Glasgow Math. J. **49** (2007) 345–356.
doi:10.1017/S001708950700359X. Printed in the United Kingdom

HARDY–BLOCH TYPE SPACES AND LACUNARY SERIES ON THE POLYDISK

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(Received 24 September, 2006; revised 29 January, 2007; accepted 12 February, 2007)

Abstract. We extend the well-known Paley and Paley-Kahane-Khintchine inequalities on lacunary series to the unit polydisk of $\mathbb{C}^n$. Then we apply them to obtain sharp estimates for the mean growth in weighted spaces $h(p,\alpha)$, $h(p,\log(\alpha))$ of Hardy–Bloch type, consisting of functions n-harmonic in the polydisk. These spaces are closely related to the Bloch and mixed norm spaces and naturally arise as images under some fractional operators.

2000 *Mathematics Subject Classification.* 32A37, 32A05.

1. Introduction and main results. Let $U^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \le j \le n\}$ be the unit polydisk in $\mathbb{C}^n$, and let $\mathbb{T}^n = \{w = (w_1, \dots, w_n) \in \mathbb{C}^n : |w_j| = 1, 1 \le j \le n\}$ be the n-dimensional torus, the distinguished boundary of U^n. We will deal with n-harmonic functions on the polydisk U^n, i.e. functions harmonic in each variable z_j separately. Denote by $H(U^n)$, $h(U^n)$ the sets of holomorphic and n-harmonic functions in U^n, respectively.

If $f(z) = f(r\zeta)$ is a measurable function in U^n, then we write

$$M_p(f;r) = \|f(r\cdot)\|_{L^p(\mathbb{T}^n;dm_n)}, \qquad r = (r_1, \dots, r_n) \in I^n, \quad 0 < p \le \infty,$$

where $I^n = (0,1)^n$, dm_n is the n-dimensional Lebesgue measure on $\mathbb{T}^n$ normalized so that $m_n(\mathbb{T}^n) = 1$. The collection of n-harmonic (holomorphic) functions $f(z)$, for which $\|f\|_{h^p} = \sup_{r\in I^n} M_p(f;r) < +\infty$, is the usual Hardy space h^p (respectively H^p).

The quasi-normed space $h(p,\alpha)\,(0 < p \le \infty, \alpha = (\alpha_1, \dots, \alpha_n), \alpha_j > 0)$ is the set of those functions $f(z)$ n-harmonic in the polydisk U^n, for which the quasi-norm

$$\|f\|_{p,\alpha} = \sup_{r\in I^n} \prod_{j=1}^{n} (1 - r_j)^{\alpha_j} M_p(f;r)$$

is finite. Corresponding little spaces $h_0(p,\alpha)$ are defined by the conditions

$$(1 - r_j)^{\alpha_j} M_p(f;r) = o(1) \qquad \text{as} \qquad r_j \to 1^-$$

for each $j \in [1,n]$ separately. For the subspaces of $h(p,\alpha)$ consisting of holomorphic functions let

$$H(p,\alpha) = H(U^n) \cap h(p,\alpha), \qquad H_0(p,\alpha) = H(U^n) \cap h_0(p,\alpha).$$

For $n = 1$ the spaces $H(p, \alpha)$ and $h(p, \alpha)$ have been studied by Flett [**9**, **10**] in the frame of mixed norm spaces. If the gradient of a function f belongs to $h(\infty, 1)$ or $h_0(\infty, 1)$ we say that f is a Bloch or little Bloch function, respectively. See [**1**, **17**] for basic properties of the Bloch space including higher dimensions.

Denote by $h(p, \log(\alpha))$ $(0 < p \leq \infty, \alpha = (\alpha_1, \ldots, \alpha_n), \alpha_j > 0)$ the set of those functions $f(z)$ n-harmonic in the polydisk U^n, for which the quasi-norm

$$\|f\|_{p,\log(\alpha)} = \sup_{r \in I^n} \left(\prod_{j=1}^{n} \log \frac{e}{1 - r_j} \right)^{-\alpha_j} M_p(f; r)$$

is finite. For the subspace of $h(p, \log(\alpha))$ consisting of holomorphic functions let $H(p, \log(\alpha)) = H(U^n) \cap h(p, \log(\alpha))$. One variable spaces $H(p, \log(\alpha))$ and more general "integrated" spaces of Hardy–Bloch type are studied in [**11**].

Recall that a sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers is said to be lacunary (or Hadamard) if there exists a constant $\lambda > 1$ such that $\frac{n_{k+1}}{n_k} \geq \lambda$ for all $k = 1, 2, \ldots$. A corresponding power series is called a lacunary series.

Lacunary series in classical function spaces such as Bloch, Bergman, Besov, Dirichlet, Q-type spaces, have been extensively studied recently ([**2**, **3**, **11**, **12**, **13**, **14**, **18**, **19**]). The purpose of the present paper is to characterize lacunary series in the weighted spaces $H(p, \alpha)$ and $H_0(p, \alpha)$ of Hardy–Bloch type (see Theorems 3 and 4) and to estimate the mean growth in $h(p, \alpha)$ and $h(p, \log(\alpha))$, see Theorem 5. To this end, we begin by extending in Theorems 1 and 2 the classical inequalities of Paley ([**20**, Ch. XII, Th. 7.8], [**8**, p. 104], [**16**, p. 170]) and Paley-Kahane-Khintchine ([**20**, Ch. V, Th. 8.20], [**16**, p. 172]) to the polydisk.

