\eta, \quad z \in \mathbb{R}^2_+, \tag{1.1}$$

where the integral converges absolutely and uniformly in each half-plane $\mathbb{R} \times (\rho, \infty)$, $\rho > 0$, and the kernel K_ω of Cauchy–Bergman type is defined in section 4.

Remark In the special case $\omega(x) = (1/\Gamma(\alpha))x^{\alpha-1}(\alpha > 0)$ the representation (1.1) coincides with that in [11,12] (see also [13] for harmonic mixed-norm spaces on $\mathbb{R}^{n+1}_+$). For tube domains and $1 \le p \le 2$, (1.1) is obtained by Karapetyan [9,10] by a different method.

There are many generalizations of the Wiener–Paley integral representation, see, e.g., [9,10] and references therein. In the next theorem we obtain another Cauchy type formula with the use of K_ω that gives a new characterization of the space H^2_ω.

THEOREM 2 *The space H^2_ω ($\omega \in W$) coincides with the set of functions $f(z)$ representable in the form*

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} K_{\omega_1}(z, t)\, \varphi(t)\, \mathrm{d}t, \quad z \in \mathbb{R}^2_+, \tag{1.2}$$

where $\varphi(t) \in L^2(\mathbb{R})$, and ω_1 is a weight function determined by the integral equation $\omega(x) = \int_0^x \omega_1(x-t)\omega_1(t)\,\mathrm{d}t$. The operator $\varphi \mapsto f$ given by (1.2) *presents an isometric isomorphism of $L^2(\mathbb{R})$ onto H^2_ω.*

The explicit formula for ω_1 is given in section 3. For the special weight $\omega(x) = (1/\Gamma(\alpha))x^{\alpha-1}$ ($\alpha > 0$) Theorem 2 is proved in [13] in the setting of the upper half-space $\mathbb{R}^{n+1}_+$.

2. Preliminaries

For weight functions $\omega \in W$ we consider their Laplace transforms

$$\mathcal{L}_\omega(t) = \int_0^\infty \omega(x) e^{-tx} \mathrm{d}x, \quad t > 0.$$

The first lemma contains some estimates for $\mathcal{L}_\omega(t)$ and its derivatives.

LEMMA 1

(i) *If the weight function ω is in $W_{\delta,\alpha}$ for some $\delta, \alpha > 0$, then for any $k = 0, 1, 2, \ldots$*

$$\left|\mathcal{L}^{(k)}_\omega(t)\right| \le C \frac{1}{t^{\delta+k}}, \quad t \ge 1, \tag{2.1}$$

$$\left|\mathcal{L}^{(k)}_\omega(t)\right| \approx \frac{1}{t^{\alpha+k}}, \quad 0 < t \le 1. \tag{2.2}$$

(ii) *Suppose that the Laplace transform* $\mathcal{L}_\omega(t)$ *of a positive and continuous function* $\omega(x)$ *converges for any* $t > 0$. *If* (2.1) *and* (2.2) *hold for* $k=0$ *and some* $\delta, \alpha > 0$, *then* $\omega \in W_{\delta,\alpha}$.

Lemma 1 follows immediately from known properties of Laplace transforms, see, e.g., [14].

Lemma 2 *Let* $0 < p < \infty$ *and* $\omega \in W$. *Then any function* $f(z) \in H^p_\omega$ *satisfies the following estimates*

(i)

$$M_p^p(f;y) \le \frac{2\|f\|_{p,\omega}^p}{y \cdot \min_{y \le \eta \le 3y} \omega(\eta)}, \quad y > 0,$$

where $M_p(f;y) = \|f(x+iy)\|_{L^p(dx)}$. *In particular, the function* $f(z)$ *is in Hardy space* H^p *over each half-plane* $\mathbb{R} \times (\rho, \infty)$, $\rho > 0$.

