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Vol. 42, No. 3, 2007
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Distributed worldwide by Springer. *Journal of Contemporary Mathematical Analysis* ISSN 1068-3623.

ISSN 1068-3623, Journal of Contemporary Mathematical Analysis, 2007, Vol. 42, No. 3, pp. 116–123. © Allerton Press, Inc., 2007.
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COMPLEX ANALYSIS

Hardy–Stein Identities and Littlewood–Paley Inequalities in Polydisc*

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Received 14 November 2006

Abstract—The paper generalizes the well-known Littlewood-Paley inequality and Hardy-Stein identity. As an application, some area inequalities and quasinorm representations in the space A^p_ω over the polydisc are obtained.

PACS numbers : 32A36

DOI: 10.3103/S1068362307030016

Key words: *Hardy–Stein identity; Littlewood–Paley inequalities.*

1. The classical Littlewood-Paley inequality (see [1]) holds for any function $f(z)$, holomorphic in the unit disc $\mathbb{D} \subset \mathbb{C}$ that belongs to the Hardy class $H^p(\mathbb{D})$, $2 \le p < \infty$:

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|)^{p-1} dm_2(z) \le C_p \|f\|^p_{H^p}, \tag{0.1}$$

where m_2 is the Lebesgue measure in $\mathbb{D}$. There are numerous generalizations of the inequality (0.1), see [2]-[13], among them the following one due to Luecking [4].

Theorem A *If* $0 < p, s < \infty$*, then the inequality*

$$\int_{\mathbb{D}} |f(z)|^{p-s} |f'(z)|^s (1-|z|)^{s-1} dm_2(z) \le C(p,s) \|f\|^p_{H^p} \tag{0.2}$$

holds for all $f \in H^p(\mathbb{D})$ *if and only if* $2 \le s < p+2$.

Our purpose is to obtain some versions of (0.2) for $0 < s < 2$ in the unit polydisc of the n-dimensional complex space $\mathbb{C}^n$. The methods used in Luecking's proof of (0.2) such as distribution of zeros, Blaschke product etc. can hardly be applied in the multidimensional case.

2. By $\mathbb{D}^n = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1, 1 \le j \le n\}$ we denote the unit polydisc with the distinguished boundary $\mathbb{T}^n = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n : |\xi_j| = 1, 1 \le j \le n\}$ ($\mathbb{T}^n$ is an n-dimensional torus), by $H(\mathbb{D}^n)$ the set of all holomorphic functions in $\mathbb{D}^n$. For a measurable in $\mathbb{D}^n$ function $f(z) = f(r\xi)$ we will consider the integral means

$$M_p(f,r) = \left[\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |f(r\xi)|^p dm_n(\xi)\right]^{1/p}, \quad r = (r_1, \dots, r_n) \in I^n,$$

where $0 < p < \infty$, and m_n is the Lebesgue measure on $\mathbb{T}^n$. We note that the ordinary Hardy space over $\mathbb{D}^n$ is the set of those functions holomorphic in $\mathbb{D}^n$, for which

$$\|f\|_{H^p} = \sup_{r \in I^n} M_p(f,r) < +\infty.$$

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For a radial weight function $\omega(r) = \prod_{j=1}^n \omega_j(r_j)$, by L^p_ω $(0 < p < \infty)$ we denote the space of those functions $f(z)$ measurable in $\mathbb{D}^n$, for which the following (quasi-)norm is finite:

$$\|f\|_{L^p_\omega} = \left(C_\omega \int_{\mathbb{D}^n} |f(z)|^p \prod_{j=1}^n \omega_j(|z_j|)\, dm_{2n}(z) \right)^{1/p},$$

where $dm_{2n}(z) = rdr\, dm_n(\xi)$ is the Lebesgue measure in $\mathbb{D}^n$ and the normalizing constant C_ω is chosen to provide $\|1\|_{L^p_\omega} = 1$. By $A^p_\omega = H(\mathbb{D}^n) \cap L^p_\omega$ we denote the holomorphic subspace of L^p_ω, and if $\omega_j(r_j) = (1 - r_j)^{\alpha_j}$ $(\alpha_j > -1,\ 1 \le j \le n)$, then we write L^p_α and A^p_α for the spaces L^p_ω and A^p_ω respectively.

