JOURNAL OF COMPUTATIONAL ANALYSIS AND APPLICATIONS,VOL.11,NO.2,239-251,2009,COPYRIGHT 2009 EUDOXUS PRESS, LLC

Holomorphic Functions on the Mixed Norm Spaces on the Polydisc II

Karen Avetisyan
Faculty of Physics, Yerevan State University,
Alex Manoogian st. 1, Yerevan, 375025, Armenia
E-mail: avetkaren@ysu.am

Stevo Stević
Mathematical Institute of the Serbian Academy of Science,
Knez Mihailova 36/III, 11000 Beograd, Serbia
E-mail: sstevic@ptt.yu; sstevo@matf.bg.ac.yu

Abstract

The paper continues the investigation of holomorphic mixed norm spaces $\mathcal{A}^{p,q}_{\vec{\omega}}$ in the unit polydisc of $\mathbb{C}^n$. We prove that a mixed norm is equivalent to a "derivative norm" for all $0 < p \le \infty, 0 < q < \infty$ and a large class of weights $\vec{\omega}$. As an application, we prove that pluriharmonic conjugation is bounded in these mixed norm spaces.

2000 AMS Subject Classification: 32A37, 32A36.

Key words and phrases: holomorphic function, polydisc, mixed norm space, weight function, pluriharmonic conjugate.

1 Introduction

Let $U^1 = U$ be the unit disc in the complex plane, U^n the unit polydisc in $\mathbb{C}^n$, and $H(U^n)$ the set of all holomorphic functions on U^n.

For the integral means of a function f given in U^n, we write

$$M_p(f,r) = \left(\frac{1}{(2\pi)^n} \int_{[0,2\pi)^n} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|^p d\theta \right)^{1/p},$$

$r = (r_1, \dots, r_n), 0 \le r_j < 1, j \in \{1, \dots, n\}, 0 < p < \infty$, $\theta = (\theta_1, \dots, \theta_n)$, $d\theta = d\theta_1 \cdots d\theta_n$ and

$$M_\infty(f,r) = \sup_{\theta \in [0,2\pi)^n} |f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})|.$$

Let $\omega(x), 0 \le x < 1$, be a weight function which is positive and integrable on (0,1). We extend ω on U by setting $\omega(z) = \omega(|z|)$, and also on U^n by $\vec{\omega} = (\omega_1, \dots, \omega_n)$.

Let $\mathcal{L}^{p,q}_{\vec{\omega}} = \mathcal{L}^{p,q}_{\vec{\omega}}(U^n), 0 < p \le \infty, 0 < q < \infty$, denote the mixed norm space, the class of all measurable functions defined on U^n such that

$$\|f\|^q_{p,q,\vec{\omega}} = \int_{(0,1)^n} M^q_p(f,r) \prod_{j=1}^n \omega_j(r_j) dr_j < \infty,$$

and $\mathcal{A}^{p,q}_{\vec{\omega}} = \mathcal{A}^{p,q}_{\vec{\omega}}(U^n)$ be the intersection of $\mathcal{L}^{p,q}_{\vec{\omega}}$ and $H(U^n)$. When $p = q$ we come to weighted Bergman spaces $\mathcal{A}^{p,p}_{\vec{\omega}} = \mathcal{A}^{p}_{\vec{\omega}}$ with general weights $\vec{\omega}$. Mixed norm, weighted Bergman and closely related spaces have been studied, for example, in [1, 2, 3, 4, 6, 7, 8, 10, 11, 13, 14, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

Following [12], for a given weight ω on U, define the *distortion function* of ω by

$$\psi(r) = \psi_\omega(r) = \frac{1}{\omega(r)} \int_r^1 \omega(t)dt, \qquad 0 \le r < 1.$$

We put $\psi(z) = \psi(|z|)$ for $z \in U$. Also, a class of *admissible weights*, a large class of weight functions ω in U is defined in [12]. For a list of examples of admissible weights, see [12, pp. 660-663].

In [20, Theorem 1] the second author, among others, proved the following result.

Theorem A. *Let $f \in H(U^n)$ and $\omega_j(z_j)$, $j = 1, \dots, n$ are admissible weights on the unit disc U, with distortion functions $\psi_j(z_j)$. If $0 < p, q < \infty$, and $f \in \mathcal{A}^{p,q}_{\vec{\omega}}$, then for all $j = 1, \dots, n$, $\psi_j(z_j)\frac{\partial f}{\partial z_j}(z) \in \mathcal{L}^{p,q}_{\vec{\omega}}$, and there is a positive constant $C = C(p, q, \vec{\omega}, n)$ such that*

$$\|f\|_{p,q,\vec{\omega}} \ge C|f(0)| + C \sum_{j=1}^n \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{p,q,\vec{\omega}}. \tag{1.1}$$

For $1 \le p, q < \infty$ the reverse inequality holds as well.

Remark 1. For all $0 < p, q < \infty$ the equivalence between the left-hand and right-hand sides of (1.1) is established in [15, 20] for standard weights $\omega_j(z_j) = (1 - |z_j|)^{\alpha_j}$, $\alpha_j > -1$. See also [13] and [14].

In [9] the authors solved an open problem posed by S. Stević ([13, 14]) regarding the reverse inequality in (1.1) for the case of the unit disk, by proving the following result:

Theorem B. *Assume $0 < p \le \infty, 0 < q < \infty$, and that ω is a differentiable weight function on U satisfying the following condition*

$$\frac{\omega'(r)}{\omega^2(r)} \int_r^1 \omega(s)ds \le L < \infty, \qquad r \in (0,1), \tag{1.2}$$

for a positive constant L. Then

$$\int_0^1 M^q_p(f,r)\omega(r)dr \asymp |f(0)|^q + \int_0^1 M^q_p(f',r)(\psi_\omega(r))^q \omega(r)dr \tag{1.3}$$

for all $f \in H(U)$.