THEOREM 1. (Paley's theorem for the polydisk)
Let a holomorphic function

$$f(z) = \sum_{k \in \mathbb{Z}_+^n} a_{k_1 \ldots k_n} z_1^{k_1} \cdots z_n^{k_n}, \qquad z \in U^n,$$

be of Hardy space H^1. Then for any lacunary sequences $\{m_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1, 2, \ldots, n$

$$\left(\sum_{k \in \mathbb{N}^n} \left| a_{m_{1,k_1} \cdots m_{n,k_n}} \right|^2 \right)^{1/2} \leq C \|f\|_{H^1}, \tag{1.1}$$

where the constant $C > 0$ is independent of f.

THEOREM 2. (Paley-Kahane-Khintchine inequalities for the polydisk)
Let $\{m_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1, 2, \ldots, n$ be arbitrary lacunary sequences and $f(z)$ be a holomorphic function in U^n given by a convergent lacunary series

$$f(z) = \sum_{k \in \mathbb{N}^n} a_{k_1 \cdots k_n} z_1^{m_{1,k_1}} \cdots z_n^{m_{n,k_n}}, \qquad z \in U^n.$$

Then for any p, $0 < p < \infty$, f *is in Hardy space* H^p *if and only if* $\{a_k\} \in \ell^2$. *Moreover, the corresponding norms are equivalent:*

$$C_1 \|f\|_{H^p} \le \left(\sum_{k \in \mathbb{N}^n} \left| a_{k_1 \cdots k_n} \right|^2 \right)^{1/2} \le C_2 \|f\|_{H^p}, \tag{1.2}$$

where the constants $C_1, C_2 > 0$ *are independent of* f.

Theorem 2 asserts in fact that if a lacunary series is in some Hardy space, then it is in all Hardy spaces on the polydisk.

In the next two theorems we characterize lacunary series in the weighted spaces $H(p, a)$ and $H_0(p, a)$ of Hardy–Bloch type.

THEOREM 3. *Let* $\{m_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1, 2, \ldots, n$ *be arbitrary lacunary sequences,* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$, *and* $f(z)$ *be a holomorphic function in* U^n *given by a convergent lacunary series*

$$f(z) = \sum_{k \in \mathbb{N}^n} a_{k_1 \cdots k_n} m_{1,k_1}^{\alpha_1} \cdots m_{n,k_n}^{\alpha_n} z_1^{m_{1,k_1}} \cdots z_n^{m_{n,k_n}}, \qquad z \in U^n.$$

Then the following statements are equivalent:

(a) $f(z) \in H(\infty, \alpha)$;
(b) $f(z) \in H(p, \alpha)$ *for some* p, $0 < p < \infty$;
(c) $f(z) \in H(p, \alpha)$ *for all* p, $0 < p < \infty$;
(d) $\{a_k\}_{k \in \mathbb{N}^n} \in \ell^{\infty}$.

Also, corresponding norms are equivalent.

The next assertion is a “little oh” version of Theorem 3.

THEOREM 4. *Let* $\{m_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1, 2, \ldots, n$ *be arbitrary lacunary sequences,* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j > 0$, *and* $f(z)$ *be a holomorphic function in* U^n *given by a convergent lacunary series*

$$f(z) = \sum_{k \in \mathbb{N}^n} a_{k_1 \cdots k_n} m_{1,k_1}^{\alpha_1} \cdots m_{n,k_n}^{\alpha_n} z_1^{m_{1,k_1}} \cdots z_n^{m_{n,k_n}}, \qquad z \in U^n.$$

the following statements are equivalent:

(a) $f(z) \in H_0(\infty, \alpha)$;
(b) $f(z) \in H_0(p, \alpha)$ *for some* p *with* $0 < p < \infty$;
(c) $f(z) \in H_0(p, \alpha)$ *for all* p *with* $0 < p < \infty$;
(d) $\lim_{k_j \to \infty} a_{k_1 \cdots k_n} = 0$ *for each* $j \in [1, n]$.

Finally, as an application, we establish in Theorem 5 sharp estimates for the mean growth in the weighted spaces $h(p, \alpha)$, $h(p, \log(\alpha))$. In particular, in (1.5)–(1.6) below we generalize and improve the well-known inequality of Clunie and MacGregor [**7**] and Makarov [**15**], and also another inequality of Girela and Peláez [**12**]. For all the inequalities we give quick and simple proofs.

Below we will write $T : X \longrightarrow Y$ if T is a bounded operator mapping X to Y, i.e. $\|Tf\|_Y \le C\|f\|_X\ \forall f \in X$.