The following notation is standard for $\mathbb{C}^n$:

$$r\zeta = (r_1\zeta_1, \dots, r_n\zeta_n), \quad dr = dr_1 \cdots dr_n,$$

$$(1 - |\zeta|)^\alpha = \prod_{j=1}^n (1 - |\zeta_j|)^{\alpha_j}, \quad \zeta^\alpha = \prod_{j=1}^n \zeta_j^{\alpha_j}, \quad \alpha q + 1 = (\alpha_1 q + 1, \dots, \alpha_n q + 1)$$

for $\zeta \in \mathbb{C}^n$, $r \in I^n$, $q \in \mathbb{R}$ and $\alpha = (\alpha_1, \dots, \alpha_n)$ for multiindices.

By $\mathbb{Z}^n_+$ we denote the set of all multiindices $k = (k_1, \dots, k_n)$ with nonnegative integers $k_j \in \mathbb{Z}_+$. We write $A \lesssim B$ if there is a constant $c > 0$ such that $A \le cB$. The notation $A \asymp B$ will mean that $A \lesssim B$ and $B \lesssim A$. For any p, $1 \le p \le \infty$ we put $p' = p/(p-1)$ (the conjugate index).

Further, for any function $f \in H(\mathbb{D}^n)$ with expansion $f(z) = \sum_{k \in \mathbb{Z}^n_+} a_k r^k \xi^k$, where $z = r\xi$, $r \in I^n$ and $\xi \in \mathbb{T}^n$, we introduce the radial fractional integro-differential operator of order $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j \in \mathbb{R}$, as follows:

$$\mathcal{D}^\alpha f(z) \equiv \mathcal{D}^\alpha_r f(z) = \sum_{k \in \mathbb{Z}^n_+} \prod_{j=1}^n (1 + k_j)^{\alpha_j} a_k r^k \xi^k.$$

Clearly, $\mathcal{D}^\alpha_r f(z) = \mathcal{D}^{\alpha_1}_{r_1} \mathcal{D}^{\alpha_2}_{r_2} \dots \mathcal{D}^{\alpha_n}_{r_n} f$, where $\mathcal{D}^{\alpha_j}_{r_j}$ means the same operator acting only in the variable r_j.

3. Theorem 1 that follows yields a family of inequalities which generalize the Littlewood-Paley type inequalities the unit disc proved by Yamashita [2] and Luecking [4].

Theorem 1. *If $0 < \alpha < s < 2$ and $s < p$, then for any $\lambda > (p - s)/\alpha$*

$$\int_{\mathbb{D}^n} |f(z)|^{p-s} |\mathcal{D}^1 f(z)|^s (1 - |z|)^{s-1} dm_{2n}(z) \lesssim \|f\|^{p-s}_{H^\lambda} \|\mathcal{D}^{\alpha/s} f\|^s_{H^s}. \tag{0.3}$$

Remark 1. *For $p = 2$, the formal limits as $s \to 2-$ and $\alpha \to +0$, reduce* (0.3) *to the classical Littlewood-Paley inequality* (0.1) *in the polydisc for $p = 2$.*

Without loss of generality, we can simplify the further proofs by considering only the case $n = 2$.

Proof of Theorem 1: We start with some additional notation and statements. For a fixed $\delta > 1$, let $\Gamma_\delta(\xi) = \{z \in \mathbb{D} : |1 - \overline{\xi} z| \le \delta(1 - |z|)\}$ denote the admissible approach domain with the vertex at $\xi \in \mathbb{T}$. Further, for any arc $I \subset \mathbb{T}$ of the length $|I|$ we define the Carleson square $\square I = \left\{ z \in \mathbb{D};\ \frac{z}{|z|} \in I,\ 1 - |z| \le \frac{1}{2\pi}|I| \right\}$ over I. Following [14], we consider the functions

$$A_p(f)(\xi) = \left(\int_{\Gamma_\delta(\xi)} \frac{|f(z)|^p}{(1 - |z|)^2} dm_2(z) \right)^{1/p}, \quad p < \infty,$$

$$C_p(f)(\xi) = \sup_{I \supset \xi} \left(\frac{1}{|I|} \int_{\square I} \frac{|f(z)|^p}{1 - |z|} dm_2(z) \right)^{1/p}, \quad p < \infty, \quad \xi \in \mathbb{T}. \tag{0.4}$$