(ii)

$$|f(x+iy)|^p \le \frac{2\|f\|_{p,\omega}^p}{y \cdot \int_{y/2}^{y} \omega(\eta)\,d\eta}, \quad z = x+iy \in \mathbb{R}^2_+.$$

In particular, every point-evaluation in $\mathbb{R}^2_+$ *is a bounded linear functional on* H^p_ω.

Lemma 3

(i) *For* $0 < p < \infty$ *and* $\omega \in W$, H^p_ω *is a closed and complete subspace of* L^p_ω, *and hence is a Banach space if* $1 \le p < \infty$.

(ii) *If* $0 < p < \infty$, $\omega \in W$, *then* $\|f_\rho - f\|_{p,\omega} \longrightarrow 0$ *as* $\rho \to +0$ *for every* $f(z) \in H^p_\omega$ *and its dilated function* $f_\rho(z) = f(z+i\rho)$.

The proof of Lemma 2 is standard, so we omit it. Lemma 3 follows from Lemma 2.

3. Operators of ω-integro-differentiation

Recall the classical Wiener–Paley theorem, see, e.g., [15].

Theorem A

(i) *If* $f(z) \in H^p(\mathbb{R}^2_+)$ *for some* $p, 1 \le p \le 2$, *then*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz} \widehat{f}(t)\,dt, \quad z \in \mathbb{R}^2_+, \tag{3.1}$$

where $\widehat{f}(t)$ *is the Fourier transform of the boundary function* $f(x)$. *Besides,* $\widehat{f}(t) = 0$ *for almost all* $t \le 0$.

(ii) *If a function* $f(z)$ *can be represented in the form*

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{itz} F(t)\,dt, \quad z \in \mathbb{R}^2_+, \tag{3.2}$$

for some $F(t) \in L^2(\mathbb{R})$, *then* $f(z)$ *is in* H^2.

Introduce now an operator of ω-integration

$$\mathfrak{I}^{\omega} f(z) = \int_0^{\infty} \omega(\eta) f(z + i\eta)\,\mathrm{d}\eta, \quad z \in \mathbb{R}^2_+.$$

In the special case $\omega(\eta) = (1/\Gamma(a))\eta^{\alpha-1}$ $(\alpha > 0)$ the operator $\mathfrak{I}^{\omega}$ reduces to the well-known fractional integration of Riemann–Liouville. In order to define the inverse operator, assume that a function $f(z)$ is representable by the convergent integral (3.2) with some $F(t)$ which is not necessarily in L^2. Then we define

$$\mathcal{D}^{\omega} f(z) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{e^{itz} F(t)}{\mathcal{L}_{\omega}(t)}\,\mathrm{d}t, \quad z \in \mathbb{R}^2_+.$$

Next we show the invertibility of the operators $\mathfrak{I}^{\omega}$ and $\mathcal{D}^{\omega}$.

We now pose a question: whether the weight function $\omega \in W$ can be represented by

$$\omega(x) = \int_0^{x} \omega_1(x - \xi)\,\omega_1(\xi)\,\mathrm{d}\xi \tag{3.3}$$

with some weight function ω_1. The following lemma solves the integral equation (3.3) in the same class W.

Lemma 4 *For $\omega \in W$ the integral equation* (3.3) *of the first kind has a solution in the class W. More precisely, if $\omega \in W_{\delta,\alpha}$ for some $\delta, \alpha > 0$, then there exists a solution* (*not necessarily positive*) $\omega_1 \in W_{\delta/2,\alpha/2}$.

Proof Assume that a weight function $\omega \in W_{\delta,\alpha}$ is represented by (3.3). A passage to Laplace transforms yields

$$\mathcal{L}_{\omega_1}(t) = \sqrt{\mathcal{L}_{\omega}(t)}. \tag{3.4}$$

The inverse Laplace formula gives for any $a > 0$

$$\omega_1(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\xi x} \mathcal{L}_{\omega_1}(\xi)\,\mathrm{d}\xi = \frac{e^{ax}}{2\pi} \int_{\mathbb{R}} e^{i\eta x} \sqrt{\mathcal{L}_{\omega}(a + i\eta)}\,\mathrm{d}\eta, \quad x > 0. \tag{3.5}$$

The equality (3.4) shows that $\mathcal{L}_{\omega_1}(t)$ satisfies the estimates (2.1) and (2.2) with $k=0$ and δ, α replaced by $\delta/2$, $\alpha/2$. By Lemma 1, $\omega_1 \in W_{\delta/2,\alpha/2}$. ■

Here we give a few examples of the weight functions ω and ω_1. We omit the routine computation of ω_1.

