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On 3D Riesz systems of harmonic conjugates

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Abstract. This note announces some results that will be presented in the forthcoming paper [10]. In continuation to these studies we discuss a constructive approach for the generation of harmonic conjugates to find nullsolutions to the Riesz system in $\mathbb{R}^3$. This class of solutions coincides with the subclass of monogenic functions with values in the reduced quaternions. The algorithm for harmonic conjugates is presented by means of an integral representation. Additionally, we discuss the weighted (monogenic) Hardy and Bergman spaces on the unit ball in $\mathbb{R}^3$ consisting of functions with values in the reduced quaternions. We end up showing the boundedness of the underlying harmonic conjugation operators in certain weighted spaces.

Keywords: Quaternion analysis, Riesz system, monogenic functions, harmonic conjugates, Hardy space, Bergman space.
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QUATERNION ANALYSIS

Let

$$\mathbb{H} := \{\mathbf{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} : q_l \in \mathbb{R}, l = 0,1,2,3\}$$

be the Hamiltonian skew field, where the imaginary units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are subject to the multiplication rules: $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$; $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$.

Consider the subset

$$\mathscr{A} := \mathrm{span}_{\mathbb{R}}\{1, \mathbf{i}, \mathbf{j}\} \subset \mathbb{H},$$

then the real vector space $\mathbb{R}^3$ may be embedded in $\mathscr{A}$ via the identification of $x := (x_0, x_1, x_2) \in \mathbb{R}^3$ with the reduced quaternion $\mathbf{x} := x_0 + x_1\mathbf{i} + x_2\mathbf{j} \in \mathscr{A}$. The conjugate of $\mathbf{x}$ is $\overline{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j}$, and the norm $|\mathbf{x}|$ of $\mathbf{x}$ is defined by $|\mathbf{x}| = \sqrt{\mathbf{x}\overline{\mathbf{x}}} = \sqrt{\overline{\mathbf{x}}\mathbf{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2}$. In the sequel, let B denote the unit ball in $\mathbb{R}^3$ centered at the origin, and S its boundary. We say that

$$\mathbf{f} : B \subset \mathbb{R}^3 \longrightarrow \mathscr{A}, \quad \mathbf{f}(x) = [\mathbf{f}(x)]_0 + [\mathbf{f}(x)]_1\mathbf{i} + [\mathbf{f}(x)]_2\mathbf{j}$$

is a reduced quaternion-valued function or, in other words, an $\mathscr{A}$-valued function, where $[\mathbf{f}]_i$ $(i = 0,1,2)$ are real-valued functions in B. Properties (like integrability, continuity or differentiability) of $\mathbf{f}$ are defined componentwise. For a real-differentiable $\mathscr{A}$-valued function $\mathbf{f}$ that has continuous first partial derivatives, the (reduced) quaternionic operators

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}, \quad \overline{D} = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2}$$

are called, respectively, generalized and conjugate generalized Cauchy-Riemann operators on $\mathbb{R}^3$.

It is worth noting that for a continuously real-differentiable scalar-valued function the application of the operator D coincides with the usual gradient, ∇.

A continuously real-differentiable $\mathscr{A}$-valued function $\mathbf{f}$ is said to be monogenic if $D\mathbf{f} = 0$, which is equivalent to the system

$$\begin{cases} \dfrac{\partial[\mathbf{f}]_0}{\partial x_0} - \dfrac{\partial[\mathbf{f}]_1}{\partial x_1} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_2} = 0 \\[2ex] \dfrac{\partial[\mathbf{f}]_0}{\partial x_1} + \dfrac{\partial[\mathbf{f}]_1}{\partial x_0} = 0, \; \dfrac{\partial[\mathbf{f}]_0}{\partial x_2} + \dfrac{\partial[\mathbf{f}]_2}{\partial x_0} = 0, \; \dfrac{\partial[\mathbf{f}]_1}{\partial x_2} - \dfrac{\partial[\mathbf{f}]_2}{\partial x_1} = 0 \end{cases}$$

or, in a more compact form:

$$\begin{cases} \mathrm{div}\, \overline{\mathbf{f}} & = & 0 \\ \mathrm{curl}\, \overline{\mathbf{f}} & = & 0. \end{cases}$$