THEOREM 5. *If $\alpha_j > 0\,(1 \le j \le n)$, then the following relations hold:*

$$\begin{array}{llll} (i) & \mathcal{D}^{-\alpha} : h(p,\alpha) \longrightarrow h(p, \log(1/p)), & 0 < p \le 2, & (1.3)\\ (ii) & \mathcal{D}^{-\alpha} : h(p,\alpha) \longrightarrow h(p, \log(1/2)), & 2 \le p < \infty, & (1.4)\\ (iii) & \mathcal{D}^{-\alpha} : h(p,\alpha) \longrightarrow h(\infty, 1/p), & 0 < p < \infty, & (1.5)\\ (iv) & \mathcal{D}^{-\alpha} : h(\infty,\alpha) \longrightarrow h(p, \log(1/2)), & 0 < p < \infty, & (1.6)\\ (v) & \mathcal{D}^{-\alpha} : h(\infty,\alpha) \longrightarrow h(\infty, \log(1)). & & (1.7) \end{array}$$

All the relations (1.3)–(1.7) *are best possible in the sense that for every relation $\mathcal{D}^{-\alpha}$: $X \longrightarrow Y$ there exists a function $f \in h(U^n)$ such that $\|\mathcal{D}^{-\alpha} f\|_Y \approx \|f\|_X$.*

REMARK 1. In the particular case $n = 1$, $\alpha = 1$ and ordinary derivatives of holomorphic functions corresponding results are known: for the relation (1.3) see [**12**, p. 461]; for the relation (1.4) see [**11**, Th. 1.1]; for (1.5) see [**12**, p. 467] ($p \ge 1/2$); for (1.6) see [**7**, p. 364] and [**15**, p. 374]; for (1.7) see, e.g., [**12**, p. 460].

2. Notation and preliminaries. We will use the conventional multi-index notation: $r\zeta = (r_1\zeta_1, \dots, r_n\zeta_n)$, $dr = dr_1 \cdots dr_n$, $(1 - |\zeta|)^\alpha = \prod_{j=1}^n (1 - |\zeta_j|)^{\alpha_j}$, $\Gamma(\alpha) = \prod_{j=1}^n \Gamma(\alpha_j)$ for $\zeta \in \mathbb{C}^n$, $r \in I^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Let $\mathbb{Z}^n$, $\mathbb{N}^n$, $\mathbb{Z}^n_+$ denote the sets of all n-tuples of integers, positive integers, nonnegative integers, respectively. Throughout the paper, the letters $C(\alpha, \beta, \lambda, \dots)$, C_α etc. stand for positive constants possibly different at different places and depending only on the parameters indicated. For $A, B > 0$, the notation $A \approx B$ denotes the two-sided estimate $c_1A \le B \le c_2A$ with some inessential positive constants c_1 and c_2 independent of the variable involved. The symbol dm_{2n} means the Lebesgue measure on the polydisk U^n normalized so that $m_{2n}(U^n) = 1$. For a function $f(z) = f(r\zeta)$, $r \in I^n$, $\zeta \in \mathbb{T}^n$, given on U^n, we will use integro-differential operators of two types: Riemann–Liouville fractional operators D^α and $\mathcal{D}^\alpha$, and also Hadamard's operator $\mathcal{F}^\alpha$ with respect to the variable $r \in I^n$:

$$D^{-\alpha} f(z) = \frac{r^\alpha}{\Gamma(\alpha)} \int_{I^n} (1-\eta)^{\alpha-1} f(\eta z)\, d\eta, \qquad D^\alpha f(z) = \left(\frac{\partial}{\partial r}\right)^m D^{-(m-\alpha)} f(z),$$

$$\mathcal{D}^{-\alpha} f(r\zeta) = r^{-\alpha} D^{-\alpha} f(r\zeta), \qquad \mathcal{D}^\alpha f(r\zeta) = D^\alpha \{r^\alpha f(r\zeta)\},$$

$$\mathcal{F}^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_{I^n} \prod_{j=1}^n \left(\log\frac{1}{\eta_j}\right)^{\alpha_j - 1} f(\eta z)\, d\eta,$$

$$\mathcal{F}^m f(z) = \left(\frac{\partial}{\partial r} \cdot r\right)^m f(z), \qquad \mathcal{F}^\alpha f(z) = \mathcal{F}^{-(m-\alpha)} \mathcal{F}^m f(z),$$

where $(\frac{\partial}{\partial r})^m = (\frac{\partial}{\partial r_1})^{m_1} \cdots (\frac{\partial}{\partial r_n})^{m_n}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_+$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > 0$, $m_j - 1 < \alpha_j \le m_j\,(1 \le j \le n)$. It is easily seen that if f is n-harmonic (or holomorphic), then so are $\mathcal{D}^\alpha f$, $\mathcal{F}^\alpha f$ for any $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{R}$, and the following inversion formulas hold

$$\mathcal{D}^{\alpha}\mathcal{D}^{-\alpha}f(z) = f(z), \qquad \mathcal{F}^{\alpha}\mathcal{F}^{-\alpha}f(z) = f(z). \tag{2.1}$$