Example 1 If $\omega(x) = (1/\Gamma(\alpha))x^{\alpha-1}$ $(\alpha > 0)$, then $\omega_1(x) = (1/\Gamma(\alpha/2))x^{\alpha/2-1}$.

Example 2 If $\omega(x) = x^{-1/2} e^{-1/x}$, then for any $a > 0$

$$\omega_1(x) = \frac{\pi^{1/4}}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta^{-1/4} e^{x\zeta - \sqrt{\zeta}}\,\mathrm{d}\zeta = \frac{2}{\pi^{3/4}} \int_0^{\infty} \sqrt{t}\, e^{-xt^2} \sin\!\left(t + \frac{\pi}{4}\right) \mathrm{d}t.$$

Example 3 If $\omega(x) = x^{\alpha-1}e^{-\beta/4x}$ ($\alpha \in \mathbb{R}, \beta > 0$), then for any $a > 0$

$$\omega_1(x) = \left(\frac{\beta}{4}\right)^{\alpha/4} \frac{1}{\pi i\sqrt{2}} \int_{a-i\infty}^{a+i\infty} \zeta^{-\alpha/4} e^{x\zeta} \sqrt{\mathbf{K}_\alpha\left(\sqrt{\beta\zeta}\right)}\, \mathrm{d}\zeta,$$

where $\mathbf{K}_\alpha$ is Macdonald's function, see, e.g., [15].

The following theorem is the main result of this section.

Theorem 3 *Suppose that* ω *is in* W *and the weight* ω_1 *is determined by* (3.3). *Then*:

(i)

$$\mathfrak{I}^\omega : H^1_\omega \longrightarrow H^1, \quad \text{that is} \quad \|\mathfrak{I}^\omega f\|_{H^1} \leq C\|f\|_{1,\omega}, \tag{3.6}$$

$$\mathfrak{I}^{\omega_1} : H^2_\omega \longrightarrow H^2, \quad \text{and moreover} \quad \|\mathfrak{I}^{\omega_1} f\|_{H^2} = \|f\|_{2,\omega}, \tag{3.7}$$

$$\mathcal{D}^{\omega_1} : H^2 \longrightarrow H^2_\omega, \quad \text{and moreover} \quad \|\mathcal{D}^{\omega_1}\varphi\|_{2,\omega} = \|\varphi\|_{H^2}. \tag{3.8}$$

(ii) *If* $1 \leq p \leq 2$ *and* $f(z) \in H^p(\mathbb{R} \times (\rho, \infty))$ *for any* $\rho > 0$, *then*

$$\mathfrak{I}^\omega \mathcal{D}^\omega f(z) = f(z), \quad z \in \mathbb{R}^2_+. \tag{3.9}$$

(iii)

$$\text{If } f(z) \in H^2_\omega, \quad \text{then} \quad \mathcal{D}^{\omega_1}\mathfrak{I}^{\omega_1} f(z) = f(z). \tag{3.10}$$

$$\text{If } f(z) \in H^1_\omega, \quad \text{then} \quad \mathcal{D}^{\omega}\mathfrak{I}^{\omega} f(z) = f(z). \tag{3.11}$$

Proof The proof of (3.6) is straightforward. Proceeding to the proof of (3.7), let $f(z) \in H^2_\omega$. By Lemma 2, $f(z) \in H^2$ $(\mathbb{R} \times (\rho, \infty))$ for any $\rho > 0$. As is well known [15],