Any monogenic $\mathscr{A}$-valued function is two-sided monogenic. This means it satisfies simultaneously the equations $D\mathbf{f} = \mathbf{f}D = 0$. We may point out that the 3-tuple $\overline{\mathbf{f}}$ is said to be a system of conjugate harmonic functions in the sense of Stein-Weiß [12, 13], and the above system is called the Riesz system [11].

For $\mathbf{f}(x) = \mathbf{f}(r\zeta)$ in B $(0 \leq r < 1,\ \zeta \in S)$, its integral means are defined by

$$\mathscr{M}_p(\mathbf{f}; r) := \left(\int_S |\mathbf{f}(r\zeta)|^p \, d\sigma(\zeta) \right)^{1/p},$$

for $0 \leq r < 1$, and $0 < p < \infty$. Here $d\sigma$ is the surface area measure on S normalized so that $\sigma(S) = 1$.

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We will denote by $h(B;\mathbb{X})$ the set of harmonic functions on B with values in $\mathbb{X}$ ($\mathbb{X} = \mathbb{R}$ or $\mathscr{A}$). As usual, the Hardy spaces of monogenic or harmonic functions are defined as follows

$$h^p(B) = \left\{u \in h(B;\mathbb{R}) \text{ or } u \in h(B;\mathscr{A}) : \|u\|_{h^p(B)} < \infty\right\}$$
$$\mathscr{H}^p(B) = h^p(B) \cap \ker D.$$

The norm in the Hardy space of $\mathbf{f}$ in B is defined by

$$\|\mathbf{f}\|_{h^p(B)} = \sup_{0<r<1} \mathscr{M}_p(\mathbf{f};r), \quad 1 \leq p < \infty.$$

Throughout this paper the letters $C(\alpha,\beta,\ldots), C_p$ etc. stand for positive different constants depending only on the parameters indicated not necessarily the same in each instance. For any $A, B > 0$ the notation $A \approx B$ denotes the two-sided estimate $c_1 A \leq B \leq c_2 A$ with some positive constants c_1 and c_2 independent of the variable involved. For any p so that $1 \leq p < \infty$, we define the conjugate index p' as $p' = p/(p-1)$.

Further, let $1 < p < \infty$, and $\alpha > -1$. We define the weighted Bergman space of $\mathbf{f}$ on B by

$$L^p_\alpha(B) = \left\{ \mathbf{f} \text{ measurable in } B : \|\mathbf{f}\|^p_{L^p_\alpha(B)} = \int_B (1-|x|)^\alpha |\mathbf{f}(x)|^p dV(x) < \infty \right\},$$

where dV is the normalized measure such that $V(B) = 1$. Let the subspaces of $L^p_\alpha(B)$ consisting of harmonic or monogenic functions be

$$h^p_\alpha(B) = L^p_\alpha(B) \cap h(B),$$

and

$$\mathscr{H}^p_\alpha(B) = L^p_\alpha(B) \cap \ker D.$$

In polar coordinates we have $dV(x) = 3r^2 dr d\sigma(\zeta)$. Therefore

$$\|\mathbf{f}\|_{L^p_\alpha(B)} = \left(3\int_0^1 (1-r)^\alpha \mathscr{M}^p_p(\mathbf{f};r)\, r^2\, dr\right)^{1/p}.$$

The norm of a monogenic function in the weighted Hardy space is defined by

$$\|\mathbf{f}\|_{h(p,\beta)(B)} = \sup_{0<r<1} (1-r)^\beta \mathscr{M}_p(\mathbf{f};r),$$

for $1 \leq p < \infty$, $\beta > 0$. We define

$$h(p,\beta)(B) = \left\{u \in h(B;\mathbb{R}) \text{ or } h(B;\mathscr{A}) : \|u\|_{h(p,\beta)(B)} < \infty\right\},$$
$$\mathscr{H}(p,\beta)(B) = h(p,\beta)(B) \cap \ker D.$$

It should be observed that for $\beta = 0$ we obviously come to the usual Hardy spaces h^p and $\mathscr{H}^p$.