Besides, we note that for any measurable in $\mathbb{D}$ functions $f(z)$ and $g(z)$

$$\int_{\mathbb{D}} \frac{|f(z)||g(z)|}{1-|z|} dm_2(z) \lesssim \int_{\mathbb{T}} A_p(f)(\xi)\, C_{p'}(g)(\xi)\, dm(\xi), \quad 1<p<\infty, \tag{0.5}$$

where $dm(\xi) = dm_1(\xi)$ is the Lebesgue measure on $\mathbb{T}$ (see [14], [7]). In addition, for any numbers $0<q<\infty$, $\alpha>0$, $\beta>0$ and any function $f(z)$ measurable in $\mathbb{D}$ (see [7])

$$\Big\| C_q\big(|f(z)|(1-|z|)^\alpha\big)\Big\|_{L^\infty}^q \asymp \sup_{w\in\mathbb{D}} (1-|w|)^\beta \int_{\mathbb{D}} \frac{|f(z)|^q(1-|z|)^{\alpha q-1}}{|1-\overline{w}z|^{\beta+1}} dm_2(z). \tag{0.6}$$

We consider a version of Luzin area integral

$$S(f)(\xi) = \left(\int_{\Gamma_\delta(\xi)} |\mathcal{D}^1 f(z)|^2 dm_2(z)\right)^{1/2}, \quad \xi\in\mathbb{T}, \quad \delta>1,$$

and recall Luzin's well-known inequality (see [1])

$$\|S(f)\|_{L^p(\mathbb{T})} \asymp \|f\|_{H^p}, \quad 0<p<\infty. \tag{0.7}$$

By L we denote the left-hand side integral in (0.3) and write it in the form

$$L = \int_{\mathbb{D}} (1-|z_2|)^{s-1} \left[\int_{\mathbb{D}} |f(z)|^{p-s} |\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-1} dm_2(z_1)\right] dm_2(z_2). \tag{0.8}$$

Further, by J we denote the inner integral in (0.8) and fixing any α, $0<\alpha<s$, we use (0.5) to estimate J:

$$\begin{aligned} J &= \int_{\mathbb{D}} |\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha} \cdot |f(z)|^{p-s}(1-|z_1|)^\alpha \frac{dm_2(z_1)}{1-|z_1|} \\ &\lesssim \int_{\mathbb{T}} A_{2/s}\Big(|\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha}\Big)(\xi_1) \\ &\qquad \times C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^\alpha\Big)(\xi_1) dm(\xi_1) \\ &\le \Big\| C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^\alpha\Big)\Big\|_{L^\infty} \\ &\qquad \times \int_{\mathbb{T}} A_{2/s}\Big(|\mathcal{D}^1 f(z)|^s (1-|z_1|)^{s-\alpha}\Big)(\xi_1) dm(\xi_1). \end{aligned} \tag{0.9}$$

Separately estimating the last integral

$$J_1 = \int_{\mathbb{T}} \left[\int_{\Gamma_\delta(\xi_1)} |\mathcal{D}^1 f(z)|^2 (1-|z_1|)^{-2\alpha/s} dm_2(z_1)\right]^{s/2} dm(\xi_1),$$

according to (0.7) and the fractional differentiation rule [6] (pp. 179 and 186) we get

$$J_1 \lesssim \int_{\mathbb{T}} \left[\int_{\Gamma_\delta(\xi_1)} |\mathcal{D}_{r_1}^{\alpha/s} \mathcal{D}^1 f(z)|^2 dm_2(z_1)\right]^{s/2} dm(\xi_1) \lesssim \big\|\mathcal{D}_{r_2}^1 \mathcal{D}_{r_1}^{\alpha/s} f\big\|_{H^s_{z_1}}^s, \tag{0.10}$$