We write $a \asymp b$ if the ratio a/b is bounded from above and below by two positive constants when the variable varies, and say that a and b are comparable. Note that condition (1.2) is weaker than that of admissible weights, see [9].

An interesting problem is to extend Theorem B to the polydisc case. This will be done by proving the next theorem.

Theorem 1. *Let $f \in H(U^n), 0 < p \leq \infty, 0 < q < \infty$, and the weights $\omega_j(z_j)$, $j = 1, \dots, n$, satisfy condition (1.2), with distortion functions $\psi_j(z_j)$, $j = 1, \dots, n$. Then $f \in \mathcal{A}^{p,q}_{\vec{\omega}}$ if and only if $\psi_j(z_j)\frac{\partial f}{\partial z_j}(z) \in \mathcal{L}^{p,q}_{\vec{\omega}}$ for all $j = 1, \dots, n$. Moreover,*

$$\|f\|_{p,q,\vec{\omega}} \asymp |f(0)| + \sum_{j=1}^{n} \left\| \psi_j \frac{\partial f}{\partial z_j} \right\|_{p,q,\vec{\omega}}. \tag{1.4}$$

Theorem 1 generalizes both Theorems A and B. In Section 2 we present several auxiliary results which will be used in the proofs of the main results of this paper. A proof of Theorem 1 is given in Section 3. In Section 4 we turn to pluriharmonic functions in U^n, that is, the real parts of holomorphic functions. As an application of Theorem 1, we prove that the operator of pluriharmonic conjugation is bounded in mixed norm spaces $\mathcal{L}^{p,q}_{\vec{\omega}}(U^n)$ for all $0 < p \leq \infty, 0 < q < \infty$.

2 Auxiliary results

In this section we collect and prove several auxiliary lemmas which we use in the proof of the main result. Throughout the paper, the letters $C(p, q, \alpha, \beta, \dots), C_\alpha$ etc. stand for positive constants depending only on the parameters indicated and which may vary from line to line.

Lemma 1. **([9])** *Let $\{A_k\}_{k=0}^\infty$ be a sequence of complex numbers, $\alpha, \gamma > 0$. Then the quantities*

$$Q_1 = \sum_{k=0}^{\infty} e^{-k\alpha}|A_k|^\gamma, \quad Q_2 = |A_0|^\gamma + \sum_{k=0}^{\infty} e^{-k\alpha}|A_{k+1} - A_k|^\gamma$$

are comparable.

Lemma 2. **([9])** *Given a function φ on $[0, 1)$ define the sequence $\{r_k\}_{k=0}^\infty \subset [0, 1)$ by $\varphi(r_k) = e^k$, $k \geq 0$.*
(a) If the function φ satisfies $\varphi(0) = 1$ and

$$\sup_{0<r<1} \frac{\varphi''(r)\varphi(r)}{\varphi'(r)^2} \leq M < \infty, \tag{2.1}$$

then for every $k \geq 0$,

$$\frac{\varphi'(y)}{\varphi'(x)} \leq e^{2M}, \qquad r_k < x < y < r_{k+2}.$$

(b) If the function φ satisfies

$$\sup_{0<r<1} \frac{|\varphi''(r)|\varphi(r)}{\varphi'(r)^2} \le M < \infty, \tag{2.2}$$

then for every $k \ge 0$,

$$e^{-2M} \le \frac{\varphi'(y)}{\varphi'(x)} \le e^{2M}, \qquad x, y \in [r_k, r_{k+2}].$$

Lemma 3. *Let $f \in H(U^n), 0 < p \le \infty, \ell = \min\{1, p\}$. Then for any r_j, ρ_j, $0 < r_j < \rho_j < 1$, $j = 1, \dots, n$,*

$$M_p^\ell(f, \rho_1, \dots, \rho_n) - M_p^\ell(f, r_1, \dots, r_n) \le C \sum_{j=1}^n (\rho_j - r_j)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_j}, \rho_1, \dots, \rho_n \right),$$

where the positive constant C depends only on p and n.

Proof. First assume that $n = 2$. Then by [20, Lemma 3] and the monotonicity of the integral means, we have that

$$\begin{aligned} M_p^\ell(f, \rho_1, \rho_2) &- M_p^\ell(f, r_1, r_2) \\ &= \Big(M_p^\ell(f, \rho_1, \rho_2) - M_p^\ell(f, r_1, \rho_2) \Big) + \Big(M_p^\ell(f, r_1, \rho_2) - M_p^\ell(f, r_1, r_2) \Big) \\ &\le C(\rho_1 - r_1)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_1}, \rho_1, \rho_2 \right) + C(\rho_2 - r_2)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_2}, r_1, \rho_2 \right) \\ &\le C(\rho_1 - r_1)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_1}, \rho_1, \rho_2 \right) + C(\rho_2 - r_2)^\ell M_p^\ell \left(\frac{\partial f}{\partial z_2}, \rho_1, \rho_2 \right). \end{aligned}$$

For $n > 2$ the proof is similar and will be omitted.