GENERATION OF $\mathscr{A}$-VALUED MONOGENIC FUNCTIONS BY CONJUGATE HARMONICS

Harmonic conjugates in the context of quaternion and Clifford analysis

There has been much recent interest in studying the classical problem of harmonic conjugates within the context of quaternion and Clifford analysis. A thorough treatment is listed in the bibliography, *e.g.* Sudbery [14], Xu [15], Brackx, Delanghe and Sommen [2], Brackx and Delanghe [3], Avetisyan, Gürlebeck and Sprößig [1], and Morais *et al.* [6, 7, 9]. The main point in the approach presented in [2, 3] as well as Sudbery's formula [14] is the construction of harmonic conjugates in $\mathbb{R}^4$ "function by function". So far no effort has been made to the question to which function spaces these conjugate harmonics and the whole monogenic function belong. In [1] this question was studied for conjugate harmonics via Sudbery's formula in the scale of Bergman spaces. These results are, however, not applicable to functions with values in the reduced quaternions.

A recent article [6] (cf. [7]) treats the problem of conjugate harmonicity also, proposing an algorithm for the generation of polynomial solutions to the Riesz system in $\mathbb{R}^3$; it uses a solid spherical monogenics expansion i.e. homogeneous monogenic polynomials which offer a refinement of the notion of solid spherical harmonics. Working with such expansion it becomes possible to overcome problems that lead in [2] and [3] to the necessity to solve a Poisson equation (resulting then in an existence theorem), so that we can express explicitly the general form of a pair of conjugate harmonic functions. Besides this, in [10] Morais *et al.* have proposed a new algorithm to the explicit construction of a "unique" pair of conjugate harmonic functions in $\mathbb{R}^3$ through its first coordinate, and we believe it is the simplest and shortest introduced so far.

Construction of a 3D Riesz system by its first component

We begin by recalling the notion of harmonic conjugates in the context of quaternion analysis.

Definition 1 (Conjugate harmonic functions) *Let U be a harmonic function defined in an open subset Ω of $\mathbb{R}^3$. A vector-valued harmonic function V in Ω is called conjugate harmonic to U if $\mathbf{f} := U + V$ is monogenic in Ω. The pair $(U;V)$ is called a pair of conjugate harmonic functions in Ω.*

We recall from [10], the following result.

Theorem 2 (Construction of a harmonic conjugate) *Let U be a scalar-valued harmonic function defined in B. Define*

$$[V(x)]_1 := -x_0 \int_0^1 \frac{\partial U(\rho x_0, x_1, x_2)}{\partial x_1}\, d\rho + W(x_1,x_2),$$

where the function $W(x_1,x_2)$ is chosen so that $\Delta_{(x_1,x_2)} W = \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0 \partial x_1}$, and

$$[V(x)]_2 := \int_0^1 \left[- \begin{vmatrix} x_0 & x_2 \\ \frac{\partial U(tx)}{\partial x_0} & \frac{\partial U(tx)}{\partial x_2} \end{vmatrix} + \begin{vmatrix} x_1 & x_2 \\ \frac{\partial [V(tx)]_1}{\partial x_1} & \frac{\partial [V(tx)]_1}{\partial x_2} \end{vmatrix} \right] dt.$$

Then the function $\mathbf{f} := U + \mathbf{i}[V]_1 + \mathbf{j}[V]_2$ is monogenic in B. Moreover, the most general monogenic function **g** *having U as its scalar part is given by*

$$\mathbf{g}(x) = \mathbf{f}(x) + \varphi(x_1,x_2),$$

where $\varphi(x_1,x_2)$ is such that $D\varphi = \overline{D}\varphi = 0$.