where $H^s_{z_1}$ means the Hardy class by the variable z_1. Then, uniting the inequalities (0.8) - (0.10) and applying Fatou's lemma and (0.5) we get

$$\begin{aligned} L &\lesssim \liminf_{r_1\to 1} \int_{\mathbb{T}}\int_{\mathbb{D}} (1-|z_2|)^{s-1} \Big\| C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^\alpha\Big)\Big\| \\ &\qquad \times \big|\mathcal{D}_{r_2}^1 \mathcal{D}_{r_1}^{\alpha/s} f\big|^s dm(\xi_1) dm_2(z_2) \\ &\lesssim \Big\| C_{(2/s)'}\Big(\Big\| C_{(2/s)'}\Big(|f(z)|^{p-s}(1-|z_1|)^\alpha\Big)\Big\|_{L^\infty} (1-|z_2|)^\alpha\Big)(\xi_2)\Big\|_{L^\infty} \end{aligned}$$

$$\times \liminf_{r_1\to 1}\int_{\mathbb{T}}\int_{\mathbb{T}} A_{2/s}\Big(\big|\mathcal{D}^1_{r_2}\mathcal{D}^{\alpha/s}_{r_1} f\big|^s(1-|z_2|)^{s-\alpha}\Big)(\xi_2)dm(\xi_2)dm(\xi_1) \equiv J_2J_3.$$

To estimate J_2 and J_3, we again use fractional differentiation rule and Fatou's lemma along with (0.7) and the equality $\mathcal{D}^{\gamma_1}_r\mathcal{D}^{\gamma_2}_r = \mathcal{D}^{\gamma_2}_r\mathcal{D}^{\gamma_1}_r$:

$$J_3 = \liminf_{r_1\to 1}\int_{\mathbb{T}}\int_{\mathbb{T}}\left[\int_{\Gamma_\delta(\xi_2)}\big|\mathcal{D}^1_{r_2}\mathcal{D}^{\alpha/s}_{r_1} f\big|^2(1-|z_2|)^{-2\alpha/s}dm_2(z_2)\right]^{s/2}dm(\xi_2)dm(\xi_1)$$
$$\lesssim \liminf_{r_1\to 1}\int_{\mathbb{T}}\int_{\mathbb{T}}\left[\int_{\Gamma_\delta(\xi_2)}\big|\mathcal{D}^{\alpha/s}_{r_2}\mathcal{D}^1_{r_2}\mathcal{D}^{\alpha/s}_{r_1} f\big|^2dm_2(z_2)\right]^{s/2}dm(\xi_2)dm(\xi_1)$$
$$\lesssim \liminf_{r_1\to 1}\int_{\mathbb{T}}\big\|\mathcal{D}^{\alpha/s}f\big\|^s_{H^s_{z_2}}dm(\xi_1) = \big\|\mathcal{D}^{\alpha/s}f\big\|^s_{H^s}.$$

To estimate J_2 with $\beta > 0$ great enough, we observe that by (0.6) the inner norm in J_2 can be estimated as follows:

$$\left\|C_{2/(2-s)}\Big(|f(z)|^{p-s}(1-|z_1|)^\alpha\Big)\right\|^{2/(2-s)}_{L^\infty} \lesssim$$
$$\lesssim \sup_{w\in\mathbb{D}}(1-|w|)^\beta\int_{\mathbb{D}}|f(z_1,z_2)|^{2(p-s)/(2-s)}\frac{(1-|z_1|)^{2\alpha/(2-s)-1}}{|1-\overline{w}z_1|^{\beta+1}}dm_2(z_1)$$
$$\le \|f\|^{2(p-s)/(2-s)}_{H^\lambda_{z_1}}\sup_{w\in\mathbb{D}}(1-|w|)^\beta$$
$$\times\int_{\mathbb{D}}\frac{(1-|z_1|)^{2\alpha/(2-s)-(2/\lambda)(p-s)/(2-s)-1}}{|1-\overline{w}z_1|^{\beta+1}}dm_2(z_1) \lesssim \|f\|^{2(p-s)/(2-s)}_{H^\lambda_{z_1}}.$$