It is evident from the definition that $\mathcal{F}^{\alpha}f = \mathcal{F}^{\alpha_1}_{r_1}\mathcal{F}^{\alpha_2}_{r_2}\cdots\mathcal{F}^{\alpha_n}_{r_n}f$, for any $\alpha = (\alpha_1, \dots, \alpha_n)$, where $\mathcal{F}^{\alpha_j}_{r_j}$ means the same operator acting in direction r_j only. There is an equivalent definition for $\mathcal{F}^{\alpha}$ suitable only for n-harmonic functions. For every function $f \in h(U^n)$ having a series expansion $f(z) = \sum_{k\in\mathbb{Z}^n} a_k r^{|k|} e^{ik\cdot\theta}$, where $r^{|k|} = r_1^{|k_1|}\cdots r_n^{|k_n|}$, $k\cdot\theta = k_1\theta_1 + \cdots + k_n\theta_n$, we can write

$$\mathcal{F}^{\alpha}f(z) = \sum_{k\in\mathbb{Z}^n}\prod_{j=1}^{n}(1+|k_j|)^{\alpha_j} a_k r^{|k|} e^{ik\cdot\theta}.$$

LEMMA 1. *If* $\alpha_j > 0\,(1 \le j \le n)$, $0 < p \le 2$, *then for all* $u \in h(U^n)$

$$\|\mathcal{D}^{-\alpha}u\|_{h^p} \le C\left(\int_{U^n}(1-|z|)^{\alpha p-1}|u(z)|^p\,dm_{2n}(z)\right)^{1/p}. \tag{2.2}$$

The one variable version of (2.2) is known and can be deduced from [**9**, Th. 2] and the fact that harmonic conjugation is bounded in Bergman spaces consisting of harmonic functions in the unit disk, see [**10**]. The inequality (2.2) can be proved by an iteration of that in one variable.

LEMMA 2. *If* $\alpha_j > 0\,(1 \le j \le n)$, $2 \le p < \infty$, *then for all* $u \in h(U^n)$

$$\|\mathcal{D}^{-\alpha}u\|_{h^p} \le C\left(\int_{I^n}(1-r)^{2\alpha-1}M_p^2(u;r)\,dr\right)^{1/2}. \tag{2.3}$$

Proof. A modification of the Littlewood–Paley type inequality ([**5**], [**6**]) gives

$$\|\mathcal{D}^{-\alpha}u\|_{h^p} \le C(p,\alpha,n)\Big\|\,\|(1-r)^{\alpha}u\|_{L^2(dr/(1-r))}\Big\|_{L^p(\mathbb{T}^n)}$$

for all $u \in h(U^n)$. An application of Minkowski's inequality immediately yields

$$\|\mathcal{D}^{-\alpha}u\|_{h^p} \le C(p,\alpha,n)\Big\|(1-r)^{\alpha}\,\|u\|_{L^p(\mathbb{T}^n)}\Big\|_{L^2(dr/(1-r))}$$

which coincides with (2.3). □

3. Proofs of Theorems 1–4. In the proofs of Theorems 1 and 2 we will use some arguments of Pavlović [**16**, Sec. 11] together with Littlewood–Paley type inequalities obtained by the author in [**5**, **6**].

Without loss of generality we may assume that $n = 2$ in proofs below.

Proof of Theorem 1. According to a Littlewood–Paley type inequality (see [**5**], [**6**])

$$\Big\|\,\|(1-r)\,\mathcal{F}^1 f\|_{L^2(dr/(1-r))}\Big\|_{L^p(\mathbb{T}^n)} \le C\|f\|_{H^p} \qquad \text{for any} \qquad 0 < p < \infty. \tag{3.1}$$

Assuming $0 < p \le 2$ we can apply Minkowski's inequality to (3.1) and get

$$\|(1-r)\,M_p(\mathcal{F}^1 f;r)\|_{L^2(dr/(1-r))} \le C\|f\|_{H^p}. \tag{3.2}$$

For two lacunary sequences $\{m_{j,k_j}\}_{k_j=1}^{\infty}$, $j = 1, 2$ there exist $\lambda_1, \lambda_2 > 1$ such that

$$\frac{m_{j,k_j+1}}{m_{j,k_j}} \geq \lambda_j \qquad \text{for all} \qquad k_j = 1, 2, \ldots, \qquad j = 1, 2.$$

Choosing two strictly increasing sequences

$$r_{1,k_1} = 1 - \frac{1}{\lambda_1^{k_1}}, \qquad r_{2,k_2} = 1 - \frac{1}{\lambda_2^{k_2}}, \qquad k_1, k_2 = 1, 2, \ldots,$$

and $p = 1$ in (3.2), we can estimate

$$\begin{aligned} \|f\|_{H^1}^2 &\geq C \int_0^1 \int_0^1 (1-r)\, M_1^2(\mathcal{F}^1 f; r) dr_1 dr_2 \\ &\geq C \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \int_{r_{1,k_1}}^{r_{1,k_1+1}} \int_{r_{2,k_2}}^{r_{2,k_2+1}} (1-r)\, M_1^2(\mathcal{F}^1 f; r) dr_1 dr_2. \end{aligned} \tag{3.3}$$