HARMONIC CONJUGATES IN WEIGHTED MONOGENIC HARDY SPACES

In this section we discuss the weighted (monogenic) Hardy space on the unit ball of $\mathbb{R}^3$ consisting of functions with values in the reduced quaternions.

The following lemma can be found, for example, in [5, p. 251].

Lemma 3 *Let $w(x)$ be a nonnegative subharmonic function in B, and*

$$\mathscr{M}(w;r) = \int_S w(r\xi)\, d\sigma(\xi), \qquad 0 \le r < 1.$$

If $\mathscr{M}(w;r)$ is bounded on $0 \le r < 1$, then $w(x)$ has a harmonic majorant $u(x) \in h^1(B)$ on B so that

$$w(x) \le u(x),\ x \in B,$$

and

$$\|u\|_{h^1(B)} \le C \sup_{0<r<1} \mathscr{M}(w;r).$$

Lemma 4 *Let $1 \le p < \infty$, $\alpha > -1$, $\beta > 0$, m be a positive integer, and $\lambda = (\lambda_0,\lambda_1,\lambda_2) \in \mathbb{N}_0^3$. Then for all $\mathscr{A}$-valued harmonic functions f it holds*

$$\begin{aligned} \|f\|_{h(p,\beta)(B)} &\approx \sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right| + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|_{h(p,\beta+m)(B)}, \\ \|f\|_{L^p_\alpha(B)} &\approx \sum_{|\lambda|<m} \left|\partial^\lambda f(0)\right| + \sum_{|\lambda|=m} \left\|\partial^\lambda f\right\|_{L^p_{\alpha+pm}(B)}, \end{aligned}$$

where ∂^λ denote the partial differential operator of the order $|\lambda| = \lambda_0+\lambda_1+\lambda_2$ with respect to x_0,x_1,x_2. In particular,

$$\begin{aligned} \|f\|_{h(p,\beta)(B)} &\approx |f(0)| + \left\|\nabla f\right\|_{h(p,\beta+1)(B)}, \\ \|f\|_{L^p_\alpha(B)} &\approx |f(0)| + \left\|\nabla f\right\|_{L^p_{\alpha+p}(B)}. \end{aligned}$$

The involved constants depend on the parameters p, m, α, β only.

We now briefly recall some basic facts about the Poisson kernel, which will be used to estimate the size of certain integrals.

Lemma 5 (see [8]) *Let Ω be a bounded domain in $\mathbb{R}^3$ with C^2-boundary $\partial\Omega$, and let $P_\Omega(x,y)$ be the Poisson kernel for Ω. Then*

$$P_\Omega(x,y) \approx \frac{\operatorname{dist}(x,\partial\Omega)}{|x-y|^3}, \qquad x \in \Omega,\ y \in \partial\Omega.$$

For any fixed $\rho, r \in (0,1)$, we also consider the following bounded domain in $\mathbb{R}^3$:

$$E_{\rho,r} := \left\{ x = (x_0,x_1,x_2) \in \mathbb{R}^3 :\ \frac{x_0^2}{\rho^2 r^2} + \frac{x_1^2}{r^2} + \frac{x_2^2}{r^2} < 1 \right\},$$

which denotes the inner domain of the oblate spheroid $\partial E_{\rho,r}$.

Now we estimate the size of the Poisson kernel for $E_{\rho,r}$.

Lemma 6 *Let $P_{E_{\rho,r}}(x,y)$ be the Poisson kernel for $E_{\rho,r}$. Then*

$$P_{E_{\rho,r}}(x,y) \approx \frac{\operatorname{dist}(x,\partial E_{\rho,r})}{|x-y|^3}, \qquad x \in E_{\rho,r},\ y \in \partial E_{\rho,r},$$

in particular,

$$P_{E_{\rho,r}}(0,y) \approx \frac{\rho\, r}{|y|^3}, \qquad y \in \partial E_{\rho,r}.$$

Before we prove the main theorem, we state two more lemmas.