Lemma 4. *Let $f \in H(U^n)$ and $0 < p \le \infty$.*
(a) Then for any $0 < r_j < \rho_j < 1$, $j, k \in \{1, \dots, n\}$

$$M_p \left(\frac{\partial f}{\partial z_k}, r_1, \dots, r_n \right) \le C \frac{M_p(f, \rho_1, \dots, \rho_n)}{\rho_k - r_k},$$

where the positive constant C depends only on p and n.
(b) If $u = Re\, f$ in U^n and $1 \le p \le \infty$, then for any $0 < r_j < \rho_j < 1$, $j, k \in \{1, \dots, n\}$

$$M_p \left(\frac{\partial f}{\partial z_k}, r_1, \dots, r_n \right) \le C \frac{M_p(u, \rho_1, \dots, \rho_n)}{\rho_k - r_k},$$

where the positive constant C depends only on p and n.

Proof. *(a)* We may assume that $k = 1$. Applying the corresponding inequality for the case $n = 1$ (with fixed $r_2, \dots, r_n$), which holds for $0 < p \le \infty$, then the monotonicity of the integral means in arguments $r_2, \dots, r_n$, we obtain

$$M_p \left(\frac{\partial f}{\partial z_1}, r_1, r_2, \dots, r_n \right) \le C \frac{M_p(f, \rho_1, r_2 \dots, r_n)}{\rho_1 - r_1} \le C \frac{M_p(f, \rho_1, \rho_2 \dots, \rho_n)}{\rho_1 - r_1}.$$

(b) The proof of this statement is similar to the proof of *(a)*, with the difference that the corresponding one-dimensional inequality holds true for $1 \le p \le \infty$.

Lemma 5. *Let $0 < p, q < \infty$. Then for any $r_j \in (0,1)$, $j, k \in \{1, \dots, n\}$,*

$$M_p^q\left(\frac{\partial f}{\partial z_k}, r_1, \dots, r_n\right) \le \frac{C(p,q)}{R^{1+q}} \int_{r_k - R}^{r_k + R} M_p^q(u, r_1, \dots, r_{k-1}, t, r_{k+1}, \dots, r_n) dt,$$

for all $f \in H(U^n)$, $u = Re\ f$, and $r_k \in (0,1)$ such that $0 < R < r_k < R + r_k < 1$.

Proof. It suffices to apply the corresponding one variable inequality, see [9, Lemma 7].

Let $Ph(U^n)$ denote the set of all (real-valued) pluriharmonic functions on U^n. For the subspace of $\mathcal{L}_{\vec{\omega}}^{p,q}(U^n)$ consisting of pluriharmonic functions let $Ph_{\vec{\omega}}^{p,q}(U^n) = Ph(U^n) \cap \mathcal{L}_{\vec{\omega}}^{p,q}(U^n)$.

Lemma 6. *For any $a \in U^n$, the point evaluation $u \mapsto u(a)$ is a bounded linear functional on $Ph_{\vec{\omega}}^{p,q}(U^n)$ for all $0 < p, q < \infty$.*

Proof. The result follows from the Hardy–Littlewood inequality (HL-property) on $|u|^p$ analogously to [20, Lemma 2] or [14, Lemma 3].

3 Proof of Theorem 1

In order to prove the main theorem, we need some more auxiliary functions.

Suppose that the weights $\omega_j(r_j)$ are differentiable on $(0,1)$ and satisfy

$$\frac{\omega_j'(r_j)}{\omega_j^2(r_j)} \int_{r_j}^1 \omega_j(t)\, dt \le C, \qquad 0 < r_j < 1, \quad j = 1, \dots, n. \tag{3.1}$$

Their distortion functions are defined by

$$\psi_j(r_j) = \psi_{\omega_j}(r_j) = \frac{1}{\omega_j(r_j)} \int_{r_j}^1 \omega_j(t)\, dt, \qquad 0 < r_j < 1, \quad j = 1, \dots, n.$$

Given a weight ω_j, and $0 < q < \infty$, define the function φ_j on $(0,1)$ by

$$\varphi_j(r_j) \equiv \varphi_{q,\omega_j}(r_j) = \left(q \int_{r_j}^1 \omega_j(t)\, dt\right)^{-1/q}, \quad 0 < r_j < 1, \quad j = 1, \dots, n. \tag{3.2}$$

Note that each of the functions φ_j is strictly increasing on $(0,1)$. Let $\psi_\omega(r) = \prod_{j=1}^n \psi_j(r_j)$ and $\varphi_\omega(r) = \prod_{j=1}^n \varphi_j(r_j)$. It is easy to check that

$$\frac{\varphi_j(r_j)}{\varphi_j'(r_j)} = q\, \psi_j(r_j), \qquad \omega_j(r_j) = \frac{\varphi_j'(r_j)}{\varphi_j(r_j)^{1+q}}, \qquad j = 1, \dots, n, \tag{3.3}$$

and that condition (3.1) is equivalent to (2.1) with $\varphi = \varphi_j$.