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t)e^{itz}\, \mathrm{d}t, \quad z \in \mathbb{R}^2_+, \tag{3.12}$$

where the function $g(t) = e^{ty}\widehat{f_y}(t)$ does not depend on y, and besides Parseval's equation yields

$$\begin{aligned} \|f\|^2_{2,\omega} &= \int_0^\infty \omega(2y)\|e^{-ty}g(t)\|^2_{L^2(\mathrm{d}t)}\, \mathrm{d}y \\ &= \int_0^\infty |g(t)|^2 \mathcal{L}_\omega(t)\, \mathrm{d}t = \|g(t)\mathcal{L}_{\omega_1}(t)\|^2_{L^2(\mathrm{d}t)}. \end{aligned}$$

Hence, for the function $\varphi(z) = \mathfrak{I}^{\omega_1} f(z)$ we have

$$\varphi(z) = \int_0^\infty \omega_1(\eta)\left(\frac{1}{\sqrt{2\pi}} \int_0^\infty g(t)e^{it(z+i\eta)}\mathrm{d}t\right)\mathrm{d}\eta = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(t)e^{itz}\mathcal{L}_{\omega_1}(t)\, \mathrm{d}t.$$

Therefore, by Theorem A, $\varphi(z) \in H^2$, and $\widehat{\varphi}(t) = g(t)\mathcal{L}_{\omega_1}(t) = 0$ for $t < 0$. So, $\|\varphi\|_{H^2} = \|\widehat{\varphi}\|_{L^2(\mathbb{R})} = \|g(t)\mathcal{L}_{\omega_1}(t)\|_{L^2(0,\infty)} = \|f\|_{2,\omega}$. The relation (3.8) can be proved in a similar way. The inversion formula (3.9) follows from definitions and Theorem A

$$\begin{aligned}\mathfrak{I}^{\omega}\mathcal{D}^{\omega}f(z) &= \int_0^\infty \omega(\eta)\left(\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\widehat{f}(t)e^{it(z+i\eta)}}{\mathcal{L}_\omega(t)}\,\mathrm{d}t\right)\mathrm{d}\eta \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \widehat{f}(t)\,e^{itz}\mathrm{d}t = f(z), \qquad (3.13)\end{aligned}$$

where $\widehat{f} \in L^{p'}(0,\infty)$. Proceeding to the proof of (3.10) note that the Fourier transform of the dilated function $f_\rho(z) \in H^2$ $(\rho > 0)$ commutes with the operator $\mathfrak{I}^{\omega_1}$. Indeed,

$$\begin{aligned}\left(\widehat{\mathfrak{I}^{\omega_1}f_\rho}\right)(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathfrak{I}^{\omega_1}f_\rho(\xi)\frac{e^{-i\xi x}-1}{-i\xi}\,\mathrm{d}\xi \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2\pi}}\int_0^\infty \omega_1(\eta)\left(\int_{\mathbb{R}} f_\rho(\xi+i\eta)\frac{e^{-i\xi x}-1}{-i\xi}\,\mathrm{d}\xi\right)\mathrm{d}\eta \\ &= \int_0^\infty \omega_1(\eta)\widehat{f_\rho}(x+i\eta)\,\mathrm{d}\eta = \left(\mathfrak{I}^{\omega_1}\widehat{f_\rho}\right)(x), \quad x > 0. \qquad (3.14)\end{aligned}$$

Here the differentiation under the integral sign is justified in view of $\omega_1 \in W_{\delta/2,\alpha/2}$ and the uniform convergence of the integral

$$\int_0^\infty \omega_1(\eta)\widehat{f_\rho}(x+i\eta)\,\mathrm{d}\eta = g(x)e^{-x\rho}\int_0^\infty \omega_1(\eta)\,e^{-x\eta}\mathrm{d}\eta$$

in $x \in [x_1, x_2]$ for any $0 < x_1 < x_2 < \infty$. Since $\mathfrak{I}^{\omega_1}f(z) \in H^2$,