Lemma 7 *For any $\alpha > 0$, and $\beta > 1$ it holds*

$$\int_0^1 \frac{t^{\alpha-1}}{(1-tr)^\beta} dt \sim \frac{1}{(\beta-1)(1-r)^{\beta-1}}$$

as $r \to 1^-$.

Lemma 8 (see [17]) *Let $w = w(x_1,x_2)$ be a nonnegative superharmonic function in the unit disk*

$$\mathbb{D} := \{x_1^2 + x_2^2 < 1\},$$

and $\gamma > -1$, $0 < p < 2+\gamma$. Then for any point $a \in \mathbb{D}$

$$\|w\|_{L^p_\gamma(\mathbb{D})} \le C(p,\gamma,a)\, w(a).$$

We now formulate the main result of this section.

Theorem 9 *Let U be a scalar-valued harmonic function defined in B. Let also $W(x_1,x_2)$ be a solution of the equation*

$$\Delta_{(x_1,x_2)}W = \frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\,\partial x_1},$$

such that $W(a)$ is finite for some point $a=(a_1,a_2)$, $a_1^2+a_2^2<1$. If $U\in h(p,\beta)(B)$ for some $\beta>0$ and $1<p<\infty$, then there exist a monogenic function $\mathbf{f}$ so that $\mathbf{f}\in\mathscr{H}(p,\beta)(B)$ and $[\mathbf{f}]_0=U$ in B, and a constant $C(p,\beta,a)<\infty$ such that

$$\|\mathbf{f}\|_{\mathscr{H}(p,\beta)(B)} \leq C(p,\beta,a)\Big(\|U\|_{h(p,\beta)(B)}+|W(a)|\Big).$$

HARMONIC CONJUGATES IN WEIGHTED MONOGENIC BERGMAN SPACES

In the present section we shall see that a similar result to Theorem 9 holds for weighted Bergman spaces $\mathscr{H}^p_\alpha(B)$ for any range $\alpha>-1$ also.

To begin with, we state the following version of the Hardy inequality [4, p.490].

Lemma 10 *If $1\leq p<\infty$, $\gamma<-1<\alpha$, and $h(r)\geq 0$, then*

$$\int_0^1 (1-r)^\alpha r^\gamma \left(\int_0^r h(t)dt\right)^p dr \leq C\int_0^1 (1-r)^{\alpha+p} r^{\gamma+p} h^p(r)dr,$$

where the constant C depends only on the parameters p,α,γ.

In the next lemma we present a useful estimate on weights.

Lemma 11 *Let $1\leq p<\infty$, and $\gamma<-1<\alpha$. Then for all $u\in h(B)$ there exists a constant $C(p,\alpha,\gamma)<\infty$ such that*

$$\left(\int_0^1 (1-r)^\alpha \mathscr{M}_p^p(u;r)\, r^\gamma\, dr\right)^{1/p} \leq C(p,\alpha,\gamma)\|u\|_{L^p_\alpha(B)}.$$

Our main tool in this section is the following theorem.

Theorem 12 *Let U be a scalar-valued harmonic function in B. Let also $W(x_1,x_2)$ be a solution of the equation $\Delta_{(x_1,x_2)}W=\frac{\partial^2 U(0,x_1,x_2)}{\partial x_0\,\partial x_1}$, such that $W(a)$ is finite for some point $a=(a_1,a_2)$, $a_1^2+a_2^2<1$. If $U\in h^p_\alpha(B)$ for some $\alpha>-1$ and $1<p<\infty$, then there exist a monogenic function $\mathbf{f}$ so that $\mathbf{f}\in\mathscr{H}^p_\alpha(B)$ and $[\mathbf{f}]_0=U$ in B, and a constant $C(p,\alpha,a)<\infty$ such that*

$$\|\mathbf{f}\|_{L^p_\alpha(B)} \leq C(p,\alpha,a)\Big(\|U\|_{L^p_\alpha(B)}+|W(a)|\Big).$$

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