Define also the measures on $(0,1)$ by

$$dm_{\varphi_j}(r_j) = \frac{\varphi_j'(r_j)}{\varphi_j(r_j)}dr_j, \qquad j = 1, \dots, n, \qquad dm_\varphi(r) = \prod_{j=1}^{n} dm_{\varphi_j}(r_j).$$

We may assume that $n = 2$. The proof for the case $n > 2$ is only technically complicated. We have to prove the inequality

$$\begin{aligned}\int_{(0,1)^2} M_p^q(f, r_1, r_2)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 &\le C|f(0,0)|^q \\ &+ C\int_{(0,1)^2} M_p^q\left(\frac{\partial f}{\partial z_1}, r_1, r_2\right)\psi_1^q(r_1)\omega_1(r_1)\omega_2(r_2)dr_1dr_2 \\ &+ C\int_{(0,1)^2} M_p^q\left(\frac{\partial f}{\partial z_2}, r_1, r_2\right)\psi_2^q(r_2)\omega_1(r_1)\omega_2(r_2)dr_1dr_2. \qquad (3.4)\end{aligned}$$

Denoting

$$F_0(r_1, r_2) = \frac{M_p(f, r_1, r_2)}{\varphi_1(r_1)\varphi_2(r_2)},$$

$$F_1(r_1, r_2) = \frac{M_p\left(\frac{\partial f}{\partial z_1}, r_1, r_2\right)}{\varphi_1'(r_1)\varphi_2(r_2)}, \qquad F_2(r_1, r_2) = \frac{M_p\left(\frac{\partial f}{\partial z_2}, r_1, r_2\right)}{\varphi_1(r_1)\varphi_2'(r_2)}, \qquad (3.5)$$

and taking into account (3.3) and (3.5), we can rewrite (3.4) in the form

$$\|F_0\|^q_{L^q(dm_\varphi)} \le C|f(0,0)|^q + C\|F_1\|^q_{L^q(dm_\varphi)} + C\|F_2\|^q_{L^q(dm_\varphi)}. \qquad (3.6)$$

Without loss of generality we may assume that $\varphi_j(0) = 1,\ j = 1, 2$.

We prove (3.6) only for $0 < p < 1$. The proof for the case $1 \le p \le \infty$ is similar and is omitted. Assuming that $F_1, F_2 \in L^q(dm_\varphi)$ and choosing two sequences $\{r_k\}_{k=0}^\infty$, $\{\rho_k\}_{k=0}^\infty$ as in Lemma 2, $\varphi_1(r_k) = e^k$, $\varphi_2(\rho_k) = e^k$, we obtain by Lemmas 1 and 3

$$\begin{aligned}\|F_0\|^q_{L^q(dm_\varphi)} &= \int_0^1\int_0^1 M_p^q(f, r, \rho)\frac{\varphi_1'(r)\,\varphi_2'(\rho)}{\varphi_1(r)^{1+q}\,\varphi_2(\rho)^{1+q}}drd\rho \\ &\le C\sum_{k=0}^\infty M_p^q(f, r_{k+1}, \rho_{k+1})\int_{r_k}^{r_{k+1}}\int_{\rho_k}^{\rho_{k+1}} \frac{\varphi_1'(r)\,\varphi_2'(\rho)}{\varphi_1(r)^{1+q}\,\varphi_2(\rho)^{1+q}}drd\rho \\ &= C\sum_{k=0}^\infty M_p^q(f, r_{k+1}, \rho_{k+1})\left(e^{-qk} - e^{-q(k+1)}\right)^2\frac{1}{q^2} \\ &\le C\sum_{k=0}^\infty e^{-2qk}\left(M_p^p(f, r_k, \rho_k)\right)^{q/p} \\ &\le C(M_p^p(f, 0, 0))^{q/p} \\ &\qquad + C\sum_{k=0}^\infty e^{-2qk}\left(M_p^p(f, r_{k+1}, \rho_{k+1}) - M_p^p(f, r_k, \rho_k)\right)^{q/p}\end{aligned}$$

$$\leq C|f(0,0)|^q + C\sum_{k=0}^{\infty} e^{-2qk}\Bigg[(r_{k+1}-r_k)^p M_p^p\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)$$

$$+ (\rho_{k+1}-\rho_k)^p M_p^p\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\Bigg]^{q/p}$$

$$\leq C|f(0,0)|^q + C\sum_{k=0}^{\infty} e^{-2qk}(r_{k+1}-r_k)^q M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)$$

$$+ C\sum_{k=0}^{\infty} e^{-2qk}(\rho_{k+1}-\rho_k)^q M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right),$$

where the involved constants $C = C(p, q, \varphi_1, \varphi_2) > 0$ depend only on p, q and the functions φ_1, φ_2. By Lagrange's theorem

$$r_{k+1} - r_k = (e-1)e^k\big(\varphi_1'(x_k)\big)^{-1}, \qquad \text{where} \qquad r_k < x_k < r_{k+1},$$

$$\rho_{k+1} - \rho_k = (e-1)e^k\big(\varphi_2'(y_k)\big)^{-1}, \qquad \text{where} \qquad \rho_k < y_k < \rho_{k+1}.$$

Hence

$$\|F_0\|^q_{L^q(dm_\varphi)} \leq C|f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_1'(x_k)\big)^{-q} e^{-qk}$$

$$+ C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_2'(y_k)\big)^{-q} e^{-qk}. \tag{3.7}$$

On the other hand,

$$\|F_1\|^q_{L^q(dm_\varphi)} = \int_0^1\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_1}, r, \rho\right)\frac{\big(\varphi_1'(r)\big)^{1-q}\varphi_2'(\rho)}{\varphi_1(r)\big(\varphi_2(\rho)\big)^{1+q}}dr d\rho$$

$$\geq \sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\int_{r_{k+1}}^{r_{k+2}}\frac{\big(\varphi_1'(r)\big)^{1-q}}{\varphi_1(r)}dr\right)\left(\int_{\rho_{k+1}}^{\rho_{k+2}}\frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}}d\rho\right).$$