Consider the intervals $I_{k_1}^{(1)} = [\lambda_1^{k_1}, \lambda_1^{k_1+1})$, $I_{k_2}^{(2)} = [\lambda_2^{k_2}, \lambda_2^{k_2+1})$, $k_1, k_2 = 1, 2, \ldots$. Each interval $I_{k_j}^{(j)}$ contains no more than one number from $\{m_{j,k_j}\}$. We may assume that each interval $I_{k_j}^{(j)}$ contains just one such number, namely $\lambda_j^{k_j} \leq m_{j,k_j} < \lambda_j^{k_j+1}$, $k_j \in \mathbb{N}$, $j = 1, 2$. We can now estimate the Taylor coefficients of the series

$$\mathcal{F}^1 f(z_1, z_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} (1+k_1)(1+k_2)\, a_{k_1 k_2} z_1^{k_1} z_2^{k_2}.$$

By Cauchy's integral formula

$$(1+k_1)(1+k_2)|a_{k_1 k_2}| \leq \frac{1}{r_1^{k_1} r_2^{k_2}} M_1(\mathcal{F}^1 f; r_1, r_2), \qquad k_1, k_2 = 0, 1, 2, \ldots.$$

So, we can continue (3.3)

$$\begin{aligned} &\int_{r_{1,k_1}}^{r_{1,k_1+1}} \int_{r_{2,k_2}}^{r_{2,k_2+1}} (1-r)\, M_1^2(\mathcal{F}^1 f; r) dr_1 dr_2 \\ &\geq \prod_{j=1}^{2} (1+k_j)^2\, |a_{k_1 k_2}|^2 \int_{r_{1,k_1}}^{r_{1,k_1+1}} \int_{r_{2,k_2}}^{r_{2,k_2+1}} (1-r_1)(1-r_2)\, r_1^{2k_1} r_2^{2k_2}\, dr_1 dr_2. \end{aligned} \tag{3.4}$$

The inner integrals can be estimated as follows ($j = 1, 2$)

$$\begin{aligned} \int_{r_{j,k_j}}^{r_{j,k_j+1}} (1-r_j)\, r_j^{2k_j}\, dr_j &\geq (1 - r_{j,k_j+1})\, r_{j,k_j}^{2k_j} (r_{j,k_j+1} - r_{j,k_j}) \\ &= \frac{1}{\lambda_j^{k_j+1}} \left(\frac{1}{\lambda_j^{k_j}} - \frac{1}{\lambda_j^{k_j+1}} \right) \left(1 - \frac{1}{\lambda_j^{k_j}} \right)^{2k_j}. \end{aligned}$$

By taking $k_j = m_{j,k_j} \geq 1,\ k_j = 1, 2, \ldots,$ we conclude that

$$\int_{r_{j,k_j}}^{r_{j,k_j+1}} (1 - r_j)\, r_j^{2m_{j,k_j}}\, dr_j \geq C(\lambda_j) \frac{1}{m_{j,k_j}^2}.$$

Thus,

$$\int_{r_{1,k_1}}^{r_{1,k_1+1}} \int_{r_{2,k_2}}^{r_{2,k_2+1}} (1 - r)\, M_1^2(\mathcal{F}^1 f; r) dr_1 dr_2$$
$$\geq \prod_{j=1}^{2} (1 + m_{j,k_j})^2 \left|a_{m_{1,k_1} m_{2,k_2}}\right|^2 \frac{C(\lambda_1, \lambda_2)}{m_{1,k_1}^2 m_{2,k_2}^2} \geq C(\lambda_1, \lambda_2) \left|a_{m_{1,k_1} m_{2,k_2}}\right|^2. \qquad (3.5)$$

A combination of inequalities (3.3)–(3.5) completes the proof of Theorem 1. □

Proof of Theorem 2. We distinguish three cases.

Case $1 \leq p \leq 2$. It is obvious that $\|f\|_{H^p} \leq \|f\|_{H^2} = \|\{a_k\}\|_{\ell^2}$. On the other side, the converse inequality $\|\{a_k\}\|_{\ell^2} \leq C\|f\|_{H^1} \leq C\|f\|_{H^p}$ follows immediately from Theorem 1.

Case $0 < p < 1$. Again the inequality $\|f\|_{H^p} \leq \|f\|_{H^2} = \left\|\{a_k\}\right\|_{\ell^2}$ is obvious. For proving the converse inequality, assume that $f(z)$ is continuous in a neighborhood of the closure of U^n. Then, by the Cauchy–Schwarz inequality,

$$\|f\|_{H^1} = \sup_{r \in I^2} \int_{\mathbb{T}^2} |f(rw)|^{p/2} |f(rw)|^{1-p/2} dm_2(w) \leq \|f\|_{H^p}^{p/2} \|f\|_{H^{2-p}}^{(2-p)/2}.$$

Since by the previous case $\|f\|_{H^2} \leq C\|f\|_{H^1}$,

$$\|f\|_{H^1} \leq C\|f\|_{H^p}^{p/2} \|f\|_{H^1}^{(2-p)/2}.$$

It follows that $\|f\|_{H^p} \geq C\|f\|_{H^1} \geq C\|f\|_{H^2} = C\|\{a_k\}\|_{\ell^2}$. For arbitrary function $f \in H(U^n)$ we apply the inequality (1.2) to the dilated function $f_\rho(z) = f(\rho z),\ \rho \in I^2$, and then the result follows by letting $\rho_1, \rho_2 \to 1$.