$$\begin{aligned}\mathcal{D}^{\omega_1}\mathfrak{I}^{\omega_1}f_\rho(z) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{\left(\widehat{\mathfrak{I}^{\omega_1}f_\rho}\right)(t)e^{itz}}{\mathcal{L}_{\omega_1}(t)}\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \omega_1(\eta)\left(\int_0^\infty \frac{\widehat{f_\rho}(t+i\eta)e^{itz}}{\mathcal{L}_{\omega_1}(t)}\,\mathrm{d}t\right)\mathrm{d}\eta \\ &= \mathfrak{I}^{\omega_1}\mathcal{D}^{\omega_1}f_\rho(z) = f_\rho(z),\end{aligned}$$

by (3.13) and (3.9). Thus, the formula (3.10) follows because $\rho > 0$ may be chosen arbitrarily. The next inversion formula (3.11) is proved simpler. This completes the proof of Theorem 3. ∎

Note that the relations (3.6)–(3.8) for the standard weight function $\omega(x) = (1/\Gamma(\alpha))x^{\alpha-1}$ can be found in [13].

4. Cauchy–Bergman type kernels and proof of main theorems

Let $\omega \in W$ and $K(z) = -1/(iz) = \int_0^\infty e^{itz}\,\mathrm{d}t$ be the usual Cauchy kernel in $\mathbb{R}^2_+$. We define Cauchy–Bergman type ω-kernel by the formula (cf [9,10])

$$K_\omega(z) = \mathcal{D}^\omega K(z), \quad z \in \mathbb{R}^2_+. \qquad (4.1)$$

Also, we set

$$K_\omega(z,\zeta) = K_\omega\big(z-\bar{\zeta}\big), \quad z,\zeta\in\mathbb{R}^2_+.$$

It is easily seen that for the standard weight $\omega(x) = (1/\Gamma(\alpha))x^{\alpha-1}(\alpha>0)$

$$K_\omega(z,\zeta) = \frac{\Gamma(\alpha+1)}{\big(i(\bar{\zeta}-z)\big)^{\alpha+1}}.$$

Theorem 3 with the following lemma containing some important estimates for K_ω, play a basic role in the proof of Theorems 1 and 2.

LEMMA 5 *Suppose that* $\omega\in W_{\delta,\alpha}$ *for some* $\delta,\alpha>0$. *Then*

$$\big|K_\omega(z)\big| \le C(\delta,\alpha)\frac{1}{y^2\,\omega(y/2)}, \quad x\in\mathbb{R}, \quad y>0, \tag{4.2}$$

$$\big|K_\omega(z)\big| \le C(\delta,\alpha)\frac{1}{|z|}\left(\frac{1}{1+y^\alpha}+\frac{1}{y^2\omega^2(y/4)}\right), \quad x\in\mathbb{R}, \quad y>0. \tag{4.3}$$

If $1<p<\infty$, *then* $K_\omega(z,\zeta)\in L^p_\omega(\mathbb{R}^2_+;dm_2(\zeta))$ *for each* $z\in\mathbb{R}^2_+$, *and*

$$\big\|K_\omega(z,\cdot)\big\|_{p,\omega}\le C(p,\omega,y), \quad x\in\mathbb{R}, \quad y>0, \tag{4.4}$$

where $C(p,\omega,y)$ *is continuous in* $y>0$ *and vanishes as* $y\to+\infty$.

The estimate (4.4) *is no longer true for* $p=1$.