Since the function $\varphi_2(\rho)$ is increasing, and

$$\int_{r_{k+1}}^{r_{k+2}}\frac{\varphi_1'(r)}{\varphi_1(r)}dr = 1, \qquad \int_{\rho_{k+1}}^{\rho_{k+2}}\frac{\varphi_2'(\rho)}{\varphi_2(\rho)}d\rho = 1,$$

by the mean value theorem for integrals, there exist numbers ξ_k, $r_{k+1} < \xi_k < r_{k+2}$, such that

$$\|F_1\|^q_{L^q(dm_\varphi)} \geq \sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_1'(\xi_k)\big)^{-q}\big(\varphi_2(\rho_{k+2})\big)^{-q}$$

$$\geq C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\big(\varphi_1'(\xi_k)\big)^{-q} e^{-qk}. \tag{3.8}$$

Similarly, there exist numbers η_k, $\rho_{k+1}<\eta_k<\rho_{k+2}$, such that

$$\|F_2\|^q_{L^q(dm_\varphi)} \ge C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_2'(\eta_k)\right)^{-q} e^{-qk}. \tag{3.9}$$

Combining inequalities (3.7)-(3.9), and using Lemma 2(a), we get

$$\begin{aligned}\|F_0\|^q_{L^q(dm_\varphi)} &\le C|f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_1'(x_k)\right)^{-q} e^{-qk}\\ &\qquad + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_2'(y_k)\right)^{-q} e^{-qk}\\ &\le C|f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_1'(\xi_k)\right)^{-q} e^{-qk}\\ &\qquad + C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_2}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_2'(\eta_k)\right)^{-q} e^{-qk}\\ &\le C|f(0,0)|^q + C\|F_1\|^q_{L^q(dm_\varphi)} + C\|F_2\|^q_{L^q(dm_\varphi)}.\end{aligned} \tag{3.10}$$

In order to obtain the reverse inequality first note that

$$\begin{aligned}\|F_0\|^q_{L^q(dm_\varphi)} &= \int_0^1\int_0^1 M_p^q(f,r,\rho)\frac{\varphi_1'(r)\,\varphi_2'(\rho)}{\varphi_1(r)^{1+q}\,\varphi_2(\rho)^{1+q}}drd\rho\\ &\ge \sum_{k=0}^{\infty} M_p^q(f,r_k,\rho_k)\int_{r_k}^{r_{k+1}}\int_{\rho_k}^{\rho_{k+1}}\frac{\varphi_1'(r)\,\varphi_2'(\rho)}{\varphi_1(r)^{1+q}\,\varphi_2(\rho)^{1+q}}drd\rho\\ &= \frac{1}{q^2}\sum_{k=0}^{\infty} M_p^q(f,r_k,\rho_k)\left(e^{-qk}-e^{-q(k+1)}\right)^2\\ &\ge C_q\sum_{k=0}^{\infty} e^{-2qk}M_p^q(f,r_k,\rho_k).\end{aligned} \tag{3.11}$$

On the other hand, employing Lemma 4, we have that

$$\begin{aligned}&\|F_1\|^q_{L^q(dm_\varphi)} = \int_0^1\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_1}, r, \rho\right)\frac{\left(\varphi_1'(r)\right)^{1-q}\varphi_2'(\rho)}{\varphi_1(r)\left(\varphi_2(\rho)\right)^{1+q}}drd\rho\\ &\le C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\int_{r_k}^{r_{k+1}}\frac{\left(\varphi_1'(r)\right)^{1-q}}{\varphi_1(r)}dr\right)\left(\int_{\rho_k}^{\rho_{k+1}}\frac{\varphi_2'(\rho)}{\left(\varphi_2(\rho)\right)^{1+q}}d\rho\right)\\ &\le C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_1'(x_k)\right)^{-q}\left(\varphi_2(\rho_k)\right)^{-q}\\ &= C\sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_1'(x_k)\right)^{-q}e^{-kq}\\ &\le C\sum_{k=0}^{\infty} M_p^q\left(f, r_{k+2}, \rho_{k+2}\right)(r_{k+2}-r_{k+1})^{-q}\left(\varphi_1'(x_k)\right)^{-q}e^{-kq}\end{aligned}$$

for some $x_k \in (r_k, r_{k+1})$. By Lagrange's theorem we have that

$$e^{k+2}(1-e^{-1}) = \varphi_1(r_{k+2}) - \varphi_1(r_{k+1}) = \varphi_1'(z_k)(r_{k+2} - r_{k+1}),$$

for some $z_k \in (r_{k+1}, r_{k+2})$. Hence by Lemma 2(a)

$$\begin{aligned} |f(0,0)|^q + \|F_1\|^q_{L^q(dm_\varphi)} &\leq |f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(f, r_{k+2}, \rho_{k+2}\right)\left(\frac{\varphi_1'(z_k)}{\varphi_1'(x_k)}\right)^q e^{-q(k+2)}e^{-qk} \\ &\leq |f(0,0)|^q + C\sum_{k=0}^{\infty} M_p^q\left(f, r_{k+2}, \rho_{k+2}\right) e^{2Mq}e^{-2q(k+1)} \\ &\leq C\sum_{k=0}^{\infty} M_p^q\left(f, r_k, \rho_k\right) e^{-2qk} \qquad (3.12) \end{aligned}$$

Similarly it can be proved that

$$|f(0,0)|^q + \|F_2\|^q_{L^q(dm_\varphi)} \leq C\sum_{k=0}^{\infty} M_p^q\left(f, r_k, \rho_k\right) e^{-2qk}. \qquad (3.13)$$

From (3.11)-(3.13) the inequality follows.