Consequently,

$$J_2 \lesssim \left\|C_{2/(2-s)}\Big(\|f\|^{p-s}_{H^\lambda_{z_1}}(1-|z_2|)^\alpha\Big)(\xi_2)\right\|_{L^\infty}$$
$$\lesssim \Bigg[\sup_{w\in\mathbb{D}}(1-|w|)^\beta\int_{\mathbb{D}}\|f(z_1,z_2)\|^{2(p-s)/(2-s)}_{H^\lambda_{z_1}}$$
$$\times\frac{(1-|z_2|)^{2\alpha/(2-s)-1}}{|1-\overline{w}z_2|^{\beta+1}}dm_2(z_2)\Bigg]^{(2-s)/2}$$
$$\lesssim \|f\|^{p-s}_{H^\lambda(\mathbb{D}^2)}\Bigg[\sup_{w\in\mathbb{D}}(1-|w|)^\beta$$
$$\times\int_{\mathbb{D}}\frac{(1-|z_2|)^{2\alpha/(2-s)-(2/\lambda)(p-s)/(2-s)-1}}{|1-\overline{w}z_2|^{\beta+1}}dm_2(z_2)\Bigg]^{(2-s)/2} \lesssim$$
$$\lesssim \|f\|^{p-s}_{H^\lambda}.$$

Thus, $L \lesssim \|f\|^{p-s}_{H^\lambda}\big\|\mathcal{D}^{\alpha/s}f\big\|^s_{H^s}$ for any $\lambda > (p-s)/\alpha$, and the proof is complete.

4. Our next theorem establishes some weighted analogs of the Hardy-Stein identity (see [15]) and the corresponding Littlewood-Paley inequalities.

Theorem 2. *If $f(z) \in H(\mathbb{D}^n)$, $0 < p < \infty$, and $\omega_j(r_j) \in C^1[0,1)$, $j = 1,\dots,n$, is such that*

$$\omega_j(r_j)\frac{\partial}{\partial r_j}M^p_p(f,r) = o(1) \quad as \quad r_j \to 1-, \tag{0.11}$$

then:

a) *The following identity is true:*

$$\int_{\mathbb{D}^n} \prod_{j=1}^{n} \omega_j(r_j) \cdot f^{\#}(z) dm_{2n}(z)$$

$$= (-1)^n \int_{\mathbb{D}^n} \prod_{j=1}^{n} \omega_j'(r_j) \frac{\partial^n}{\partial r_1 \cdots \partial r_n} |f(z)|^p dm_{2n}(z), \tag{0.12}$$

where $f^{\#}(z) = \Delta_{z_1}\Delta_{z_2}\ldots\Delta_{z_n}|f(z)|^p$ *and* Δ_{z_j} *is the ordinary Laplacian in the variable* z_j. *The conditions (11) can be omitted in case of standard weight functions* $\omega_j(r_j) = (1-r_j)^{\alpha_j}$ $(\alpha_j > 0)$.

b) *For* $n = 1$, (0.12) *is refined by the identity*

$$\int_{\mathbb{D}} (1-|z|)^{\alpha} f^{\#}(z) dm_2(z) = \alpha \int_{\mathbb{D}} (1-|z|)^{\alpha-1} \frac{\partial}{\partial r} |f(z)|^p dm_2(z), \tag{0.13}$$

which is true for any $p > 0$ *and* $\alpha > 0$, *if at least one side integral exists. Here*

$$f^{\#}(z) = \Delta|f(z)|^p = p^2|f(z)|^{p-2}|f'(z)|^2. \tag{0.14}$$

c) *The integrals*

$$A(f;p,\alpha) = \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2(1-|z|)^{\alpha} dm_2(z),$$