Proof The estimate (4.2) follows immediately from Lemma 1. To prove (4.3) we integrate by parts:

$$K_\omega(z) = \frac{1}{iz}\left(\int_0^1+\int_1^\infty\right)e^{itz}\,\frac{\mathcal{L}'_\omega(t)}{\mathcal{L}^2_\omega(t)}\,\mathrm{d}t.$$

Then

$$\int_0^1 e^{-ty}\,\frac{\big|\mathcal{L}'_\omega(t)\big|}{\mathcal{L}^2_\omega(t)}\,\mathrm{d}t \approx \int_0^1 e^{-ty}t^{\alpha-1}\mathrm{d}t \approx \frac{1}{1+y^\alpha}, \quad y>0,$$

and

$$\int_1^\infty e^{-ty}\,\frac{\big|\mathcal{L}'_\omega(t)\big|}{\mathcal{L}^2_\omega(t)}\mathrm{d}t \le C\frac{1}{\omega^2(y/4)}\int_1^\infty te^{-ty/2}\,\mathrm{d}t, \quad y>0.$$

A further estimation as $y\to+\infty$ and $y\to+0$ gives (4.3).

To prove (4.4), we split the integral into three parts

$$\|K_\omega(z,\cdot)\|^p_{p,\omega}=\int_{|\xi-x|<1}\int_0^\infty+\int_{|\xi-x|>1}\int_0^1+\int_{|\xi-x|>1}\int_1^\infty$$

and then estimate by (4.2) and (4.3). Finally, for $p=1$ it is enough to consider the power weight $\omega(y)=y^{\alpha-1}$ $(\alpha>0)$ to get a contradiction with (4.4). ■

Now we are in a position to prove the main theorems.

Proof of Theorem 1 For $p=1$ the convergence of the integral (1.1) is evident in view of Lemma 5. For $1<p<\infty$ we apply Hölder's inequality and use Lemma 5:

$$\iint_{\mathbb{R}^2_+} |f(\zeta)|\,|K_\omega(z,\zeta)|\,\omega(2\eta)\,d\xi\,d\eta \le \|f\|_{p,\omega}\,\|K(z,\cdot)\|_{p',\omega} \le C\|f\|_{p,\omega}, \tag{4.5}$$

where $C=C(p,\omega,y)$ is continuous in $y>0$ and vanishes as $y\to+\infty$, and the integral in the left-hand side of (4.5) converges uniformly in each half-plane $\mathbb{R}\times(\rho,\infty)$, $\rho>0$.

Let us now prove (1.1) for $p=1$. By Theorem 3, we have for $\rho>0$,

$$\begin{aligned} f(z+i\rho) &= \mathfrak{I}^\omega\mathcal{D}^\omega f(z+i\rho) = 2\int_0^\infty \omega(2\eta)\mathcal{D}^\omega f(z+i2\eta+i\rho)\,d\eta \\ &= \frac{1}{\pi}\int_0^\infty \omega(2\eta)\mathcal{D}^\omega\left\{\int_{\mathbb{R}} f(\xi+i\eta+i\rho)K(z,\zeta)\,d\xi\right\}d\eta \\ &= \frac{1}{\pi}\int_0^\infty \omega(2\eta)\left(\int_{\mathbb{R}} f(\xi+i\eta+i\rho)K_\omega(z,\zeta)\,d\xi\right)d\eta, \end{aligned} \tag{4.6}$$

where the ω-differentiation under the integral sign is justified because the function $g(z)=\int_{\mathbb{R}} f(\xi+i\eta+i\rho)\,K(z,\zeta)\,d\xi$ is in H^2 for fixed $\eta>0$. By Lebesgue's dominated convergence theorem, Lemma 3 and

$$\iint_{\mathbb{R}^2_+} |f(\zeta+i\rho)-f(\zeta)|\,|K_\omega(z,\zeta)|\,\omega(2\eta)\,d\xi\,d\eta \le C(p,\omega,y)\|f_\rho - f\|_{1,\omega} = o(1),$$

we may take the limit as $\rho\to+0$ in (4.6). Suppose now $1<p<\infty$, $f(z)\in H^p_\omega$. Then we apply the previous part of this theorem to the function

$$F_\lambda(z) = \frac{ie^{i\lambda z}}{i+\lambda z} f(z) \in H^1_\omega, \quad \lambda>0,$$