4 Pluriharmonic conjugates

In this section we discuss pluriharmonic functions in mixed norm spaces $Ph^{p,q}_{\vec{\omega}}(U^n)$. The problem of harmonic conjugation in mixed norm and Bergman spaces is classical and goes back to Hardy and Littlewood [5]. For pluriharmonic conjugation on the unit ball, unit polydisc and more general bounded symmetric domains in $\mathbb{C}^n$, see [8, 10, 11, 21], where standard weight functions were considered. For harmonic conjugation in mixed norm spaces on the unit disc, with general weights see [9, 14].

Theorem 2. *Let $1 \leq p \leq \infty, 0 < q < \infty$, and each of the weight functions $\omega_j(z_j)$, $j = 1, \ldots, n$, satisfies (3.1). Then $Ph^{p,q}_{\vec{\omega}}(U^n)$ is a self-conjugate space. Moreover, if $f \in H(U^n)$, $f = u + iv$, $u \in Ph^{p,q}_{\vec{\omega}}(U^n)$, and v is the pluriharmonic conjugate of u normalized so that $v(0) = 0$, then*

$$\|f\|_{p,q,\vec{\omega}} \leq C(p, q, \vec{\omega}, n)\|u\|_{p,q,\vec{\omega}}. \qquad (4.1)$$

Proof. Denoting

$$F_0(r_1, r_2) = \frac{M_p(f, r_1, r_2)}{\varphi_1(r_1)\varphi_2(r_2)} \qquad \text{and} \qquad F_3(r_1, r_2) = \frac{M_p(u, r_1, r_2)}{\varphi_1(r_1)\varphi_2(r_2)}, \qquad (4.2)$$

we can easily see that (4.1) is equivalent to

$$\|F_0\|_{L^q(dm_\varphi)} \le C(p,q,\vec{\omega},n)\|F_3\|_{L^q(dm_\varphi)}. \tag{4.3}$$

Since $1 \le p \le \infty$, the method of the proof of Theorem 1 works for this case as well. Indeed, similar to (3.11), we obtain

$$\|F_3\|^q_{L^q(dm_\varphi)} \ge C_q \sum_{k=0}^{\infty} e^{-2qk} M_p^q(u, r_k, \rho_k). \tag{4.4}$$

On the other hand, employing Lemma 4(b), we have that

$$\begin{aligned}\|F_1\|^q_{L^q(dm_\varphi)} &\le C \sum_{k=0}^{\infty} M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho_{k+1}\right)\left(\varphi_1'(x_k)\right)^{-q}\left(\varphi_2(\rho_k)\right)^{-q} \\ &\le C \sum_{k=0}^{\infty} M_p^q\left(u, r_{k+2}, \rho_{k+2}\right)\left(r_{k+2}-r_{k+1}\right)^{-q}\left(\varphi_1'(x_k)\right)^{-q} e^{-kq}\end{aligned}$$

for some $x_k \in (r_k, r_{k+1})$. By Lagrange's theorem and Lemma 2(a) we obtain

$$|f(0,0)|^q + \|F_1\|^q_{L^q(dm_\varphi)} \le C \sum_{k=0}^{\infty} M_p^q\left(u, r_k, \rho_k\right) e^{-2qk} \tag{4.5}$$

Similarly, (4.5) can be stated for F_2 instead of F_1. Thus,

$$\|F_0\|_{L^q(dm_\varphi)} \le C|f(0,0)| + C\|F_1\|_{L^q(dm_\varphi)} + C\|F_2\|_{L^q(dm_\varphi)} \le C\|F_3\|_{L^q(dm_\varphi)},$$

as desired.

An interesting question is whether Theorem 2 holds true for $0 < p < 1$. In this case we are able to prove a slightly weaker result.

Theorem 3. *Let $0 < p \le \infty, 0 < q < \infty$, and the weight functions $\omega_j(z_j)$, $j = 1, \ldots, n$, together with their corresponding functions $\varphi_j = \varphi_{\omega_j}$ defined by (3.2), satisfy (2.2). Then $Ph^{p,q}_{\vec{\omega}}(U^n)$ is a self-conjugate space. Moreover, if $f \in H(U^n), f = u + iv$, $u \in Ph^{p,q}_{\vec{\omega}}(U^n)$, and v is the pluriharmonic conjugate of u normalized so that $v(0) = 0$, then*

$$\|f\|_{p,q,\vec{\omega}} \le C(p,q,\vec{\omega},n)\|u\|_{p,q,\vec{\omega}}. \tag{4.6}$$

Proof. Again we have to prove the inequality (4.3). The proof is now based on Lemmas 2(b), 5 and 6. Note that in view of (3.10) it suffices to prove the inequality

$$|f(0,0)| + \|F_1\|_{L^q(dm_\varphi)} + \|F_2\|_{L^q(dm_\varphi)} \le C\|F_3\|_{L^q(dm_\varphi)}.$$