$$B(f;p,\alpha) = \int_{\mathbb{D}} |f(z)|^{p-1}|f'(z)|(1-|z|)^{\alpha-1} dm_2(z)$$

satisfy the inequalities: if $p > 0$ *and* $\alpha > 0$, *then*

$$A(f;p,\alpha) \le \frac{\alpha}{p} B(f;p,\alpha), \tag{0.15}$$

where α/p *is the best constant; if* $p > 0$ *and* $\alpha > 1$, *then there exists some constant* $C_{\alpha,p} > 0$ *such that*

$$B(f;p,\alpha) \le C_{\alpha,p} A(f;p,\alpha). \tag{0.16}$$

Remark 2. *For* $p = 2$, *the inequalities* (0.15) *and* (0.16) *are proved in [10]. Their analogs for integer values of* p $(p \ge 2)$, *the unit disc* $\mathbb{D}$ *and the ball in* $\mathbb{C}^n$ *are proved in [11], [12] by some other method.*

Below we will need the following generalization of the Hardy-Stein identity (see [15]) in the polydisc.

Lemma 1. *If* $f(z) \in H(\mathbb{D}^n)$, $0 < p < \infty$, *then for any* $r = (r_1, \ldots, r_n) \in I^n$

$$\prod_{j=1}^{n} r_j \frac{\partial^n}{\partial r_1 \ldots \partial r_n} M_p^p(f,r) = \frac{1}{(2\pi)^n} \int_{|z_1|<r_1} \cdots \int_{|z_n|<r_n} f^{\#}(z) dm_{2n}(z). \tag{0.17}$$

Proof: by successive application of Green's formula.

Remark *Due to* (0.14), *for* $n = 1$ *the formula* (0.17) *coincides with the well-known Hardy-Stein identity [15].*

Proof of Theorem 2: An application of Lemma 1 leads to the identity

$$r_1 r_2 \int_{\mathbb{T}^2} \Delta_{z_1}\Delta_{z_2}|f(z_1,z_2)|^p dm(\xi_1)dm(\xi_2) =$$

$$= \frac{\partial^2}{\partial r_1 \partial r_2} \int_0^{r_1} \int_0^{r_2} \int_{\mathbb{T}^2} \Delta_{z_1} \Delta_{z_2} |f(z_1, z_2)|^p \rho_1 \rho_2 dm(\xi_1) dm(\xi_2) d\rho_1 d\rho_2 =$$

$$= (2\pi)^2 \frac{\partial^2}{\partial r_1 \partial r_2} \left[r_1 r_2 \frac{\partial^2}{\partial r_1 \partial r_2} M_p^p(f, r_1, r_2) \right]. \quad (0.18)$$

To prove the identity (0.13), we integrate by parts the left-hand side integral in (0.13) and use (0.18):

$$\frac{1}{2\pi} \int_{\mathbb{D}} (1-|z|)^\alpha f^{\#}(z) dm_2(z) = \frac{1}{2\pi} \int_0^1 (1-r)^\alpha \left[\int_{-\pi}^{\pi} \Delta |f(re^{i\theta})|^p d\theta \right] r dr =$$

$$= \int_0^1 (1-r)^\alpha \left[\frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} M_p^p(f, r) \right) \right] dr =$$

$$= \lim_{r \to 1-} (1-r)^\alpha \, r \frac{\partial}{\partial r} M_p^p(f, r) + \alpha \int_0^1 (1-r)^{\alpha - 1} \, r \frac{\partial}{\partial r} M_p^p(f, r) dr. \quad (0.19)$$

If the right-hand side integral in (0.13) or (0.19) exists, then the limit (0.19) is zero. Indeed, by the Hardy-Stein identity $r \frac{\partial}{\partial r} M_p^p(f, r)$ is an increasing function for $r \in (0, 1)$. Hence, for any $\rho \in (0, 1)$

$$\int_\rho^{(1+\rho)/2} (1-r)^{\alpha-1} \, r \frac{\partial}{\partial r} M_p^p(f, r) dr \geq C_\alpha \, \rho \, (1-\rho)^\alpha \frac{\partial}{\partial \rho} M_p^p(f, \rho).$$