Case $2 < p < \infty$. The inequality $\|\{a_k\}\|_{\ell^2} \leq \|f\|_{H^p}$ is clear. So it remains to prove the converse inequality. Consider the identity operator $(If)(z) = f(z)$. If $q = p/(p-1)$ is the conjugate index of p, then $1 < q < 2 < p < \infty$ and by the first case $\|If\|_{H^2} \leq C\|f\|_{H^q}$. In view of the self-conjugacy of the identity operator, we finally get $\|f\|_{H^p} = \|If\|_{H^p} \leq C\|f\|_{H^2}$. □

Proof of Theorem 4. The implication $(a) \Rightarrow (b)$ is obvious because of the elementary inclusion $H(\infty, \alpha) \subset H(p, \alpha)$.

The implication $(b) \Rightarrow (c)$ follows from Theorem 2 which says that $M_p(f; r) \approx M_s(f; r)$ for any $s, 0 < s < \infty$.

For proving the implication $(c) \Rightarrow (d)$, let $f(z) \in H_0(p, \alpha)$ for any $p, 0 < p < \infty$. In particular, $(1 - r)^\alpha M_1(f; r_1, r_2) = o(1)$ as $r_1 \to 1^-$ or $r_2 \to 1^-$. By Cauchy's integral formula

$$|a_{k_1k_2}|m_{1,k_1}^{\alpha_1}m_{2,k_2}^{\alpha_2} = \left|\frac{1}{(2\pi i)^2}\int_{|\zeta_1|=r_1}\int_{|\zeta_2|=r_2}\frac{f(\zeta_1,\zeta_2)\,d\zeta_1 d\zeta_2}{\zeta_1^{1+m_{1,k_1}}\,\zeta_2^{1+m_{2,k_2}}}\right|$$
$$\leq \frac{1}{r_1^{m_{1,k_1}}r_2^{m_{2,k_2}}}M_1(f;r_1,r_2) = \frac{(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}M_1(f;r_1,r_2)}{(1-r_1)^{\alpha_1}(1-r_2)^{\alpha_2}\,r_1^{m_{1,k_1}}r_2^{m_{2,k_2}}}$$

for any $r = (r_1, r_2) \in I^2$ and $k_1, k_2 = 1, 2, \ldots$. Taking $r_j = 1 - 1/m_{j,k_j}$, $j = 1, 2$, we conclude that

$$|a_{k_1k_2}| \leq \left(1 - \frac{1}{m_{1,k_1}}\right)^{-m_{1,k_1}}\left(1 - \frac{1}{m_{2,k_2}}\right)^{-m_{2,k_2}}$$
$$\times \left(\frac{1}{m_{1,k_1}}\right)^{\alpha_1}\left(\frac{1}{m_{2,k_2}}\right)^{\alpha_2} M_1\left(f; 1 - \frac{1}{m_{1,k_1}}, 1 - \frac{1}{m_{2,k_2}}\right) = o(1)$$

as $k_1 \to \infty$ or $k_2 \to \infty$.

We now turn to the proof of the implication $(d) \Rightarrow (a)$. Let $a_{k_1k_2} = o(1)$ as $k_1 \to \infty$ or $k_2 \to \infty$. Given $\varepsilon > 0$ there exists a number $k_1^0 \in \mathbb{N}$ such that

$$|a_{k_1k_2}| < \varepsilon \qquad \text{for all} \qquad k_1 > k_1^0 \qquad \text{and fixed} \qquad k_2.$$

Applying Hadamard's operator $\mathcal{F}^{1-\alpha}$ to the function $f(z)$ we get

$$\mathcal{F}^{1-\alpha}f(z_1,z_2) = \sum_{k_1,k_2=1}^{\infty}\prod_{j=1}^{2}(1+m_{j,k_j})^{1-\alpha_j}\,m_{j,k_j}^{\alpha_j}\cdot a_{k_1k_2}\,z_1^{m_{1,k_1}}z_2^{m_{2,k_2}},$$

which implies that

$$\left|\mathcal{F}^{1-\alpha}f(z_1,z_2)\right| \leq C(\alpha_1,\alpha_2)\sum_{k_2=1}^{\infty}\left(\sum_{k_1=1}^{\infty}|a_{k_1k_2}|\,m_{1,k_1}\,r_1^{m_{1,k_1}}\right)m_{2,k_2}\,r_2^{m_{2,k_2}}.$$

Next, we break the inner sum into two sums

$$\sum_{k_1=1}^{\infty}|a_{k_1k_2}|\,m_{1,k_1}\,r_1^{m_{1,k_1}} = \left(\sum_{k_1=1}^{k_1^0} + \sum_{k_1=k_1^0+1}^{\infty}\right)|a_{k_1k_2}|\,m_{1,k_1}\,r_1^{m_{1,k_1}}. \tag{3.6}$$