By the monotonicity of the integral means and the mean value theorem for integrals, we deduce that

$$\begin{aligned}
\|F_1\|^q_{L^q(dm_\varphi)} &= \int_0^1 \left[\int_0^1 M_p^q\left(\frac{\partial f}{\partial z_1}, r, \rho\right) \frac{\big(\varphi_1'(r)\big)^{1-q}}{\varphi_1(r)} dr\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho\right) \int_{r_k}^{r_{k+1}} \frac{\big(\varphi_1'(r)\big)^{1-q}}{\varphi_1(r)} dr\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&= C \int_0^1 \left[\sum_{k=0}^\infty M_p^q\left(\frac{\partial f}{\partial z_1}, r_{k+1}, \rho\right) \big(\varphi_1'(x_k)\big)^{-q}\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty M_p^q\left(\frac{\partial f}{\partial z_1}, \frac{r_{k+1}+r_{k+2}}{2}, \rho\right) \big(\varphi_1'(x_k)\big)^{-q}\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho
\end{aligned}$$

for some $x_k \in (r_k, r_{k+1})$. An application of Lemma 5 with $R = \frac{1}{2}(r_{k+2} - r_{k+1})$ and $r_1 \mapsto \frac{1}{2}(r_{k+1} + r_{k+2})$, $k \ge 0$, yields

$$\|F_1\|^q_{L^q(dm_\varphi)} \le C \int_0^1 \left[\sum_{k=0}^\infty \frac{\big(\varphi_1'(x_k)\big)^{-q}}{(r_{k+2}-r_{k+1})^{1+q}} \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)dt\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho.$$

Next, we apply Lagrange's theorem and Lemma 2(b) to obtain

$$\begin{aligned}
&\|F_1\|^q_{L^q(dm_\varphi)} \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty \frac{\big(\varphi_1'(x_k)\big)^{-q}\big(\varphi_1'(y_k)\big)^{q}}{(r_{k+2}-r_{k+1})\, e^{q(k+2)}} \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)dt\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty \frac{e^{-q(k+2)}}{r_{k+2}-r_{k+1}} \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)dt\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty (r_{k+2}-r_{k+1})^{-1} \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\big(\varphi_1(t)\big)^{-q}dt\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty \frac{\varphi_1'(y_k)}{\varphi_1(r_{k+2})-\varphi_1(r_{k+1})} \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\big(\varphi_1(t)\big)^{-q}dt\right] \frac{\varphi_2'(\rho)\, d\rho}{\big(\varphi_2(\rho)\big)^{1+q}},
\end{aligned}$$

where $r_{k+1} < y_k < r_{k+2}, \quad \varphi_1(r_k) = e^k$. Since the function $\varphi_1(t)$ is increasing, we get by Lemma 2(b)

$$\begin{aligned}
\|F_1\|^q_{L^q(dm_\varphi)} &\le C \int_0^1 \left[\sum_{k=0}^\infty \varphi_1'(y_k) \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho)\big(\varphi_1(t)\big)^{-1-q}dt\right] \frac{\varphi_2'(\rho)\, d\rho}{\big(\varphi_2(\rho)\big)^{1+q}} \\
&\le C \int_0^1 \left[\sum_{k=0}^\infty \int_{r_{k+1}}^{r_{k+2}} M_p^q(u,t,\rho) \frac{\varphi_1'(t)}{\big(\varphi_1(t)\big)^{1+q}} dt\right] \frac{\varphi_2'(\rho)}{\big(\varphi_2(\rho)\big)^{1+q}} d\rho \\
&\le C\|F_3\|^q_{L^q(dm_\varphi)}.
\end{aligned}$$

Similarly it can be proved that

$$\|F_2\|_{L^q(dm_\varphi)} \le C\|F_3\|_{L^q(dm_\varphi)}.$$

Finally, by Lemma 6,

$$|f(0,0)| = |u(0,0)| \le C\|F_3\|_{L^q(dm_\varphi)}.$$

This completes the proof of Theorem 3.

Note that although condition (2.2) is stronger than (2.1), the class of weight functions $\omega(z)$ satisfying (2.2) is still rather wide. For example,

$$\omega(r) = \left(\log \frac{1}{1-r}\right)^{\gamma} (1-r)^{\beta} \exp\left(\frac{-c}{(1-r)^{\alpha}}\right), \quad \alpha > 0, c > 0, \beta \in \mathbb{R}, \gamma \in \mathbb{R},$$

is a typical weight function satisfying (2.2), see [9].

Pluriharmonic conjugation makes it possible to extend Theorem 1 to pluriharmonic functions. The partial differential operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \overline{z}_j}$ are defined by

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \qquad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \qquad z_j = x_j + iy_j.$$

Theorem 4. *Let $u \in Ph(U^n)$ and one of the following two conditions holds:*

(a) $1 \le p \le \infty, 0 < q < \infty$, *and the weights $\omega_j(z_j)$, $j = 1, \dots, n$, satisfy condition* (3.1), *with distortion functions $\psi_j(z_j)$, $j = 1, \dots, n$.*

(b) $0 < p \le \infty, 0 < q < \infty$, *and the weight functions $\omega_j(z_j)$, $j = 1, \dots, n$, together with their corresponding functions $\varphi_j = \varphi_{\omega_j}$ defined by* (3.2), *satisfy* (2.2). *Then*

$$\|u\|_{p,q,\vec{\omega}} \asymp |u(0)| + \sum_{j=1}^{n} \left\|\psi_j \frac{\partial u}{\partial z_j}\right\|_{p,q,\vec{\omega}} \asymp |u(0)| + \sum_{j=1}^{n} \left\|\psi_j \frac{\partial u}{\partial \overline{z}_j}\right\|_{p,q,\vec{\omega}}. \tag{4.7}$$

Proof. Since the function u is real–valued, the second equivalence in (4.7) is obvious. Let now $f \in H(U^n)$, $f = u + iv$, and v be the pluriharmonic conjugate of u normalized so that $v(0) = 0$. Then by Theorems 1-3 and Cauchy-Riemann equations

$$|u(0)| + \sum_{j=1}^{n} \left\|\psi_j \frac{\partial u}{\partial z_j}\right\|_{p,q,\vec{\omega}} = |f(0)| + C\sum_{j=1}^{n} \left\|\psi_j \frac{\partial f}{\partial z_j}\right\|_{p,q,\vec{\omega}} \asymp \|f\|_{p,q,\vec{\omega}} \asymp \|u\|_{p,q,\vec{\omega}},$$

as desired.