By the Cauchy criterion, $\lim_{\rho \to 1-} (1-\rho)^\alpha \, \frac{\partial}{\partial \rho} M_p^p(f, \rho) = 0$, and by (19)

$$\frac{1}{2\pi} \int_{\mathbb{D}} (1-|z|)^\alpha f^{\#}(z) dm_2(z) = \alpha \int_0^1 (1-r)^{\alpha-1} \, r \frac{\partial}{\partial r} M_p^p(f, r) dr.$$

Thus, the statement b) is proved.

Turning to the proof of the inequality (0.15), observe that the example $f(z) = z$ shows that the constant α/p is the best possible. Further, the identity (0.13) can be written in the form

$$A(f; p, \alpha) = \frac{\alpha}{p} \int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|^{p-1} \left(\frac{\partial}{\partial r} \big| f(re^{i\theta}) \big| \right) (1-r)^{\alpha-1} r dr d\theta.$$

It remains to check that $\left| \frac{\partial}{\partial r} |f(re^{i\theta})| \right| \leq \left| f'(re^{i\theta}) \right|$ since $\big| |f(re^{i\theta})| - |f(\rho e^{i\theta})| \big| \leq \left| f(re^{i\theta}) - f(\rho e^{i\theta}) \right|$.

To prove the inequality (0.16), we observe that by the Cauchy-Schwarz inequality

$$B(f; p, \alpha) \leq \sqrt{A(f; p, \alpha)} \left(\int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\alpha-2} dm_2(z) \right)^{1/2}.$$

Hence, it remains only to verify that

$$\int_{\mathbb{D}} |f(z)|^p (1-|z|)^{\alpha-2} dm_2(z) \lesssim A(f; p, \alpha), \quad p > 0, \quad \alpha > 1.$$

Integration by parts yields

$$\frac{p^2}{2\pi\alpha} A(f; p, \alpha) = \int_0^1 (1-r)^{\alpha-1} \, r \frac{\partial}{\partial r} M_p^p(f, r) dr$$

$$= \lim_{r \to 1-} (1-r)^{\alpha-1} \, r \, M_p^p(f, r) + \frac{1}{2\pi} \int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|^p (1-r)^{\alpha-2} (\alpha r - 1) dr d\theta.$$

This equality shows that if $A(f; p, \alpha)$ exists, then $f(z) \in A^p_{\alpha-2}(\mathbb{D})$. Consequently, $\lim_{r \to 1-} (1 - r)^{\alpha-1} M_p^p(f, r) = 0$, and hence

$$A(f; p, \alpha) = \frac{\alpha}{p^2} \int_0^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|^p (1-r)^{\alpha-2} (\alpha r - 1) \, dr \, d\theta \geq$$

$$\geq \frac{\alpha(\alpha-1)}{2p^2} \int_{(\alpha+1)/(2\alpha)}^1 \int_{-\pi}^{\pi} |f(re^{i\theta})|^p (1-r)^{\alpha-2} dr \, d\theta \geq$$

$$\geq C(\alpha,p)\int_{\mathbb{D}}|f(z)|^p(1-|z|)^{\alpha-2}dm_2(z),$$

which leads to the desired result. Thus, the statement c) is proved. The statement a) is proved similarly, by using (0.18).

5. Our last theorem gives a characterization of the weighted spaces A^p_ω with general weights in the bidisc and some representations of (quasi-)norms in A^p_ω by means of Luecking type integrals (0.2).