For the finite sum in (3.6) we can find $r_1^0 < 1$ such that

$$(1-r_1)\sum_{k_1=1}^{k_1^0}|a_{k_1k_2}|\,m_{1,k_1}\,r_1^{m_{1,k_1}} < \varepsilon \qquad \text{for all} \qquad r_1 \in (r_1^0, 1). \tag{3.7}$$

The last sum in (3.6) can be estimated as follows. It is easily seen that

$$m_{1,k_1+1} \leq \frac{\lambda_1}{\lambda_1 - 1}(m_{1,k_1+1} - m_{1,k_1}).$$

Consequently

$$m_{1,k_1+1} r_1^{m_{1,k_1+1}} \le \frac{\lambda_1}{\lambda_1 - 1}\left[r_1^{1+m_{1,k_1}} + r_1^{2+m_{1,k_1}} + \cdots + r_1^{m_{1,k_1+1}}\right].$$

It follows that

$$\sum_{k_1=k_1^0+1}^{\infty} |a_{k_1k_2}|\, m_{1,k_1}\, r_1^{m_{1,k_1}} < \varepsilon \frac{\lambda_1}{\lambda_1 - 1} \sum_{k_1=1}^{\infty} r_1^{k_1} = \varepsilon\, C(\lambda_1) \frac{1}{1 - r_1}. \tag{3.8}$$

Combining (3.6)–(3.8), we obtain that for all $r_1 \in (r_1^0, 1)$

$$(1 - r_1) M_\infty\big(\mathcal{F}^{1-\alpha} f; r_1, r_2\big) \le \varepsilon\, C(\alpha_1, \alpha_2, \lambda_1) \sum_{k_2=1}^{\infty} m_{2,k_2}\, r_2^{m_{2,k_2}}.$$

Hence

$$(1 - r_1) M_\infty\big(\mathcal{F}^{1-\alpha} f; r_1, r_2\big) = o(1) \qquad \text{as} \qquad r_1 \to 1^-.$$

One can show in the same manner that

$$(1 - r_2) M_\infty\big(\mathcal{F}^{1-\alpha} f; r_1, r_2\big) = o(1) \qquad \text{as} \qquad r_2 \to 1^-.$$

Thus, $\mathcal{F}^{1-\alpha} f \in H_0(\infty, 1)$. Since Hadamard's operator is invertible, we can now twice apply the rule of fractional integro-differentiation in mixed norm spaces (see [**10**, Th. 6]) in each variable r_1 and r_2, and obtain

$$f(z) = \mathcal{F}^{\alpha-1} \mathcal{F}^{1-\alpha} f(z) \in H_0(\infty, 1 + (\alpha - 1)) = H_0(\infty, \alpha).$$

This completes proof of Theorem 4. □

Theorem 3 can be proved more easily and so we omit the details.

4. Proof of Theorem 5. *Proof of* (*i*). Let $u \in h(p, \alpha)$ for some $0 < p \le 2$ and $\alpha_j > 0$. We first apply Lemma 1 to the dilated function $u_\rho(z) = u(\rho z)$, $\rho \in I^n$,

$$M_p(\mathcal{D}^{-\alpha} u; \rho r) \le C \left(\int_{U^n} (1 - |z|)^{\alpha p - 1} |u(\rho z)|^p\, dm_{2n}(z) \right)^{1/p}, \qquad \rho, r \in I^n.$$

Fatou's lemma and further estimation yield

$$\begin{aligned} M_p^p(\mathcal{D}^{-\alpha} u; \rho) &\le C \int_{I^n} (1 - r)^{\alpha p - 1} M_p^p(u; \rho r)\, dr \\ &\le C \|u\|_{p,\alpha}^p \int_{I^n} \frac{(1 - r)^{\alpha p - 1}}{(1 - \rho r)^{\alpha p}}\, dr \le C \|u\|_{p,\alpha}^p \prod_{j=1}^{n} \log \frac{e}{1 - \rho_j} \end{aligned}$$

for any $\rho \in I^n$. Thus,

$$\|\mathcal{D}^{-\alpha} u\|_{p,\log(1/p)} \le C \|u\|_{p,\alpha}, \qquad 0 < p \le 2. \tag{4.1}$$

The inequality (4.1) is sharp because of the example

$$f_1(z) = \prod_{j=1}^{n} \frac{1}{(1 - z_j)^{\alpha_j + 1/p}}, \qquad z \in U^n. \tag{4.2}$$

It is easy to compute that

$$(1 - r)^{\alpha} M_p(f_1; r) \approx 1, \qquad M_p(\mathcal{D}^{-\alpha} f_1; r) \approx \left(\prod_{j=1}^{n} \log \frac{e}{1 - r_j} \right)^{1/p}.$$

Proof of (*ii*). Let $u \in h(p, \alpha)$ for some $2 \le p < \infty$ and $\alpha_j > 0$. Lemma 2, together with Fatou's lemma, yields

$$\begin{aligned} M_p^2(\mathcal{D}^{-\alpha} u; \rho) &\le C \int_{I^n} (1 - r)^{2\alpha - 1} M_p^2(u; \rho r) dr \\ &\le C \|u\|_{p,\alpha}^2 \int_{I^n} \frac{(1 - r)^{2\alpha - 1}}{(1 - \rho r)^{2\alpha}} dr \le C \|u\|_{p,\alpha}^2 \prod_{j=1}^{n} \log \frac{e}{1 - \rho_j} \end{aligned}$$

for any $\rho \in I^n$. Thus,

$$\|\mathcal{D}^{-\alpha} u\|_{p, \log(1/2)} \le C \|u\|_{p,\alpha}, \qquad 2 \le p < \infty. \tag{4.3}$$