and then it only remains to pass to the limit as $\lambda\to+0$. ■

COROLLARY 6 *If* $1\le p<\infty$, $\omega\in W$ *and* $f\in H^p_\omega$, *then*

$$0 = \frac{1}{\pi}\iint_{\mathbb{R}^2_+} \overline{f(\zeta)}K_\omega(z,\zeta)\omega(2\eta)\,d\xi d\eta, \quad z\in\mathbb{R}^2_+,$$

$$\operatorname{Re} f(z) = \frac{2}{\pi}\iint_{\mathbb{R}^2_+} \operatorname{Re} f(\zeta)\operatorname{Re} K_\omega(z,\zeta)\omega(2\eta)\,d\xi\,d\eta, \quad z\in\mathbb{R}^2_+.$$

Proof of Theorem 2 Let $f(z)\in H^2_\omega$ be any function, $\omega\in W$, and ω_1 is deduced from (3.3), (3.5). By Lemma 4, the weight ω_1 (perhaps not positive) belongs to W. Hence the kernel K_{ω_1} satisfies the conditions of Lemma 5, and the integral (1.2) converges for every $y>0$. By Theorem 3, the functions $\varphi(z)=\mathfrak{I}^{\omega_1}f(z)$ and $\varphi_\rho(z)$ are in H^2.

Differentiation of the Cauchy integral formula for φ_ρ by means of the operator $\mathcal{D}^{\omega_1}$ leads to

$$\mathcal{D}^{\omega_1}\mathfrak{I}^{\omega_1}f_\rho(z)=\mathcal{D}^{\omega_1}\varphi_\rho(z)=\mathcal{D}^{\omega_1}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}K(z,t)\,\varphi_\rho(t)\,\mathrm{d}t\right\}.$$

Consequently, (3.10) and an argument similar to that in the relation (4.6) imply

$$f_\rho(z)=\frac{1}{2\pi}\int_{\mathbb{R}}\mathcal{D}^{\omega_1}K(z,t)\,\varphi_\rho(t)\,\mathrm{d}t,\quad z\in\mathbb{R}^2_+.$$

It remains to pass to the limit as $\rho\to+0$ which is justified in view of Lebesgue's dominated convergence theorem and

$$\begin{aligned}\int_{\mathbb{R}}\left|K_{\omega_1}(z,t)\right|\left|\varphi_\rho(t)-\varphi(t)\right|\mathrm{d}t&\le\left\|K_{\omega_1}(z,t)\right\|_{L^2(\mathrm{d}t)}\left\|\varphi_\rho-\varphi\right\|_{L^2(\mathrm{d}t)}\\&\le C(\omega_1,y)\left\|\varphi_\rho-\varphi\right\|_{L^2(\mathrm{d}t)}=o(1)\quad\text{as } \rho\to+0.\end{aligned}$$

Conversely, let $f(z)$ be a function representable in the form (1.2) with some $\varphi\in L^2(\mathbb{R})$. Integration by means of the operator $\mathfrak{I}^{\omega_1}$ yields

$$\mathfrak{I}^{\omega_1}f(z)=\mathfrak{I}^{\omega_1}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}K_{\omega_1}(z,t)\,\varphi(t)\,\mathrm{d}t\right\}=\frac{1}{2\pi}\int_{\mathbb{R}}K(z,t)\,\varphi(t)\,\mathrm{d}t=\varphi(z),$$

where $\varphi(z)\in H^2$. By Theorem 3, $\mathcal{D}^{\omega_1}\mathfrak{I}^{\omega_1}f(z)=\mathcal{D}^{\omega_1}\varphi(z)\in H^2_\omega$. On the other hand, by Theorem A, $f(z+i\rho)\in H^2$. Thus, $f(z)=\mathcal{D}^{\omega_1}\mathfrak{I}^{\omega_1}f(z)\in H^2_\omega$ by the inversion formula (3.10). ∎

In conclusion, note that Theorems 1 and 2 suggest many problems concerning the projection, duality, interpolation and operators in general weighted Bergman spaces H^p_ω with $\omega\in W$. Some results will appear in a forthcoming publication.

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