Theorem 3. *Let $0<p<\infty$, $f(z)\in H(\mathbb{D}^2)$, $\omega_j(r_j)\in L^1(0,1)$, $\omega_j(r_j)>0$, $j=1,2$. Then the following representations are valid:*

$$\|f\|^p_{A^p_\omega(\mathbb{D}^2)}\asymp|f(0,0)|^p+\int_{\mathbb{D}^2}\Big(\Delta_{z_1}\Delta_{z_2}|f(z_1,z_2)|^p+\Delta_{z_1}|f(z_1,0)|^p+$$
$$+\Delta_{z_2}|f(0,z_2)|^p\Big)\prod_{j=1}^2h_{\omega_j}(|z_j|)\,dm_4(z), \tag{0.20}$$

$$\|f\|^p_{A^p_\omega(\mathbb{D}^2)}+|f(0,0)|^p=\|f(\cdot,0)\|^p_{A^p_{\omega_1}}+\|f(0,\cdot)\|^p_{A^p_{\omega_2}}+$$
$$+C_\omega\int_{\mathbb{D}^2}f^\#(z_1,z_2)\prod_{j=1}^2h_{\omega_j}(|z_j|)\,dm_4(z), \tag{0.21}$$

where $A^p_{\omega_j}$ are one-dimensional spaces by the variables z_j, while h_{ω_j} are the weight functions

$$h_{\omega_j}(|z_j|)=\int_{|z_j|}^1\left(\int_{\rho_j}^1\omega_j(x)xdx\right)\frac{d\rho_j}{\rho_j}.$$

In particular, $f\in A^p_\alpha(\mathbb{D}^2)$ if and only if $f^\#\in L^1_{\alpha+2}(\mathbb{D}^2)$ ($\alpha_j>-1$).

Remark *If $n=1$ and $\omega(r)=(1-r)^\alpha$ ($\alpha>-1$), then by (0.14) the limit of the relation (0.20) as $\alpha\to-1$ coincides with Yamashita's characterization [2] of the Hardy classes $H^p(\mathbb{D})$. Some analogs of (0.20) and (0.21) for the ball in $\mathbb{C}^n$ can be found in [3], [5] and [8].*

Proof of Theorem 3: The integrated Hardy-Stein identity (see Lemma 1)

$$M^p_p(f,r_1,r_2)+|f(0,0)|^p=M^p_p(f,0,r_2)+M^p_p(f,r_1,0)+$$
$$+\frac{1}{(2\pi)^2}\int_0^{r_1}\int_0^{r_2}\left(\int_{|z_1|<\rho_1}\int_{|z_2|<\rho_2}f^\#(z_1,z_2)dm_4(z)\right)\frac{d\rho_1d\rho_2}{\rho_1\rho_2}$$

can be integrated with the weight $(2\pi)^2C_{\omega_1}C_{\omega_2}\omega_1(r_1)\omega_2(r_2)r_1r_2dr_1dr_2$. This gives

$$\|f\|^p_{A^p_\omega}+|f(0,0)|^p\equiv J_1+J_2+J_3,$$

where

$$J_1=\big\|f(z_1,0)\big\|^p_{A^p_{\omega_1}(\mathbb{D})}$$
$$=|f(0,0)|^p+2\pi C_{\omega_1}\int_0^1M_1\Big(\Delta_{z_1}|f(z_1,0)|^p,r_1\Big)h_{\omega_1}(r_1)r_1dr_1,$$
$$J_2=\big\|f(0,z_2)\big\|^p_{A^p_{\omega_2}(\mathbb{D})}$$
$$=|f(0,0)|^p+2\pi C_{\omega_2}\int_0^1M_1\Big(\Delta_{z_2}|f(0,z_2)|^p,r_2\Big)h_{\omega_2}(r_2)r_2dr_2,$$
$$J_3=C_{\omega_1}C_{\omega_2}\int_{I^2}\left[\int_0^{r_1}\int_0^{r_2}\left(\int_{|z_1|<\rho_1}\int_{|z_2|<\rho_2}f^\#(z_1,z_2)dm_4(z)\right)\frac{d\rho_1d\rho_2}{\rho_1\rho_2}\right]$$

$$\times\, \omega(r) r dr$$
$$= (2\pi)^2 C_{\omega_1} C_{\omega_2} \int_0^1 \int_0^1 M_1\big(f^{\#}(z_1, z_2), r_1, r_2\big) h_{\omega_1}(r_1) h_{\omega_2}(r_2) r_1 r_2 dr_1 dr_2,$$

and the proof is complete.

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