The function given by the lacunary series

$$f_2(z) = \sum_{k \in \mathbb{Z}_+^n} 2^{\alpha_1 k_1} \cdots 2^{\alpha_n k_n} z_1^{2^{k_1}} \cdots z_n^{2^{k_n}}, \qquad z \in U^n, \tag{4.4}$$

provides an example showing the sharpness of the inequality (4.3). Indeed, by Theorem 2

$$M_p(f_2; r) \approx \left(\sum_{k \in \mathbb{Z}_+^n} 2^{2\alpha k} r^{2^{k+1}} \right)^{1/2} \approx \frac{r}{(1 - r)^{\alpha}} \equiv \prod_{j=1}^{n} \frac{r_j}{(1 - r_j)^{\alpha_j}}$$

whenever $r \in I^n$. The last estimate can be found for instance in [**8**, p. 66]. On the other hand,

$$\mathcal{D}^{-\alpha} f_2(z) = \frac{1}{\Gamma(\alpha)} \sum_{k \in \mathbb{Z}_+^n} 2^{\alpha k} \left(\int_{I^n} (1 - \eta)^{\alpha - 1} \eta^{2^k} d\eta \right) z^{2^k},$$

and

$$M_p(\mathcal{D}^{-\alpha} f_2; r) \approx \left(\prod_{j=1}^{n} \log \frac{e}{1 - r_j} \right)^{1/2}. \tag{4.5}$$

Proof of (iii). Let $u \in h(p, \alpha)$ for some $0 < p < \infty$ and $\alpha_j > 0$. Then

$$M_\infty(\mathcal{D}^{-\alpha}u; r) \le \frac{1}{\Gamma(\alpha)} \int_{I^n} (1 - r)^{\alpha - 1} M_\infty(u; \eta r) d\eta$$
$$\le \|u\|_{\infty, \alpha + 1/p} \frac{1}{\Gamma(\alpha)} \int_{I^n} \frac{(1 - \eta)^{\alpha - 1}}{(1 - \eta r)^{\alpha + 1/p}} d\eta$$
$$\le C(\alpha, p, n) \|u\|_{\infty, \alpha + 1/p} \frac{1}{(1 - r)^{1/p}}.$$

Consequently $\|\mathcal{D}^{-\alpha}u\|_{\infty, 1/p} \le C\|u\|_{\infty, \alpha + 1/p}$. According to the continuous inclusion $h(p, \alpha) \subset h(\infty, \alpha + 1/p)$, see [**4**, p. 733], we deduce that

$$\|\mathcal{D}^{-\alpha}u\|_{\infty, 1/p} \le C\|u\|_{p, \alpha}.$$

The inequality is sharp because of example (4.2), which can easily be checked.

Proof of (iv). Let $u \in h(\infty, \alpha)$ for some $0 < p < \infty$ and $\alpha_j > 0$. By (1.4) and the increasing property of M_p in p,

$$\|\mathcal{D}^{-\alpha}u\|_{p, \log(1/2)} \le \|\mathcal{D}^{-\alpha}u\|_{\max\{2, p\}, \log(1/2)} \le C\|u\|_{\max\{2, p\}, \alpha} \le C\|u\|_{\infty, \alpha}.$$

The inequality is sharp because of example (4.4). Indeed, estimating as in the proof of (ii), we obtain (4.5) and

$$M_\infty(f_2; r) \le \sum_{k \in \mathbb{Z}_+^n} 2^{\alpha k} r^{2^k} \approx \frac{r}{(1 - r)^\alpha}, \qquad r \in I^n.$$

Hence, $\|\mathcal{D}^{-\alpha} f_2\|_{p, \log(1/2)} \approx \|f_2\|_{\infty, \alpha}$.

Proof of (v). Let $u(z) \in h(\infty, \alpha)$ be any function. Then

$$M_\infty(\mathcal{D}^{-\alpha}u; r) \le \frac{1}{\Gamma(\alpha)} \int_{I^n} (1 - \eta)^{\alpha - 1} M_\infty(u; \eta r) d\eta$$
$$\le \frac{1}{\Gamma(\alpha)} \|u\|_{\infty, \alpha} \int_{I^n} \frac{(1 - \eta)^{\alpha - 1}}{(1 - \eta r)^\alpha} d\eta \le C_\alpha \|u\|_{\infty, \alpha} \prod_{j=1}^n \log \frac{e}{1 - r_j}.$$

Thus, $\|\mathcal{D}^{-\alpha}u\|_{\infty, \log(1)} \le C\|u\|_{\infty, \alpha}$. The inequality is sharp because of the example $f_3(z) = 1/(1 - z)^\alpha$, $\alpha_j > 0$. This completes the proof of Theorem 5. □

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