Remark 2. It is not difficult to see that Theorem B holds for the case of holomorphic functions on the unit ball $B \subset \mathbb{C}^n$, where ∇f appears instead of f' in (1.3). Note that by the maximal theorem the inequality in Lemma 3 becomes

$$M_p^{\ell}(f,\rho) - M_p^{\ell}(f,r) \le C(\rho - r)^{\ell} M_p^{\ell}(\nabla f, \rho),$$

$0 < r < \rho < 1$, $f \in H(B)$, where $\ell = \min\{1, p\}$, $p \in (0, \infty]$.

References

[1] K. Avetisyan, Continuous inclusions and Bergman type operators in n-harmonic mixed norm spaces on the polydisc, *J. Math. Anal. Appl.* **291** (2) (2004), 727-740.

[2] K. Avetisyan and S. Stević, Equivalent conditions for Bergman space and Littlewood-Paley type inequalities, *J. Comput. Anal. Appl.* **9** (1) (2007), 15-28.

[3] G. Benke and D. C. Chang, A note on weighted Bergman spaces and the Cesàro operator, *Nagoya Math. J.* **159** (2000), 25-43.

[4] T. M. Flett, The dual of an inequality of Hardy and Littlewood and some related inequalities, *J. Math. Anal. Appl.* **38** (1972), 746-765.

[5] G. H. Hardy and J. E. Littlewood, Some properties of conjugate functions, *J. Reine Angew. Math.* **167** (1932), 405–423.

[6] S. Li, Derivative free characterization of Bloch spaces, *J. Comput. Anal. Appl.* **10** (2) (2008), 253-258.

[7] S. Li and S. Stević, Integral type operators from mixed-norm spaces to α-Bloch spaces, *Integral Transform. Spec. Funct.* **18** (7) (2007), 485-493.

[8] J. Mitchell, Lipschitz spaces of holomorphic and pluriharmonic functions on bounded symmetric domains in $\mathbb{C}^N (N > 1)$, *Annales Polon. Math.* **39** (1981), 131–141.

[9] M. Pavlović and J. A. Peláez, An equivalence for weighted integrals of an analytic function and its derivative, *Math. Nachr.* (to appear).

[10] J. H. Shi, On the rate of growth of the means M_p of holomorphic and pluriharmonic functions on bounded symmetric domains of $\mathbb{C}^n$, *J. Math. Anal. Appl.* **126** (1987), 161–175.

[11] J. H. Shi, Inequalities for the integral means of holomorphic functions and their derivatives in the unit ball of $\mathbb{C}^n$, *Trans. Amer. Math. Soc.* **328** (1991), 619–637.

[12] A. Siskakis, Weighted integrals of analytic functions, *Acta Sci. Math.* **66** (2000), 651-664.

[13] S. Stević, A note on weighted integrals of analytic functions, *Bull. Greek Math. Soc.* **46** (2002), 3-9.

[14] S. Stević, Weighted integrals and conjugate functions in the unit disk, *Acta Sci. Math.* **69** (2003), 109-119.

[15] S. Stević, Weighted integrals of holomorphic functions on the polydisc, *Zeit. Anal. Anwen.* **23** (2004), 577–587.

[16] S. Stević, Weighted integrals of holomorphic functions on the unit polydisk II, *Z. Anal. Anwendungen* **23** (4) (2004), 775-782.

[17] S. Stević, Boundedness and compactness of an integral operator on mixed norm spaces on the polydisc, *Sibirsk. Mat. Zh.* **48** (3) (2007), 694-706.

[18] S. Stević, On Ren-Kähler's paper "Hardy Littlewood inequalities and Q_p-spaces *Z. Anal. Anwendungen* **24** (2) (2005), 375-388", *Z. Anal. Anwendungen* **26** (4) (2007), 473-480.

[19] S. Stević, Weighted composition operators between Mixed norm spaces and H_α^∞ spaces in the unit ball, *J. Inequal. Appl.* Vol. 2007, Article ID 28629, (2007), 9 pages.

[20] S. Stević, Holomorphic functions on the mixed norm spaces on the polydisc, *J. Korean Math. Soc.* **45** (1) (2008), 63-78.

[21] M. Stoll, On the rate of growth of the means M_p of holomorphic and pluriharmonic functions on the ball, *J. Math. Anal. Appl.* **93** (1983), 109–127.

[22] K. Zhu, The Bergman spaces, the Bloch spaces, and Gleason's problem, *Trans. Amer. Math. Soc.* **309** (1988), 253-268.

[23] K. Zhu, Duality and Hankel operators on the Bergman spaces of bounded symmetric domains, *J. Funct. Anal.* **81** (1988), 260-278.

[24] X. Zhu, Generalized weighted composition operators from Bloch-type spaces to weighted Bergman spaces, *Indian J. Math.* **49** (2) (2007), 139-149.

[25] X. Zhu, Products of differentiation, composition and multiplication from Bergman type spaces to Bers type spaces, *Integral Transform. Spec. Funct.* **18** (3) (2007), 223-231.